Surjective/Injective/Bijective

Aim

To introduce and explain the following properties of functions: "surjective", "injective" and "bijective".

Learning Outcomes

At the end of this section you will be able to:

- Understand what is meant by surjective, injective and bijective,
- Check if a function has the above properties.

Surjective Functions

Let $f : A \to B$ be an arbitrary function with domain A and codomain B. Part of the definition of a function is that every member of A has an image under f and that all the images are members of B; the set R of all such images is called the range of the function f. Thus $R = f(A)$ and clearly $R \subseteq B$. If it should happen that $R = B$, that is, that the range coincides with the codomain, then the function is called a *surjective* function.
Definition : A function $f : A \to B$ is an **surjective**, or **onto**, function if the range of f equals the codomain of f.

In every function with range R and codomain B, $R \subseteq B$. To prove that a given function is surjective, we must show that $B \subseteq R$; then it will be true that $R = B$. We must therefore show that an arbitrary member of the codomain is a member of the range, that is, that it is the image of some member of the domain. On the other hand, if we can produce one member of the codomain that is not the image of any member of the domain, then we have proved that the function is not surjective.

To show that a function is surjective pick an arbitrary element in the codomain and show that it has a preimage in the domain.

Graph the following function and check is it surjective?

$$f : \mathbb{R} \to \{x \mid x > 0\}, \quad f(x) = e^x$$

The codomain is $x > 0$. By looking at the graph of the function $f(x) = e^x$ we can see that $f(x)$ exists for all non-negative values, i.e. for all values of $x > 0$. Hence the range of the function is $x > 0$. This means that the codomain and the range are identical and so the function is surjective.

Graphically speaking, if it is possible to draw a horizontal line across the graph of a function without making contact with the curve representing the function then the function is not surjective.

Graph the following two functions

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$;
2. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$.

and check to see if they are surjective. The answers are (1) yes, (2) no. Can you see why?

Injective Functions

The definition of a function guarantees a unique image of every member of the domain. A given member of the range may have more that one preimage, however. If this is the case then the function is not *injective*.

Definition : A function $f : A \to B$ is **injective**, or **one-to-one**, if no member of B is the image under f of two distinct elements of A.

To show that a function is injective, we assume that there are elements a_1 and a_2 of A with $f(a_1) = f(a_2)$ and then show that $a_1 = a_2$.

Graphically speaking, if a horizontal line cuts the curve representing the function at most once then the function is injective.

Test the following functions to see if they are injective.

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$;
2. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$;
3. $f : [0, \infty) \to \mathbb{R}, \quad f(x) = x^2$;

Solutions:

1. Injective
2. Not Injective
3. Injective

Bijective Function

Definition : A function $f : A \rightarrow B$ is **bijective** (a **bijection**) if it is both *surjective* and *injective*.

If $f : A \rightarrow B$ is injective and surjective, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and visa versa.

Which of the following functions are surjective, injective and bijective ?

1. $f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^3;$
2. $f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = 2^x;$
3. $f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^3 - 2x^2 - 5x + 6;$

Plotting the above functions with Maple may help.

Related Reading

Gersting, J.L. 2007. *Mathematical Structures for Computer Science.* 6^{th} Edition. Freeman & Company.