INTRODUCTION

A number in the form of a + ib, where a, b are real numbers and i = $\sqrt{-1}$ is called a complex number. A complex number can also be defined as an ordered pair of real numbers a and b and may be written as (a, b), where the first number denotes the real part and the second number denotes the imaginary part. If z = a + ib, then the real part of z is denoted by Re (z) and the imaginary part by Im (z). A complex number is said to be purely real if Im(z) = 0, and is said to be purely imaginary if Re(z) = 0. The complex number 0 = 0 + i0 is both purely real and purely imaginary.

Symbol *i* : We define positive square root of –1 as imaginary unit, denoted by *i*. Thus, $i = \sqrt{-1}$ $\Rightarrow$ $i^2 = -1$.

Properties of *i*

(i) For any integer *n*, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -i$, $i^{4n+3} = -i$.

For example : $i^{2004} = i^{4 \times 501} = 1$, $i^{497} = i^{4 \times 124 + 1} = i$

Also $i = -\frac{1}{i}$

(ii) For any integer *n* , $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0$

That is, the sum of four consecutive powers of *i* is zero.

For example : $i^{93} + i^{94} + i^{95} + i^{96} = 0$

Complex number : A number of the form *x* + *iy*, where *x* and *y* are real numbers, is called a complex number, denoted by *z*. Thus $z = x + iy$, $x \in R$, $y \in R$ is a complex number. We define

x = Real part of *z*, denoted by Re(*z*)

y = Imaginary part of *z*, denoted by Im(*z*)

$\sqrt{x^2 + y^2}$ = Modulus or absolute value of *z*, denoted by |*z*|

Properties of *z* :

(i) If Re(*z*) = 0, then *z* = *iy* is called a purely imaginary number.

(ii) If Im(*z*) = 0, then *z* = *x* is called a purely real number.

(iii) *z* = 0 = 0 + *i*0 is both purely real as well as purely imaginary.

(iv) Order relation (> or <) is not defined on complex numbers, which are not purely real.

(v) $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

(vi) The number *x* – *iy* is called *complex conjugate* of the number *z* = *x* + *iy*, denoted by $\bar{z}$ or *z**. Thus if *z* = *x* + *iy*, then $\bar{z} = x - iy$ $\Rightarrow$ Re(*z*) = $\text{Re}(\bar{z})$ and Im(*z*) and $-\text{Im}(\bar{z})$.

(vii) The property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ holds good only if at least one of a and b is a positive number.

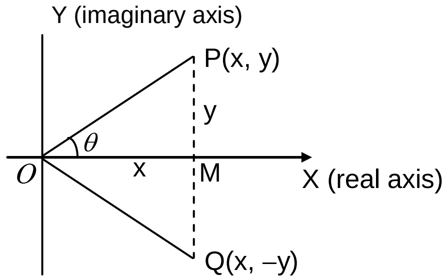
EXAMPLE 1 : Find the sum and product of the two complex numbers
$Z_1 = 2 + 3i$ and $Z_2 = -1 + 5i$

SOLUTION : $Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = \mathbf{1 + 8i}$

$$\left(\frac{x-1}{-2}\right)^3 = 1 \ \frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right)$$

Geometrical representation of complex numbers

A complex number $z = x + iy$ can be represented by a point P, whose Cartesian coordinates are (x, y) referred to axes OX and OY, usually called real and imaginary axes respectively. Point P is called the **image** of the complex number z and the z is called the **affix** of the point P. The conjugate $\bar{z}$ of the number z is the affix of image Q of the point P in the real axis. Now, the modulus of z, *i*.e., $|z| = \sqrt{x^2 + y^2} = OP$.



The angle XOP is called the **argument** or **amplitude** of z, denoted by arg(z) or amp(z).

Thus $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

If we take $OP = R$, then $x = R\cos\theta$, and $y = R\sin\theta$. Then $z = x + iy = R(\cos\theta + i\sin\theta)$.

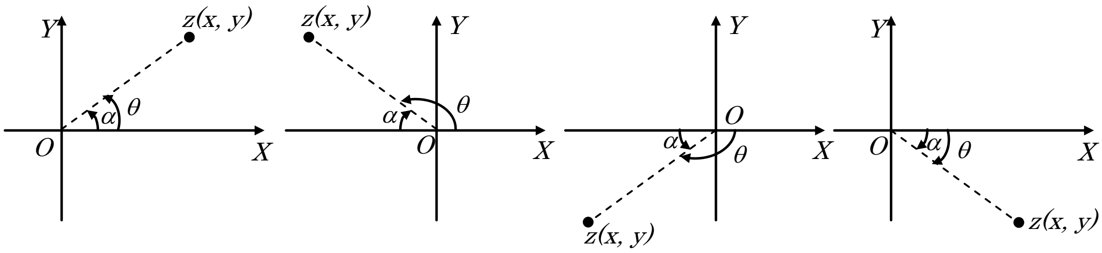
This is known as trigonometric or polar form of the complex number z.

Also $z = R(\cos\theta + i\sin\theta) = re^{i\theta}$. This is known as **Euler's formula**. Again if z_1 and z_2 represent two points P and Q in the Argand plane, then $|z_1 - z_2|$ represents the distance PQ.

Principal value of Argument : In general the arg(z) of a complex number z is the solution of the simultaneous equation

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Clearly the argument (z), i.e., θ cannot be unique. $2n\pi + \theta$, n is an integer, is also an argument of z. The value of θ such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument. The argument of the complex number 0 is not defined. The principal value of argument (θ) of the complex number $z = x + iy$ for different combinations of x and y are shown in following figures:



$x > 0, y > 0, \theta = \alpha$ $\quad$ $x < 0, y > 0, \theta = \pi - \alpha$ $\quad$ $x < 0, y < 0, \theta = \alpha - \pi$ $\quad$ $x > 0, y < 0, \theta = -\alpha$

In each case $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$, and $0 \le \alpha < \frac{\pi}{2}$.

EXAMPLE 2 : Represent the given complex numbers in polar form:

(i) $(1 + i\sqrt{3})^2 / 4i(1 - i\sqrt{3})$ (ii) $\sin\alpha - i\cos\alpha$ (α acute) (iii) $1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

SOLUTION : (i) $i(1 - i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$$\therefore \frac{(1 + i\sqrt{3})^2}{4i(1 - i\sqrt{3})} = \frac{(1 + i\sqrt{3})^2}{4(\sqrt{3} + i)} = \frac{-2 + 2i\sqrt{3}}{4(\sqrt{3} + i)} = \frac{(-1 + i\sqrt{3})(\sqrt{3} - i)}{2(\sqrt{3} + i)(\sqrt{3} - i)}$$

$$= \frac{-\sqrt{3} + \sqrt{3} + 4i}{2(3 + 1)} = \frac{i}{2}$$

and $\frac{i}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

Hence $\dfrac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})} = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) = \dfrac{1}{2}\ e^{i\pi/2}$

(ii) Real part > 0; Imaginary part < 0

argument of $\sin\alpha - i\cos\alpha$ is in the nature of a negative acute angle.

$\therefore\ \sin\alpha - i\cos\alpha = \cos\left(\alpha - \dfrac{\pi}{2}\right) + i\sin\left(\alpha - \dfrac{\pi}{2}\right) = e^{i\left(\alpha - \frac{\pi}{2}\right)}$

(iii) $1 + \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = 2\cos^2\dfrac{\pi}{6} + i\cdot 2\sin\dfrac{\pi}{6}\cos\dfrac{\pi}{6}$

$= 2\cos\dfrac{\pi}{6}\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right) = 2\cos\dfrac{\pi}{6}e^{i\pi/6}$

Properties of conjugate of a complex number

(i) $|z| = |\bar{z}|$ (ii) $z + \bar{z} = 2\text{Re}(z)$ (iii) $z - \bar{z} = 2i\text{Im}(z)$

(iv) $z\bar{z} = |z|^2$ (v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (vi) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

Note: The properties (v) and (vi) can be extended to any number of complex number.

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$ (ix) $\overline{(\bar{z})} = z$

(x) $\overline{z^n} = (\bar{z})^n$ (xi) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(\bar{z}_1 z_2) = 2\text{Re}(z_1\bar{z}_2)$

(xii) $z = \bar{z} \Leftrightarrow z$ is purely real.

(xiii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary.

EXAMPLE 3 : If $|z - 2 + i| \le 2$ then find the greatest and least value of $|z|$.

SOLUTION : Given that

$|z - 2 + i| \le 2$ …(i)

$\because\ |z - 2 + i| \ge \big|\,|z| - |2 - i|\,\big|$

$\therefore\ |z - 2 + i| \ge \big|\,|z| - \sqrt{5}\,\big|$ …(ii)

From (i) and (ii)

$\big|\,|z| - \sqrt{5}\,\big| \le |z - 2 + i| \le 2$

$\therefore\ \big|\,|z| - \sqrt{5}\,\big| \le 2$

$\Rightarrow\ -2 \le |z| - \sqrt{5} \le 2$

$\Rightarrow\ \sqrt{5} - 2 \le |z| \le \sqrt{5} + 2$

Hence greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

Properties of Modulus of a Complex Number

(i) $|z| = 0 \Leftrightarrow z = 0$ (ii) $|z| \ge 0$ for any complex number z,

(iii) $|z_1 z_2| = |z_1||z_2|$, can be extended to any number of complex numbers.

(iv) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|},\ z_2 \neq 0$ (v) $\left|\dfrac{z}{|z|}\right| = 1$, *i.*e. $\dfrac{z}{|z|}$ is a unimodular complex number.

(vi) $|z_1 \pm z_2| \le |z_1| + |z_2|$ (vii) $|z_1 \pm z_2| \ge \big||z_1| - |-z_2|\big|$ (viii) $-|z| \le \text{Re}(z) \le |z|$

(ix) $-|z| \le \text{Im}(z) \le |z|$ (x) $|z| \le |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}|z|$

(xi) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2) = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\bar{z}_2)$

(xii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$

Properties of Argument of Complex Numbers

(i) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$, can be extended to any number of complex numbers.

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ (iii) $\arg(\bar{z}) = -\arg(z)$

(iv) $\arg(z^n) = n\arg(z)$ (v) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z)$

(vi) $\arg(z) = 0$ iff z is purely real. (vii) $\arg(z) = \pm\frac{\pi}{2}$ iff z is purely imaginary.

EXAMPLE 4 : Find out the principal arguments of the following complex numbers.

(i) 3 + 4i (ii) 3 –4i (iii) –3 + 4i (iv) –3 – 4i

SOLUTION : (i) $\tan^{-1}4/3$

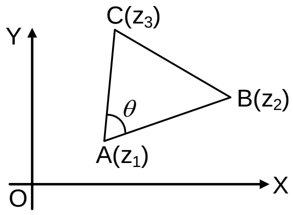
(ii) $\tan^{-1}\left(-\frac{4}{3}\right)$

(iii) $\pi + \tan^{-1}(-4/3)$

(iv) $-\pi + \tan^{-1}\frac{4}{3}$

Concept of Rotation in Complex Plane

Let z_1, z_2, z_3 represent points A, B, C respectively on the complex plane. Then $AB = |z_2 - z_1|$, $AC = |z_3 - z_1|$ and $BC = |z_3 - z_2|$. Let θ be the counter clockwise angle $\angle BAC$, then $\theta = \arg\frac{z_3 - z_1}{z_2 - z_1}$. We may write



$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|(\cos\theta + i\sin\theta) = \frac{AC}{AB}(\cos\theta + i\sin\theta) = \frac{AC}{AB}e^{i\theta}$$

(i) Multiplying a complex number by i represents a rotation of angle $\frac{\pi}{2}$ counter-clockwise about origin.

(ii) Multiplying a complex number by ω represents a rotation of angle $\frac{2\pi}{3}$ about origin clockwise or anticlockwise.

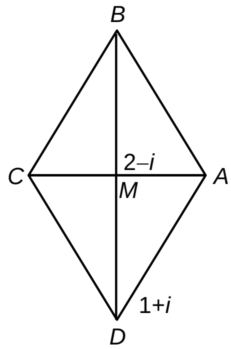
EXAMPLE 5 : ABCD is a rhombus. Its diagonals AC and BD intersect at M such that BD = 2AC. If the points D and M represent the complex number 1 + i and 2 – i respectively, find the complex number(s) representing A.

SOLUTION : Let A be z. The position MA can be obtained by rotating MD anticlockwise through an angle $\frac{\pi}{2}$; simultaneously length gets halved.

$$\therefore\ z - (2 - i) = \frac{1}{2}\left((1+i) - (2-i)\right)e^{i\pi/2}$$

$$= \frac{1}{2}(-2 - i) = -1 - \frac{1}{2}i$$

$$z = -1 - \frac{1}{2}i + 2 - i = 1 - \frac{3i}{2}$$



Another position of A corresponds to A and C getting interchanged and in that the complex number of A is $1 + \frac{1}{2}i + 2 - i = 3 - \frac{1}{2}i$

$\therefore$ The complex number of A is either $1 - \frac{3i}{2}$ or $3 - \frac{1}{2}i$

De Moivre theorem

(i) If $n \in I$, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$

(ii) If $n \in Q$, say $n = \frac{p}{q}, q \neq 0$, then $(\cos\theta + i\sin\theta)^n$ will have Q values one of which is given by $\cos n\theta + i \sin n\theta$. ($P$ and Q are integers)

EXAMPLE 6 : If n be a positive integer, prove that

$$(1+i)^{2n} + (1-i)^{2n} = \begin{cases} 0 & \text{if n be odd} \\ 2^{n+1} & \text{if } \frac{n}{2} \text{ be even} \\ -2^{n+1} & \text{if } \frac{n}{2} \text{ be odd} \end{cases}$$

SOLUTION : $(1+i)^{2n} = 2^n \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2n} = 2^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)$

$(1-i)^{2n} = 2^n \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{2n} = 2^n\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$

$\therefore\ (1+i)^{2n} + (1-i)^{2n} = 2^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$

$= 2^{n+1}\cos\left(\frac{n\pi}{2}\right)$

If n be odd = $2m + 1$, then RHS = $2\cos(2m+1)\frac{\pi}{2}$

$= 0$

If n be even and $\frac{n}{2}$ also even so that $n = 4k$, then RHS = $2^{n+1}\cos(2k\pi) = 2^{n+1}$

else RHS = $2^{n+1}\cos\left(\frac{n\pi}{2}\right) = -2^{n+1}$

Cube Roots of Unity

Let $z^3 = 1 \Rightarrow z^3 - 1 = 0 \Rightarrow (z-1)(z^2 + z + 1) = \theta \Rightarrow z = 1$ or $z = \frac{-1 \pm i\sqrt{3}}{2}$.

$z = \frac{-1 \pm i\sqrt{3}}{2}$ are called imaginary cube roots of unity and one the roots of $z^2 + z + 1 = 0$.

$\because \left(\frac{-1 \pm i\sqrt{3}}{2}\right)^2 = \frac{-1 - i\sqrt{3}}{2}$, we generally represent $\omega = \frac{-1 + i\sqrt{3}}{2}$ and $\omega^2 = \frac{-1 - i\sqrt{3}}{2}$.

Also, $\omega = \frac{-1 + i\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i\frac{2\pi}{3}}$

and $\omega^2 = \frac{-1 - i\sqrt{3}}{2} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = e^{i\frac{4\pi}{3}}$

EXAMPLE 7 : If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is cube roots of unity), then find the value of $\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1}$.

SOLUTION : We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore \quad (x-1)^3 + 8 = 0 \quad \therefore \; (x-1)^3 = (-2)^3$

$\Rightarrow \quad \left(\frac{x-1}{-2}\right)^3 = 1 \quad \Rightarrow \quad \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$ (cube roots of unity)

$\therefore \quad x = -1, 1-2\omega, 1-2\omega^2$

Here $\alpha = -1, \beta = 1-2\omega, \gamma = 1-2\omega^2$

$\therefore \quad \alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$.

Then $\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right)+\left(\frac{-2\omega}{-2\omega^2}\right)+\left(\frac{-2\omega^2}{-2}\right)$

$$= \frac{1}{\omega}+\frac{1}{\omega}+\omega^2$$

$$= \omega^2+\omega^2+\omega^2 = 3\omega^2$$

Properties of ω and ω^2

(i) $1 + \omega + \omega^2 = 0$, in general $1 + \omega^n + 2^{2n} = 3$ or 0 according as n is a multiple of 3 or not ($n \in I$).

(ii) $\omega^3 = 1$; in general $\omega^{3n} = 1$, $\omega^{3n+1} = \omega$ and $\omega^{3n+2} = \omega^2$

(iii) $\omega^2 = \bar{\omega}$ and $\omega = \bar{\omega}^2$

(iv) The cube roots of unity represent the vertices of an equilateral triangle inscribed in a unit circle with centre at origin on the complex plane. One vertex is always on positive real axis.

(v) If α is a real cube root of a real number then its other roots are $\alpha\omega$ and $\alpha\omega^2$.

(vi) If a complex number z is such that $|\text{Re}(z)| : |\text{Im}(z)| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$, then z can be expressed in terms of i, ω or ω^2.

(vii) For any real a, b, c; $a + b\omega + c\omega^2 = 0 \;\Rightarrow\; a = b = c$.

The n^{th} Roots of Unity

Let $z^n = 1 = \cos 2k\pi + i \sin 2k\pi$, $k \in I$

$\therefore \quad z = (\cos 2k\pi + i \sin 2k\pi)^{1/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0, 1, 2, ..., n-1$

If we represent $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ by α, then the n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

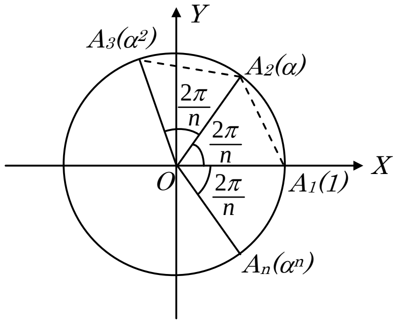
Properties of n^{th} Roots of Unity

(i) $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0 \quad \Rightarrow \quad \sum_{k=0}^{n-1}\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = 0$

$\Rightarrow \sum_{k=0}^{n-1}\cos\frac{2k\pi}{n} = 0 \quad$ and $\quad \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n} = 0$

(ii) $1.\alpha.\alpha^2\ldots\alpha^{n-1} = (-1)^{n+1}$.

(iii) The points represented by the n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle with centre at the origin. One vertex being on the positive real axis.



EXAMPLE 8 : Find the cube roots of $4 - 4\sqrt{3}\,i$.

SOLUTION : Let $z = (4 - 4\sqrt{3}\,i)^{1/3}$, $\rho = \sqrt{16+48} = 8$ $\cos\alpha = 1/2$, $\sin\alpha = -\frac{\sqrt{3}}{2}$

$\therefore$ Cube roots of $4 - 4\sqrt{3}\,i$ are given by

$z = \rho^{1/3}\,\text{cis}\,\frac{2k\pi+\alpha}{3}$, $k = 0, 1, 2$ and $\rho^{1/3} = 8^{1/3} = 2$ (positive real cube root of 8)

Thus $z = 2\,\text{cis}\,\frac{\alpha}{3}$, $2\,cis\frac{2\pi+\alpha}{3}$, $2\,cis\,\frac{4\pi+\alpha}{3}$ are the required roots.

Here α is given by $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = -\frac{\sqrt{3}}{2}$ i.e. $\alpha = -\frac{\pi}{3}$.

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Let $z = (4 - 4\sqrt{3}i)^{1/3}$

or, $z = (8e^{-i\pi/3})^{1/3}$

or, $z = 2e^{-i\pi/9}\ (1)^{1/3}$

$\Rightarrow$ $z = 2e^{-i\pi/9}$, $2e^{-i\pi/9}.\omega$ and $2e^{-i\pi/9}.\omega^2$

since $\omega = e^{i2\pi/3}$, $\omega^2 = e^{i4\pi/3}$

Therefore, $z = 2e^{-i\pi/9}$, $2e^{i5\pi/9}$ and $2e^{i11\pi/9}$.

Geometrical Applications

(i) Distance between two points A and B represented by complex numbers z_1 and z_2 is $AB = |z_2 - z_1|$.

(ii) Affix of a point P dividing the join of point A and B with affices z_1 and z_2 in the ratio $m : n$, internally is $\frac{mz_2+nz_1}{m+n}$; externally is $\frac{mz_2-nz_1}{m-n}$.

(iii) Affix of mid point of $A(z_1)$ and $B(z_2)$ is $\frac{z_1+z_2}{2}$.

(iv) Affix of centroid of ΔABC, with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is $\frac{z_1+z_2+z_3}{3}$.

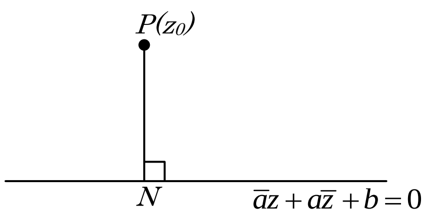
(v) Equation of straight line passing through two points $A(z_1)$ and $B(z_2)$ in complex form is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

or $\frac{z-z_1}{\bar{z}-\bar{z}_1} = \frac{z_2-z_1}{\bar{z}_2-\bar{z}_1}$.

(vi) General equation of a straight line in complex plane is $\bar{a}z + a\bar{z} + b = 0$, where a is a constant complex number and b is a constant real number.

Slope of this line = $\frac{a+\bar{a}}{i(a-\bar{a})} = -\frac{\text{Re}(a)}{\text{Im}(a)}$.

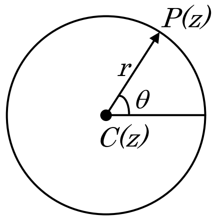
(vii) Distance of a given point $P(z_0)$ from the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\frac{|\bar{a}z_0 + a\bar{z}_0 + b|}{2|a|}$.



(viii) Equation of a circle of radius R and centre at point $C(z_0)$ is $|z - z_0| = R$.

$|z - z_0| > R$ represents the points lying outside the circle.

$|z - z_0| < R$ represents the points lying inside the circle.



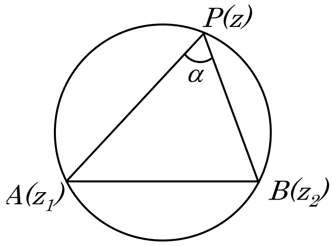
(ix) Any point on the circle $|z - z_0| = R$ can be given by $z = z_0 + re^{i\theta}$.

(x) General equation of a circle in complex plane is given by $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where $b \in R$. Its center is at the point C with affix $-a$ and radius $\sqrt{|a|^2 - b}$. The circle is real iff $|a|^2 - b \geq 0$.

(xi) Equation of a circle described on a line segment AB, as diameter is $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$, where z_1 and z_2 are affices of points A and B.

(xii) Let z_1 and z_2 be two given complex numbers. Then $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha, 0 < \alpha < \pi$ represents all points z lying on the arc of a circle.

If $\alpha \in \left(0, \frac{\pi}{2}\right)$, z lies on the major arc (excluding points A and B).



If $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, z lies on the minor arc (excluding points A and B).

(xiii) Four points $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ taken in order are concyclic if $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is purely real.

(xiv) $|z - z_1| + |z - z_2| = a$, $a \in R^+$ represents an ellipse if $|z_1 - z_2| < a$. Points z_1 and z_2 represent the foci of ellipse.

(xv) $|z - z_1| + |z - z_2| = a$, $a \in R - \{0\}$ represents an hyperbola if $|z_1 - z_2| > |a|$. Points z_1 and z_2 represents the foci of hyperbola.

(xvi) The triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 is equilateral if and only if

$$\frac{1}{z_2 - z_3} - \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0 \iff z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0 .$$

EXAMPLE 9 : Interpret Geometrically the complex number 'z' which satisfied the following inequality $\log_{1/2} \frac{|z-1|+4}{|z-1|-2} < 1$.

SOLUTION : In order the log is to be defined, $|z - 1| - 2 > 0$

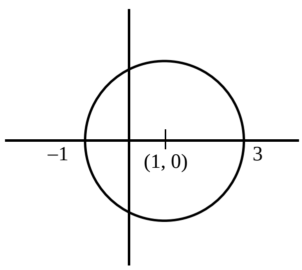
$\Rightarrow |z - 1| > 2$.

Also, $\frac{|z-1|+4}{|z-1|-2} > \frac{1}{2}$

$\Rightarrow |z - 1| > -10$ which is always true.

Hence the inequality will hold for all 'z' satisfying the condition that $|z - 1| > 2$.

Geometrically, it represents the exterior of a circle with center (1 + 0i) and radius '2'.



EXAMPLE 10 : If $||z + 2| - |z - 2|| = a^2$, $z \in C$ representing a hyperbola for $a \in R$, then find the values of a.

SOLUTION : Here foci are at –2 and 2 at a distance at 4. Hence the given equation represents a hyperbola if $a^2 < 4$ i.e. $a \in (-2, 2)$.