

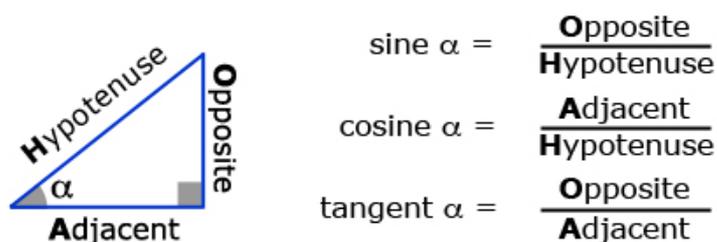
## Basic numerical skills: TRIANGLES AND TRIGONOMETRIC FUNCTIONS

### 1. Introduction and the three basic trigonometric functions (simple)

Triangles are basic geometric shapes comprising three straight lines, each of which is joined to the two others at its ends. A triangle has three included angles (angles defined by two of the sides) and together these angles sum to 180°. We will start with a special sort of triangle, where one of the included angles is a right-angle (90°). The lengths of the sides of a right-angled triangle can be calculated using Pythagoras' Theorem, and this is described graphically in the Appendix to this document.

Here, the main aim is to introduce a set of relationships between the sides and angles of triangles which form the basis of **trigonometry**. Most people encounter trigonometric functions in the context of geometry. However, some trigonometric functions are used in other applications, including the construction of regular cycles in models of physical and biological systems.

The three basic trigonometric functions relate an angle to the ratio between pairs of sides of a right-angled triangle. In this diagram, the angle  $\alpha$  (Greek 'alpha') is at the junction between the hypotenuse, the longest side of the right-angled triangle, and a side designated the 'adjacent'. The side opposite angle  $\alpha$  is designated the 'opposite'. Three ratios between pairs of sides are the sine, cosine and tangent:



You will notice that for the third angle of the triangle, which we can designate  $\beta$ :

$$\begin{aligned} \beta &= 90 - \alpha \\ \text{sine } \beta &= \text{cosine } \alpha \\ \text{cosine } \beta &= \text{sine } \alpha \\ \text{tangent } \beta &= 1/\text{tangent } \alpha \end{aligned}$$

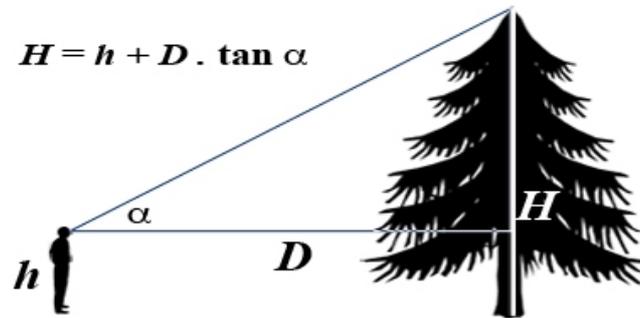
Given a value for one of the two non-right angles in a right-angled triangle, and the length of any side, it is possible to calculate the length of either of the other two sides. Conversely, given the length of any two sides, it is possible to calculate the value of either non-right angle from one of the trigonometric functions shown in the figure.

The three functions are usually written in their abbreviated forms, which are 'sin', 'cos' and 'tan'. These terms are used in spreadsheet calculations, for instance as SIN(A12), where the value contained in cell A12 (in brackets) is termed the 'argument' of the function.

### 2. How high is that tree? (simple)

A simple application of trigonometry is to calculate the height of an object that cannot be measured directly. For instance, it is seldom practical to measure the height of a tree with a tape measure, but it can be accomplished easily by using an instrument called a clinometer to measure the angle of a line between the horizontal and a line from the observer to the top of the tree, when the distance of the observer from the base of the tree is known:

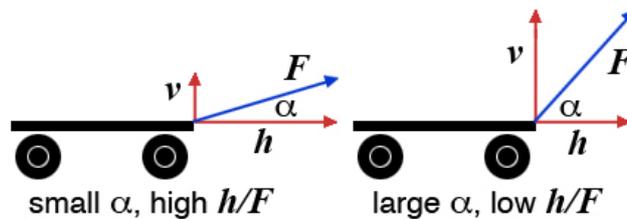
then,  $\tan \alpha = (\text{height of tree} - \text{height of observer})/\text{distance to base of tree}$   
 $\text{height of tree} = (\tan \alpha/\text{distance to base of tree}) + \text{height of observer}$



Obviously, if the clinometer were set on the ground, level with the base of the tree, there would be no need to include the height of the observer in the calculation.

### 3. Triangle of forces (intermediate)

Any force has both magnitude and direction. Imagine someone pulling along a loaded trolley, attached to them by a rope and harness. If the rope makes a very shallow angle with the direction of travel, most of the force generated by the person pulling is applied to moving the trolley. Conversely, if the rope is at a steep angle, some of the force is trying to lift the trolley rather than pulling it along. Look at these two diagrams, where the force along the rope,  $F$ , has been divided into a horizontal component  $h$  and a vertical component  $v$ :



In both cases, the magnitude and direction of the pulling force are represented by the hypotenuse of a right-angled triangle. We can use trigonometry to determine the proportion of the force that acts horizontally, pulling the trolley along. If we assume that the force is the same in each case, this is shown as the length of the hypotenuse,  $F$ . The horizontal force,  $h$ , is the side of the triangle that is adjacent to the angle between the rope and the horizontal, which we can again represent as  $\alpha$ . (For simplicity, the third side of the triangle, which is equal to  $v$ , has been omitted from the diagram.) The trigonometric function that relates the hypotenuse and adjacent is the cosine, so that:

$$\cos \alpha = h/F$$

$$h = F \cdot \cos \alpha$$

If the rope were horizontal, the angle  $\alpha$  would be zero and its cosine would have the value 1, so that the entire towing force acts to pull the trolley along. As the rope is inclined at a greater angle to the horizontal, the value of the cosine decreases. When the rope is vertical,  $\cos \alpha = 0$  and there is no horizontal force – all the person's effort is going into trying to lift the trolley off the ground! At an angle of  $60^\circ$ , half of the force along the rope is acting horizontally.

Angle of rope above horizontal	0°	15°	30°	45°	60°	75°
Proportion of force along rope acting to tow the trolley	100%	97%	87%	71%	50%	26%

#### 4. Other types of triangles (intermediate)

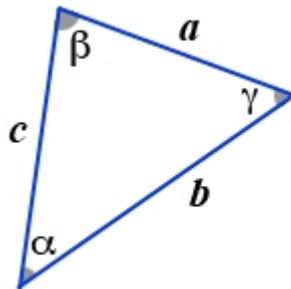
So far, we have looked at trigonometric functions in the context of a right-angled triangle, where one of the angles is 90°. There are other types of triangles:

- Equilateral triangles have all three sides of equal length, and all angles are 60°
- Isosceles triangles have two sides of equal length, so that two angles are also equal
- Scalene triangles have no sides or angles equal

For all triangles:

- The sum of the three angles of a triangle is always 180°.
- The longest side is opposite to the largest angle.
- A triangle can only have one angle that is equal to 90° (a right angle) or greater (an obtuse angle).

There are two important trigonometric rules that apply to all triangles. First, look at this scalene triangle with sides  $a$ ,  $b$  and  $c$  opposite angles  $\alpha$ ,  $\beta$  and  $\gamma$  respectively:



The **sine rule** states that:

$$a/\sin\alpha = b/\sin\beta = c/\sin\gamma$$

This means that all of the sides and angles of any triangle can be calculated if one angle and its opposite side, plus one other angle or side are known.

The **cosine rule** states that:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos\alpha \\ b^2 &= a^2 + c^2 - 2ac \cos\beta \\ c^2 &= a^2 + b^2 - 2ab \cos\gamma \end{aligned}$$

This allows the length of one side of a triangle to be calculated if the lengths of the other two sides and the size of the included angle are known, or any angle if the lengths of all three sides are known.

#### 5. Degrees and radians (intermediate)

We are used to measuring angles in degrees, where a right angle is 90°, the angle subtended by a straight line is 180° and a full revolution of a circle is 360°. We can also measure angles using a unit called a 'radian', which is based on the relationship between the circumference of a circle and its radius. Imagine an arc of a circle whose length is equal to the circle's radius. If you draw the

radii at each end of the arc, the angle that they make at the centre of the circle is 1 radian. A full revolution of a circle is expressed as  $2\pi$  radians, so:

$$90^\circ = 0.5\pi \text{ radians}$$

$$180^\circ = 1\pi \text{ radians}$$

$$270^\circ = 1.5\pi \text{ radians}$$

$$360^\circ = 2\pi \text{ radians}$$

Generally, if we have an angle measured in degrees, it can be converted to radians using the following formula:

$$\alpha^\circ = (\alpha \cdot \pi/180) \text{ radians}$$

and conversely:

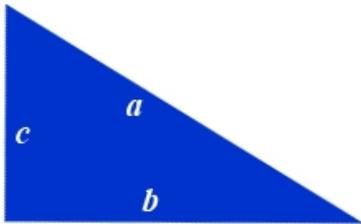
$$\alpha \text{ radians} = (\alpha \cdot 180/\pi)^\circ$$

Note spreadsheets and many calculators use radians as the default measurement when calculating trigonometric functions. Thus  $\text{SIN}(2)$  returns the result 0.91, which is the sine of 2 radians, whereas the sine of  $2^\circ$  is 0.035.

## A geometrical proof of Pythagoras' theorem

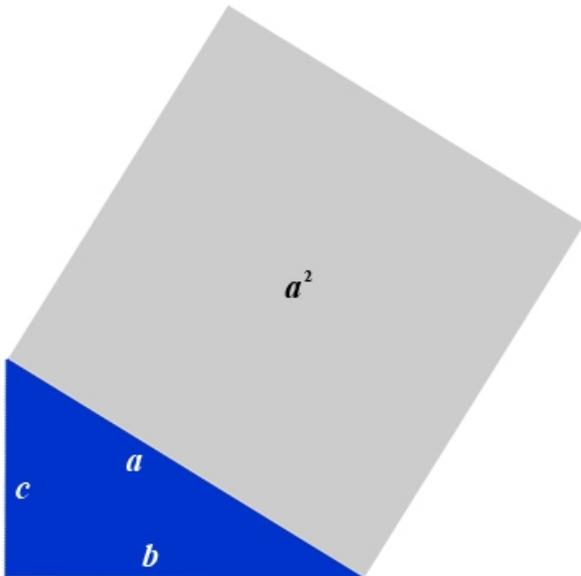
These seven diagrams illustrate a simple geometric proof of Pythagoras' theorem. This states that for a right-angled triangle, the square of the length of the hypotenuse (the longest side of the triangle, opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

(This illustration is based on an animation produced by the University of Waterloo, Ontario. Visit: <http://www.math.uwaterloo.ca/navigation/ideas/grains/pythagoras.shtml>)

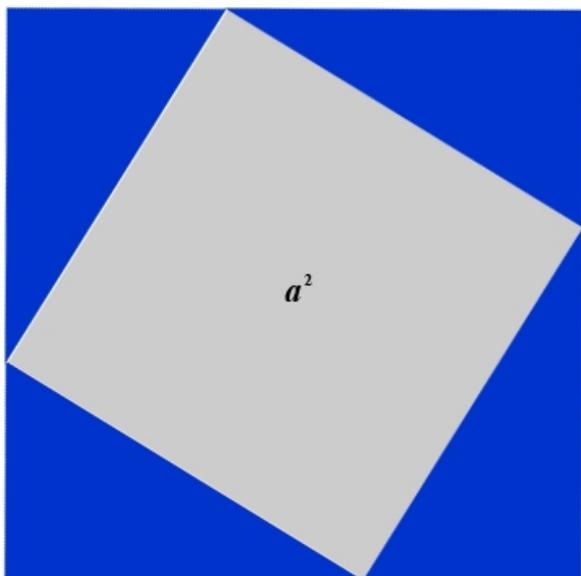


**Figure 1.** This right-angled triangle has three sides  $a$ ,  $b$  and  $c$ . Side  $a$  is the hypotenuse, and Pythagoras' theorem states that:

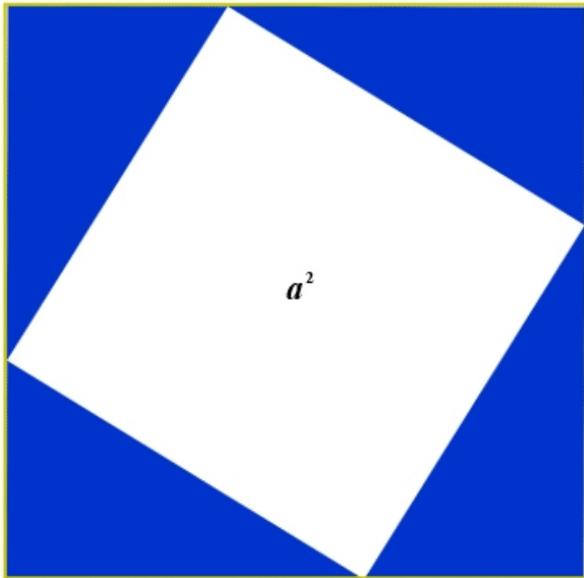
$$a^2 = b^2 + c^2$$



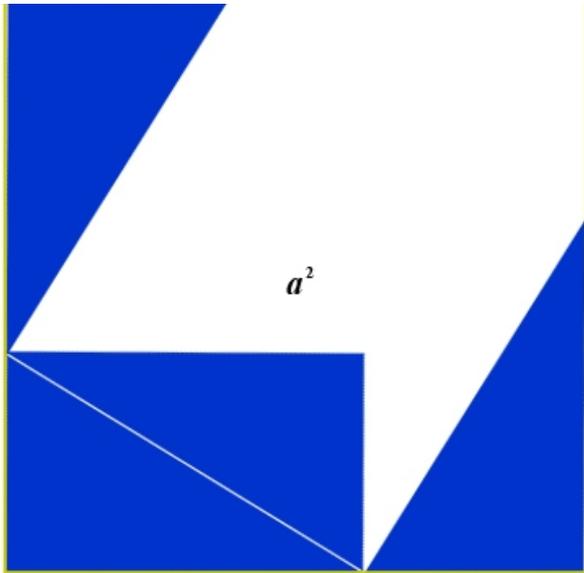
**Figure 2.** Draw a square with side  $a$  along the hypotenuse. The square has an area  $a^2$ .



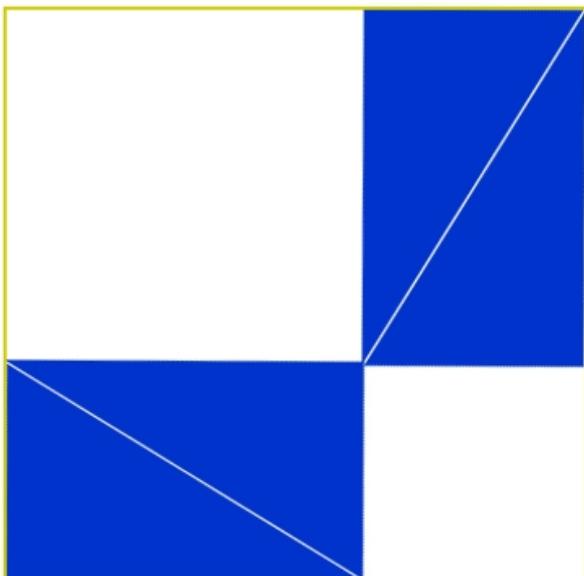
**Figure 3.** Construct a larger square by arranging copies of the triangle around the square on the hypotenuse as shown.



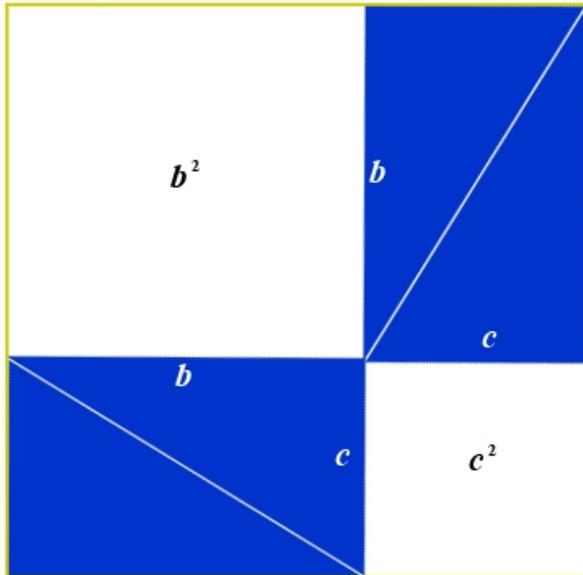
**Figure 4.** The square contains four copies of the triangle. The area not occupied by the four triangles (unshaded) is still  $a^2$ .



**Figure 5.** Wherever the triangles are placed in the new square, the unshaded area will always be equal to  $a^2$  (so long as the triangles do not overlap).



**Figure 6.** This arrangement of the four triangles creates two rectangles. The unshaded area, now broken into two smaller squares, must still be equal to  $a^2$ .



**Figure 7.** The sides of the two unshaded squares are equal to the lengths of the non-hypotenuse sides of the original triangle,  $b$  and  $c$ . Therefore, the areas of the two squares that occupy the non-shaded area are  $b^2$  and  $c^2$ .

The four triangles still account for the same area within the larger square, so these two unshaded squares must occupy the same area as the original unshaded area before the triangles were re-arranged. So:

$$a^2 = b^2 + c^2$$