Topic: Higher Order Derivatives

Higher order derivatives are derivatives greater than the first derivative. The derivative of a differentiable function is called *first derivative.* This first derivative of the given function is also a function and the derivative of this first derivative is called *second derivative.* The derivative of the second derivative is called *third derivative.* The derivative of the third derivative is called *fourth derivative and so on.*

NOTATIONS TO BE USED FOR HIGHER ORDER DERIVATIVES

First Derivative	Second Derivative	Third Derivative	Fourth derivative
$f'(x)$	$f''(x)$	$f'''(x)$	$f^{IV}(x)$
y'	y''	y'''	Y^{IV}
dy/dx	d^2y/dx^2	d^3y/dx^3	d^4y/dx^4
$D_x(y)$	$D^2{}_x(y)$	$D^3{}_x(y)$	$D^4{}_x(y)$

Note: The process of finding the 2nd derivative, 3rd derivative, 4th derivative and so on is the same as as the process of determining the first derivative.

Example 1
Find the first, second, third, fourth and fifth derivatives of $y = 3x^4 - 5x^3 + 4x^2 - x + 8$

Solutions:
$y = 3x^4 – 5x^3 + 4x^2 – x + 8$
$y' = 12x^3 – 15x^2 + 8x - 1$
$y'' = 36x^2 = 30x + 8$
$y''' = 72x – 30$
$y^{IV} = 72$
$y^{V} = 0$

Example 2
Find the first, second and third derivatives of $y = 5\cos(x) + 4/(x+6)^2 – 3x$

Solutions:
$y = 5 \cos (x) + 4/(x+6)^2 – 3x$
$y' = 5(-\sin x) + 4 (-2) (X+6)^{-3} - 3$
$= -8 (x+6)^{-3} – 3 – 5\sin x$
$Y'' = -8 (-3) (x+6)^{-4} – 5\cos x$
$= 24 (x + 6)^{-4} – 5\cos x$
$Y''' = -4 (24) (x +6)^{-5} – (- 5 \sin x)$
$= -96 (x +6)^{-5} + 5 \sin x$

Example 3
Determine the first, second, third and fourth derivatives of $1/x^2$

Solutions:

$y = 1/x^2$
$y = x^{-2}$
$y' = -2x^{-2-1}$
$y' = -2x^{-3}$ or $y = -2/x^3$
$y'' = -2(-3)x^{-3-1}$
$= 6x^{-4}$ or $6/x^4$
$Y'' = -4(6)x^{-4-1}$
$= -24x^{-5}$ or $-24/x^5$
$Y^{IV} = -24(-5)x^{-5-1}$
$= 120x^{-6}$ or $120/x^6$

Example 4
Determine the first, second, third and fourth derivatives of $f(x) = x^4 + 2x^3 - 3x^2 - x + 5$

Solutions:
$f(x) = x^4 + 2x^3 - 3x^2 - x + 5$
$f'(x) = 4x^3 + 6x^2 - 6x - 1$
$f''(x) = 12x^2 + 12x - 6$
$f'''(x) = 24x + 12$
$f^{IV}(x) = 24$

STUDENT TASK

A. Directions: Find the first, second, and third derivatives for each of the following functions.

1. $Y = 3x^3 + 3x^2 = 3x = 3$

2. $Y = 6x^4 + 5x^3 - 4x^2 - 3x - 2$

3. $y = (2x + 3)^4$

4. $y = (2 - x^2)^3$

5. $y = (x+3)^2 (x - 2)$

B. Directions: Differentiate the following functions up to the third order.

1. $F(x) = 3\cos(x) + (x - 2)^3$

2. $F(X) = \tan(x) + (x - 2)^4$

3. $F(x) = 2 \sin(x) + 3 (x+5)^3$

4. $F(x) = \cot (x) – 3x^5$

5. $F(x) = 5 \cos (x) + (X+5)^{-2} +$

Topic: Implicit Differentiations

An *implicit function* is not explicitly defined. Consider the two problems:

1. Find the equation of the line tangent to the parabola $y = x^2 – 4x + 2$ at point (3, -1)
2. Determine the equation of the line tangent to the circle $x^2 + y^2 = 100$ at point (6, 8)

In the first problem, the given equation is explicitly defined. The dependent variable y is expressed in terms of the independent variable of x. The equation is of the form y = f(x) where $f(x) = x^2 – 4x + 2$.

To solve the problem, the first tdyhing to do is to get the first derivative

$$y = x^2 – 4x + 2$$
$$dy/dx = 2x – 4$$

To find the slope, replace x by the abscissa of the given point of tangency

$$dy/dx = 2x = 4$$
$$= 2 (3) – 4$$
$$= 6 – 4$$
$$= 2$$

To find the equation of the tangent line, use the Point-Slope Formula

$$y – y_1 = m (x – x_1)$$
$$y – (-1) = 2 (x – 3)$$
$$y + 1 = 2x – 6$$
$$y = 2x – 7$$

In the second problem, the given equation is not explicitly defined. The dependent variable y is not expressed in terms of the independent variable x. Before solving for the first derivative, write it first in the form y = f(x).

$$x^2 + y^2 = 100$$
$$y^2 = 100 – x^2$$
$$y = \sqrt{100-x2}$$
$$= (100 – x^2)^{1/2}$$
$$dy/dx = ½ (100 – x_2)_{-1/2} (- 2x)$$
$$= -x / (100 – x_2)_{1/2}$$
$$= -x / \sqrt{100-x^2}$$

To find the slope, replace x by the abscissa of the point of tangency.

$$\text{slope} = -x / \sqrt{100-x^2}$$
$$= - 6 /\sqrt{100-6^2}$$
$$= - 6 /\sqrt{100-36}$$

$= -6 / \sqrt{64}$

$= -6/8$

$= -3/4$

To find the equation of the line tangent to the circle at (6, 8), use the Point-Slope Formula

$y - y_1 = m (x - x_1)$

$y - 8 = -3/4 (x - 6)$

$y = -\frac{3}{4}x + \frac{18}{4}$

$y = -\frac{3}{4}x + \frac{18}{4} + 8$

$y = -\frac{3}{4}x + \frac{25}{2}$

There is another method in solving for the first derivative in the second problem. It is called implicit differentiation. In this method, there is no need to write the equation in the form y = f(x). Here are the steps:

1. Differentiate both sides with respect to x.
2. Isolate all dy/dx terms on one side (preferably on the left side)

$x^2 + y^2 = 100$

$dy/dx\ (x^2) + dy^2/dx = d/dx\ (100)$

$2x + 2y\ dy/dx = 0$

$2y\ dy/dx = - 2x$

$= -2x/2y$

$= -x/y$

Slope $= -x/y$

$= -6/8$

$= -3/4$

Illustrative Examples:

1. Differentiate $4x^2 + y^2 = 16$ Solution: $4x^2 + y^2 = 16$

$d/ dx\ (4x^{2)}) + d/dx\ (y^2) = d/dx\ (16)$

$8x + 2y\ dy/dx = 0$

$2y\ dy/dx = - 8x$

$dy/dx = -8x/2y$

$= -4x/y$

2. Differentiate $y - x^2 = -7$

Solutions: $y - x^2 = -7$

$d/dx\ (y) - d/dx\ (x^2) = d/dx\ (-7)$

$1\ dy/dx - 2x = 0$

$dy/dx = 2x$

3. Differentiate $x^2 - 2x + y^2 = 24$

Solution: $d/dx\ (x^2) - d/dx\ (2x) + d/dx\ (y^2) = d/dx\ (24)$

$2x - 2 + 2y\ dy/dx = 0$

$2y\ dy/dx = 2 - 2x$

$dy/dx = (2 - 2x)/2y$

$= (1 - x)/y$

4. Find the derivative of $x^2 + y^2 - 8x + 2y = 16$

Solution : $x^2 + y^2 - 8x + 2y = 16$

$d/dx\ (x^2) + d/dx\ (y^2) - d/dx\ (8x) + d/dx\ (2y) = d/dx(16)$

$2x + 2y\ dy/dx - 8 + 2\ dy/dx = 0$

$2y\ dy/dx + 2\ dy/dx = 8 – 2x$

$dy/dx\ (2y + 2) = 8 - 2x$

$dy/dx = (8 – 2x)/(2y + 2)$

$= 2(4 – x)/2(y + 1)$

$= (4 – x)/(y + 1)$

5. Find the derivative of $y – (x – 3)^2 = 4$

   Solution: $y – (x – 3)^2 = 4$

   $d/dx\ (y) – d/dx\ (x-3)^2 = d/dx\ (4)$

   $1\ dy/dx - 2\ (x - 3) = 0$

   $dy/dx = 2x – 6$

STUDENT TASK

A. Directions: Differentiate the following without solving for y in terms of x.

1. $x^2 + y^2 = 81$
2. $2x + 9y = 12$
3. $5x – 3y = 15$
4. $3x^2 + y^2 = 9$
5. $x^2 + 4y = 16$
6. $x^2 + y = 1$
7. $y – x^2 + 2x = o$
8. $4x^2 + 9y^2 = 36$
9. $16x^2 + 4y^2 = 64$
10. $x^2 - 4x + y^2 - 2y = 7$

B. Directions: Determine the first and second derivatives of each given function without solving for y in terms of x.

1. $x + 2y = 5$
2. $x^2 - x – y = 6$
3. $x^2/4 + y^2 = 1$
4. $x^2 + y^2/9 = 1$
5. $x^2 – 7x + y^2 – 7y + 1$
6. $X^2 + 5xy + 3y^2 = 4$
7. $Y + x^2 - 18x = 49$
8. $y^2 + 3xy + x^2 = 9$
9. $x^2/9 + y^2/4 = 1$
10. $x^2 + 2xy – y^2 = 0$

Topic: Optimization Problems

Problems about maximum and minimum values are problems about *optimization* which will be taken in consideration in this lesson. In this problems, there are two equations involved:

1. The optimization equation
2. The constraint

In solving optimization problem, the first thing to do is to determine what is to be optimized or maximized. Then determine the constraint.

A *constraint* is a quantity that I true regardless of the solution. It can be viewed as a condition the problem must satisfy. In every problem there is a fixed value which is a part of the constraint. Carefully analyze the constraint equation and solve for one of its variables. The value of this variable can be used to replace the same variable appearing in the optimization equation. Simplify the optimization equation and find the first derivative. Then find the critical value by setting the first derivative to zero. Test the critical value/s either by using the First Derivative Test or the Second Derivative Test.

Example 1: The sum of two positive numbers is 12. What are these numbers if the product is a maximum?

Solution:
Let x – be one of the two positive numbers
Y – be the other positive number

$x + y = 12$ (this is the constraint)
$P = xy$ (this is the optimization equation)

Solve for one of the variables in the constraint. Then, substitute the value of this variable in the optimization equation.

$x + y = 12$
$y = 12 - x$
$p = xy$
$P = x(12 - x)$
$= 12x - x^2$

Find the first derivative

$P = 12x - x^2$
$P' = 12 - 2x$

Set the first derivative equal to zero

$12\ 2x = 0$
$-2x = -12$
$x = 6$

Test

$P' = 12 - 2x$
$P'' = -2$ (negative)
$P'' < 0$

The maximum value occurs when $x = 6$

$y = 12 - x$
$= 12 - 6$
$= 6$

Hence, the two numbers are 6 and 6.

Example 2: Find two positive numbers whose product is 100 and whose sum is a minimum.

Solution:
Let x – be one of the two positive numbers
Y – the other positive numbers

$x \cdot y = 100$
$y = 100/x$

$S = x + y$
$= x + 100/x$

Find the first derivative
$S' = 1 + (-1)(100)x^{-2}$
$= 1 - 100\ x_{-2}$

Set the first derivative to zero and solve for x

$1 - 100x^{=2} = 0$

Example $1 - 100/x^2 = 0$

$X^2 - 100 = 0$
$(x + 10)(x - 10) = 0$
$x + 10 = 0$; $x - 10 = 0$
$x = -10$; $x = 10$

Disregard -10 because the problem is asking for two positive numbers.

Test
$S' = 1 - 100\ x^{-2}$
$S'' = -100(-2)x_{-3}$
$= 200\ x^{-3}$
$= 200/x_3$
$= 200/10^3$
$= 200/1000$
$= 0.2$
$S'' > 0$

The minimum value occurs when x = 10.

$y = 100/x$
$= 100/10$
$= 10$ (the other number)

Therefore, the two positive numbers are 10 and 10.

Example 3: You are asked to fence a rectangular region and maximize the area. You are given 260 meters of fencing materials. What should be the dimensions of the rectangular region?

Solution:

Let x – be the width of the rectangle
Y – be the length of the rectangle

$P = 2x + 2y$ (the constraint)
$260 = 2x + 2y$
$2y = 260 - 2x$
$y = (260 - 2x)/2$
$y = 130 - x$

$A = x \cdot y$ (the optimization equation)

$= x\,(130 - x)$

$= 130x - x^2$

Find the first derivative

$$A = 130x - x^2$$
$$A' = 130 - 2x$$

Set the first derivative equal to 0

$$130 - 2x = 0$$
$$130 = 2x$$
$$2x = 130$$
$$x = 65$$

Test

$$A' = 130 - 2x$$
$$A'' = -2$$
$$A'' < 0$$

The maximum value occurs when $x = 65$.

$y = 130 - x$

$= 130 - 65$

$= 65$

The length is 65 meters and the width is 65 meters.

STUDENT TASK

Directions: On a yellow paper, solve the following optimization problem.

1. A rectangular garden is to be fenced off along the side of a building. No fence is required along the side. There are 120 meters of fencing materials to be used. Find the dimensions of the garden with the largest area.

2. An open box is to be made from a 24 cm square cardboard by cutting equal squares out of the corners and turning up the sides. Find the height of the box that will give a maximum volume.

3. The sum of two positive numbers is 42. Find these numbers if their product is the maximum.

4. The sum of two positive numbers is 10. Find these numbers if the sum of their squares is a minimum.

5. Find two positive numbers whose sum is 15 if the product of the first number and the square of the second number is to be a maximum.

6. Determine the dimensions of a rectangle with an area of 22 cm^2, whose perimeter is as small as possible.

7. John build a rectangular pig pen with two parallel partitions using 150 meters of fencing materials. What should be the dimensions of the pig pen so that it would enclose the largest possible area?

8. The principal of private elementary school wants to construct a rectangular garden. She wants it to be built next to the wall of the school building. There are 90 meters of fencing materials available for the three sides. What should the dimensions of the garden be to maximize the area?

9. An open box with a square base is to be made from 1200 cm^2 of materials. What will be its dimensions so that it will have the largest possible volume?

Make an open box with a square base from 1728 cm^2 of materials. What should be the dimensions of the box in order to have the