MATHEMATICAL INQUIRY: PART 2

CONJECTURES, PROOFS, THEOREMS, AND CLASSIFICATION

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What is a Mathematical Theory?

A mathematical theory is composed of

(a) axioms,

(b) definitions, and

(c) theorems.

Axioms and definitions are premises in mathematical proofs, while theorems are conclusions.

Of these, the concept of theorems is perhaps what most people are familiar with. Examples of theorems in geometry include: “The sum of angles in a triangle is two right angles;” “Every triangle can be circumscribed by a circle;” and “The square of the hypotenuse of a right-angled triangle is equal to the sum of the sqaures of the other two sides.” Familiar examples of definitions include statements like: “A triangle is a three-sided polygon;” “An equilateral triangle is one in which all sides are equal;” and “A circle is the set of all and only points equidistant from a centre.” What is less known, perhaps, is the concept of axioms. Examples of axioms in Euclidean geometry (the geometry that students learn in school) include: “Every finite line has infinitely many points;” and “Given any two distinct points, there is one and only one straight line between them.”

In Mathematical Inquiry Part 1, we came across axioms that distinguish between the geometry of flat surfaces (e.g., Euclidean geometry) and the geometry of spherical surfaces. Thus, “A straight line can never meet itself,” is an axiom that characterizes flat surfaces; and “Every straight line when extended will meet itself,” is an axiom that characterizes spherical surfaces.

To show that a given statement is a theorem in a given theory, we need to prove it. In mathematics, we prove something by demonstrating that it is a logical consequence of the axioms and definitions of the theory. A statement that we wish to establish as a theorem is called a conjecture. Once a conjecture is proved, and the proof is accepted as valid by the community of mathematicians, it is called a theorem.

In Mathematical Inquiry Part 1, we went through examples of:

coming up with clear and precise definitions,

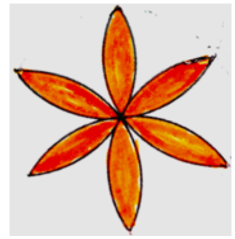
evaluating definitions, and

coming up with conjectures.

We did not go into the details of proving conjectures to establish them as theorems, even though we hinted at proofs. In what follows, we will explore the challenges of conjecturing and proving in mathematics. We will also peep into the challenge of classifying mathematical objects and look at how classifying is an important part of theory construction.

PROVING: SIX-PETAL FLOWER

Here is a dialogue between a student and a teacher:



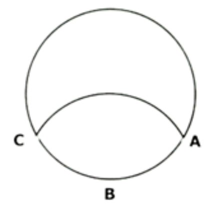
T: Look at this picture. It's a flower with six identical petals. Did you ever learnt to draw this?

S: No, but how did you do it?

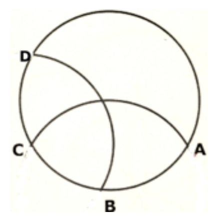
T: When I was in Class 8, art class was compulsory in our school. I learnt it in that class. Would you like to learn how to do it?

S: Of course! Show us!

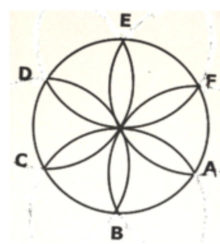
T: Well, first draw a circle, using a compass. Then, without changing the angle of the compass, place its metal point on any point on the circle, call it B, and draw an arc AC such that A and C are also points on the circle:



Now place the metal point of the compass on C, and draw an arc that touches point B and a new point D on the circle.



Draw a third arc, with the point of the compass on D. This arc touches points C and a new point E on the circle. A fourth arc, starting with the compass on E, would touch D and a new point F. A fifth one would start on F, with the arc touching E and A'. The sixth and final arc, starting on A', would touch F. But it also touches B. This means that A' coincides with A.



And voila! you have the flower!

S: Wow, so simple! and so elegant! But this is supposed to be a Maths class, not art. So how is the flower related to Maths?

T: Good question! Can you guess the answer?

S: Mmmm, no. You'll have to tell us.

T: Okay. Here's a clue. Suppose we join the tips of all the adjacent petals of the flower, what do we get?

S: (Pause) A hexagon.

T: Just a hexagon?

S: A hexagon circumscribed by the circle.

T: Indeed! Now this means that each line of the hexagon is a chord of the circle.

S: Yes.

A SCIENTIFIC PROOF

T: What is a hexagon?

S: A six-sided polygon.

T: Do you notice anything special about the hexagon you get using the procedure for the six-petal flower?

S: (Pause) All the sides are equal.

T: What is the name of a polygon whose sides are equal?

S: An equilateral polygon.

T: How do you know that this hexagon is equilateral? Can you prove it?

S: Yes. We can measure the sides. (measuring) We've measured them. Yes, Teacher, they are all equal!

T: Aha! Would this happen with any circle we draw, no matter what the radius? Whether the radius is three centimeters, six centimeters or ten centimeters? What do you think?

S: (drawing more circles and hexagons, and measuring the sides) The sides are equal in all the hexagons we've measured.

T: So, would this be the case for any circle? What if the radius is three kilometers? Even if it is as big as the earth? Are you sure that if the radius is ten thousand kilometers, the hexagon drawn this way will still be equilateral? Will you be able to measure it?

S: I guess not.

T: So what you have done is to (i) make measurements in a sample from a population of hexagons, (ii) identify what is true of the sample, and (iii) conclude that it is true of the population as well. This would be a legitimate form of reasoning in scientific inquiry.

S: But Teacher, why do we have to prove it? Didn't we use a procedure of using the radius of a circle to draw each side in a hexagon? The sides HAVE to be equal. We got the vertices of the hexagon by drawing arcs on the circle. Since the sides of the hexagon are chords of the circle, and the chords are all equal to the radius of the circle, the sides must all be equal.

T: Hang on. The first five chords that you drew are equal. But how do you know that the last chord will not end up shorter or longer?

You have shown that:
  if we construct a hexagon using the procedure for six-petal flowers,
  no matter which circle,
  the first five sides will be equal.

You have also shown, through measurement, that:
  in a finite sample of circles
  using this procedure results in the sixth side of the hexagon being equal to the other sides.

But can you prove that:

no matter which circle you pick from the population of circles,
using this procedure results in the sixth side of the hexagon being equal to the other sides?

The assumption that what is true of the sample is true of the population is an acceptable form of justification or proof in scientific inquiry, but not in mathematical inquiry, because mathematical inquiry demands greater certainty. Using the procedure above, we cannot rule out the possibility that there exists some circle we have not looked at where this is not true.

How do we know that A’ will coincide with A every time, no matter what the size of the circle is?

S: Oh, I see!

T: There is another problem with your proof. Strictly speaking, what we have drawn, and can see, is the picture of a hexagon. The sides of a hexagon are line segments, and since lines do not have breadth, we cannot see them, and so we cannot measure them. What you have measured is the length of a *picture* of a line segment, not a line segment itself. Line segments and hexagons exist only in our imagination. So you cannot use measurement to prove that all the sides of the hexagon constructed this way are equal.

S: (thinking) But it has to be that way!

T: But why should it be so? Does it follow from something?

S: (thinking): Don’t know...

T: Do you notice that you are now right in the heart of mathematics?

S: Yeah!

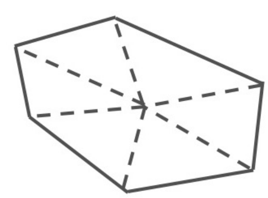
A MATHEMATICAL PROOF

T: So, using the radius of the circle to draw arcs, we got six petals whose tips touch the circumference of the circle at equidistant points. That is what you are saying.

Question: How is it that the procedure we used manages to cut a circle into six equal parts? If it will happen every time, can we prove that this procedure will cut the circumference of any circle into six equal parts?

To put it differently, given the way we drew the arcs, we know that AB, BC, CD, DE, EF, FA’ are equal. But how do we know that, in every instance, A will coincide with A’? Can we prove that this is so?

For that, it would be useful to conceptualize a hexagon as combination of six triangles. Like this:

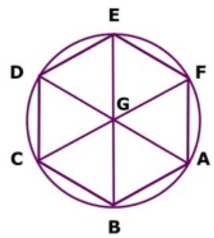


In this picture, the hexagon is not equilateral. But what we need to prove is that the given procedure for six petal flowers can be used to construct a polygon made up of six equilateral triangles.

To do this, you will first have to prove that, given an equilateral hexagon, and a circle that circumscribes it, each side of the hexagon would be equal to the radius of the circle. This, then, is our conjecture.

Conjecture: Given an equilateral hexagon, and a circle that circumscribes it, each side of the hexagon would be equal to the radius of the circle.

Proof: Take an equilateral hexagon, and circumscribe it with a circle.



Connect the opposite vertices of the hexagon: AD, BE, CF.

Assumption 1: The diagonals of an equilateral even-sided polygon intersect at its center, G. [This results in the hexagon being divided into six triangles that meet at the center. The diagonals of the hexagon are diameters of the circle. Hence the center of the circle coincides with the center of the hexagon.]

The sum of the six angles at G is 4 right angles [by definition of 'right angle' discussed in part 1].

Hence each angle is 1/6 of 4 right angles (= 2/3 right angle).

If one of the angles of a triangle is 2/3 of a right angle, the sum of the remaining angles must be 2 right angles minus 2/3 right angle: 4/3 right angles (1 plus 1/3 right angles).

Assumption 2: The sum of the angles of a triangle is 2 right angles.

As two of the sides of each triangle is a radius of the circle, the six triangles are isosceles triangles. Hence, the angles opposite the two radii must be equal.

As the sum of the two angles is 4/3 right angles, and the angles are equal, each of the two angles must be 2/3 right angles.

Given that all three angles of the triangles are 2/3 right angles, they are equiangular.

Assumption 3: Equiangular triangles are equilateral.

Therefore the triangles in our activity are equilateral.

Since two of the sides are the radii of the circle, the third side, which is the chord, is equal to the radius.

[QED]

(QED: abbreviation of the Latin phrase "*Quod Erat Demonstrandum,*" which, when translated, means "that which was to be demonstrated".)

What we have proved:

Given any circle that circumscribes a regular hexagon, the sides of the hexagon would be equal to the radius of the circle.

Conjecture that we needed to prove:

Any circle can be divided into six arcs such that the chords will be equal to the radius.

We can prove this conjecture from the previously established theorem.

Proof:

Beginning with point A, draw chords AB, BC, CD, DE, EF, FA' are equal to the radius.

Assume, contrary to what we wish to prove, that A' does not coincide with A. This means that FA' is either longer or shorter than FA.

We now take the same circle and inscribe a regular polygon in it, with sides ABCDEF.

Superimpose the sides of the regular polygon on the sides of the chords we drew earlier.

We have proved that FA is the radius of the circle. We have drawn FA' as equal to the radius. Since both FA and FA' are equal to the radius, FA and FA' are also equal. Hence, contrary to our starting point, A does coincide with A'. (QED)

[We have made three assumptions (1, 2 & 3) without defending or proving them. For our proof to be complete, we would need to establish the truth of those assumptions as well. But we will not do that here.]

CONJECTURING, PROVING & GENERALIZING: CONSECUTIVE NUMBERS

[Recall from LT1 the idea of a consecutive number theorem. What is given below is an expanded version of the discovery and proving of that conjecture.]

Here is a dialogue between a student and a teacher:

T: Give me two consecutive numbers.

S: 4 and 5.

T: Add them. What do you get?

S: 9.

T: Give me a few more consecutive number pairs.

S: 9 and 10, 18 and 19, and 43 and 44.

T: Add each of them.

S: You get 19, 37 and 87 respectively.

T: Do you see a pattern?

S: Not sure.

T: Try a few more then.

S: 12 and 13, 145 and 146, 33 and 34.

T: Add each of them.

S: 25, 191 and 67.

T: Let's write all of them together:

Consecutive Numbers	Sum
4 and 5	9
9 and 10	19
18 and 19	37
43 and 44	87
12 and 13	25
145 and 146	291
33 and 34	67

T: Now do you see a pattern?

S: Yes! All the sums are odd numbers.

T: That is true here, but is it true for all pairs of consecutive numbers?

S: Seems like it is. But we can't be sure. We can't try out every number!

T: What you have is a conjecture. A conjecture is a statement that is either true or false, but we don't know which one it is currently. If you prove it to be true, it becomes a theorem.

S: How do we prove it?

T: What you need here is an argument that will work for all numbers

S: How do we go about creating such an argument?

T: In this case, one way we can go about it is to notice that if we add 1 to an odd number, you get an even number, and if we add 1 to an even number, we get an odd number.

S: So, if we can show that 1 less than the sum of two consecutive numbers is even, we automatically get that the sum of two consecutive numbers is odd.

T: Right!

S: How do we do this now?

T: What is one less than the sum of two consecutive numbers?

S: Hmm... that would be the sum of the smaller number with itself, wouldn't it?

T: Yes!

S: How does that help us?

T: You want to show that the sum of a number with itself is even, right?

S: Right

T: So, what is an even number?

S: Now I get it!

An even number is a number that can be broken into two equal parts.
If we add a number to itself, we can break the result into two equal parts.
This means that the result is even!
Add one more and it is odd. So,
the sum of two consecutive numbers is not divisible by two, it is odd.

T: Good job! You now have your first theorem. But this is only the start. Don't rest on your laurels.

S: What do we do next?

T: Now, that you have proved something, you shouldn't stop here. You should look at the theorem you have created and try asking questions related to it.

S: Such as?

T: Now that you have answered a question about two consecutive numbers, how about trying to add three consecutive numbers?

S: Can we try that out?

Consecutive Numbers	Sum
3, 4 and 5	12
8, 9 and 10	27
17, 18 and 19	54
42, 43 and 44	129
11, 12 and 13	36
144, 145 and 146	435
32, 33 and 34	99

S: Some seem to be odd and some seem to be even, so that's not a pattern here.

T: What about other patterns?

S: Are all of the results divisible by 3?

T: I think all the ones you have tried out so far are. But do check it yourself.

S: Yes, they are all divisible by 3. But I'm not sure if that pattern will continue.

T: Try out a few more examples in order to figure out whether your conjecture — that the pattern will continue — is plausible or not.

S: What do you mean by plausible?

T: A plausible conjecture is one which you are convinced is true but have not proved so far. To do that, we look for counterexamples. If we haven't found any counterexamples after serious search in which we have made an honest effort to disprove ourselves, then it is a plausible conjecture.

S: After trying some more examples, we are satisfied that this conjecture is plausible.

T: Now you need a proof!

S: This time, let us try to make the argument. Ok, so, let's try a similar method to the last one. Adding three consecutive numbers is like adding the smaller number to itself three times and then adding 1 for the second number, and 2 for the third number.

T: This seems like a promising approach. Can you put it down as an equation?

S: Ok, so:

Sum of three consecutive numbers = 3 x smallest number + 1 + 2.

Oh! that is the same as saying:

Sum of three consecutive numbers = 3 x smallest number + 3.

By the distributive law of multiplication over addition, we can say that:

Sum of three consecutive numbers = 3 x (smallest number + 1).

Since the right hand side of the equation is divisible by 3, the left hand side must also be divisible by 3! So, *the sum of three consecutive numbers is divisible by 3!*

T: You have your second theorem now!

S: How do we move forward? Should we try four, five and six?

T: Yes you should. However, first, reflect on what you have done so far.

Your first theorem showed that the sum of two consecutive numbers is odd. In other words, the sum of 2 consecutive numbers is not divisible by 2.

The second theorem showed that the sum of 3 consecutive numbers is divisible by 3.

So, as you go forward, you should ask: 'Is the sum of 4 consecutive numbers divisible by 4?', 'Is the sum of 5 consecutive numbers divisible by 5?' and so on. Now that you know what you are looking for, you no longer need to spend time finding patterns like you did earlier.

S: Good point. Now, let's try 4, 5 and 6!

T: Do that. And try the same method.

— After some work —

S: Using the same methods, we found that the sum of 4 consecutive numbers is not divisibly by 4. The sum of 5 consecutive numbers is divisible by 5. The sum of 6 consecutive numbers is not divisible by 6.

T: Nice! Now that you have a bunch of theorems, can you see a pattern amongst those theorems? Now you are looking for patterns in theorems rather than patterns in numbers!

S: How do we look for those?

T: How about writing out what you have in the form of a table.

Number of Consecutive Numbers	Divisible by the Number of Consecutive Numbers?
2	No
3	Yes
4	No
5	Yes
6	No

T: Do you see a pattern?

S: It seems to alternate. For even numbers, it is not divisible, but for odd numbers it is.

T: So, let me state what you are saying. You are saying that *the sum of n consecutive numbers is divisible by n if n is odd, but not divisible by n if n is even*.

S: What is n?

T: n is a variable. A variable is like a place-holder. Here, by placing n for a number, you are just saying, 'The sum of any number of consecutive numbers is divisible by the number of consecutive numbers if there are an odd number of consecutive numbers, and is not divisible by the number of consecutive numbers if there are an even number of consecutive numbers.'

S: That is what we are saying. Though using n does make it a lot shorter!

T: Can you attempt to prove this?

S: Let's try using the same method. The sum of n consecutive numbers = smallest number x n + 1 + 2 + 3 + … + (n-1). The first part is always divisible by n. The second part seems really complicated!

T: Separate it out and write only the second part. Since the first part is always divisible by n, all you need to do is to show that the second part is divisible by n if n is odd, and not divisible by n if n is even.

S: Here's the second part: 1+2+3+4+…+(n-1)

T: Let's leave this here for now. Moving forward from here is quite hard. Spend some time thinking about it. However, if you are unable to get it, google Gauss. He was a German mathematician who came up with a way to add sequences like the one you have. He did this while he was in school!

CONSTRUCTING CLASSIFICATORY SYSTEMS: CLASSIFICATION OF SHAPES

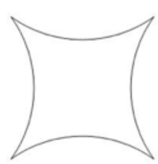
T: What is a square?

S: A square is a shape with equal sides.

T: So, an equilateral triangle is a square?

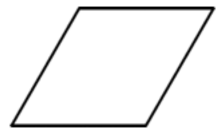
S: No, a square is a shape with 4 sides all of which are equal.

T: So, this is a square:



S: No. A square is a shape with 4 straight-line sides, all of which are equal.

T: So, is this a square:



S: No, the angles have to be 90 degrees. Sorry, no, I mean each angle is a right angle.

T: Okay, so given our earlier definitions of right angles, this is a picture of a square? It has four equal sides and all angles are right angles.

S: No. It has to be closed.

T: Okay, so let us recap the properties which distinguish a square from other shapes:

1. It is a closed shape.
2. It has 4 sides.
3. All its sides are straight lines.
4. All angles are right angles.
5. All sides are equal

Anything which has all these properties is a square, and anything missing even one of these is not. So, this counts as a definition of a square.

Let us move to another question: Is a square a rectangle?

S: No. In a square, all sides must be equal. In a rectangle, only the opposite sides need to be equal.

T: Since a square has all sides equal, it also has opposite sides equal. So, why is a square not a rectangle?

S: Well, when we look at a square table, we don't call it a rectangular table, and when somebody tells us that a table is rectangular, we don't consider that it might be square. However, I remember reading in a textbook that a square is a rectangle.

T: For now, let us ignore textbooks and you try to answer the question. Do you think a square should be a rectangle or not? To put it differently, we have to choose between these two options:

Option 1: Squares are special case of rectangles.
(All squares are rectangles, but there are some rectangles that are not squares.)

Option 2: Squares and rectangles are distinct.
(Squares are not rectangles, and no rectangle is a square.)

Which option would you pick, and why?

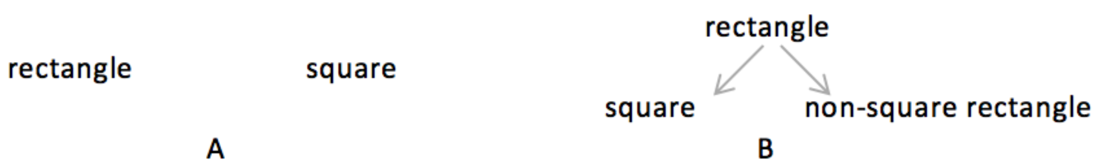
S: For the reasons I gave earlier, clearly a square should not be a rectangle.

T: Let us write down the definition of a rectangle:

1. It is a closed shape.
2. It has 4 sides.
3. Its sides are straight lines.
4. All angles are right angles.
5. Opposite sides are equal.
6. Adjacent sides are not equal.

Notice, that if we keep 6, a square is not a rectangle. However, if we remove 6, a square is a rectangle.

We are now choosing between the following classificatory systems:



S: As we said before, in order to describe things around us A is probably better.

T: I agree. However, let us look at the classification in another way — if we accept B, everything that is true about a rectangle is automatically inherited by squares. For instance, you may have learnt that the diagonals of a rectangle are equal. Well that is also true about squares. If we choose classification A, we have to prove this separately for squares. In classification B, if we prove it for rectangles, it is automatically inherited by the square. So, even though A is better for **descriptive** purposes, B is better for '**academic**' purposes.

Also, notice that the definition of square is a lot shorter in B — a square is a rectangle with adjacent sides equal. All the other properties we listed earlier, that:

It is closed;

It has 4 sides;

Its sides are straight lines; and

All angles are right angles;

would follow from the statement that it is a rectangle.

If this doesn't convince you, let me try something else. You probably know what parallelograms and quadrilaterals are. I will provide two classifications for quadrilaterals, parallelograms, rectangles and squares. I want you to come up with their definitions like we did for squares and rectangles.

Classification A:

Polygon

Square | Rectangle | Parallelogram | Quadrilateral

Classification B:

Polygon

↓

Quadrilateral

↓

Parallelogram

↓

Rectangle

↓

Square

Let us assume that all of these are types of Polygons. Polygons are closed shapes with straight lines as sides. Start with B.

S: A Quadrilateral is a 4-sided polygon.

A Parallelogram is a Quadrilateral with opposite sides parallel.

A Rectangle is a Parallelogram with all right angles.

A Square is a Rectangle with adjacent sides equal.

T: That was pretty easy. Now try A. Start with a square. Then you will have to define a rectangle in such a way that it is not a square; a parallelogram such that it is not a rectangle, nor a square; and a quadrilateral such that it is none of these: a square, rectangle or parallelogram. Don't get disheartened if this takes a while.

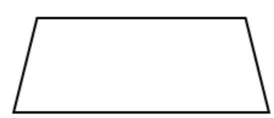
S: A Square is a 4-sided polygon with all right angles, and all sides equal.

A Rectangle is a 4-sided polygon with all right angles, opposite sides equal, and adjacent sides not equal.

A Parallelogram is a 4-sided polygon with opposite sides parallel and no right angle.

A Quadrilateral is a 4-sided polygon with no pair of parallel sides.

T: I think all of your definitions work except for the last one. Look at this quadrilateral. It has one pair of parallel sides, but is not a parallelogram, rectangle or square:



S: Okay, a Quadrilateral is a 4-sided polygon with at most one pair of parallel sides.

T: Notice that A was a lot harder than B, and the definitions were longer. So, we have two reasons for choosing B over A:

a) Inheritance of properties
b) Simpler definitions

Note: This can be generalized to areas outside of mathematics. For instance, consider the choice between the Linnaean classificatory system of living organisms vs. Aristotle's classification. Linnaeus classified humans under mammals, great apes, and so on, while Aristotle chose to separate humans from all other living creatures. If our goal is to make predictions about humans and other living things, the reasons for choosing Linnaeus over Aristotle are similar to the ones we have for choosing the system in B over the one in A above. [Also see LT4, Reading A, sections 4.4 and 4.5.]

Summary

In Part 1 of mathematical inquiry, we made a distinction between school math and real math, 'real math' being the kinds of mental activities that professional mathematicians engage in. Focusing on pure mathematics, we described theory construction, and identified some of the activities that go into theory construction as:

a. Defining objects
b. Creating classificatory systems
c. Creating representations for objects
d. Unearthing axioms and definitions
e. Putting together axioms and definitions as a coherent theory
f. Finding Patterns
g. Stating those Patterns as Conjectures
h. Proving those Conjectures from the Axioms, Definitions and existing Theorems of the Theory
i. Evaluating Proofs
j. Generalizing and Extending Theorems

We illustrated some of these activities in Part 1 and Part 2.

This is by no means a comprehensive account of mathematical thinking or an in-depth account of some of the strands of mathematical thinking. All that we have done is give you a feel for what it is like to think like a mathematician.