NUMERICAL SOLUTION OF BOUNDARY LAYER EQUATION FOR NON-NEWTONIAN FLUID FLOW

A Thesis submitted to Gujarat Technological University

for the Award of

Doctor of Philosophy

in

Science–Maths

By
PATEL JAYSHRIBEN RAMJIBHAI
Enrollment No.: 139997673017

under the supervision of

Prof. (Dr.) MANISHA P. PATEL

GUJARAT TECHNOLOGICAL UNIVERSITY
AHMEDABAD

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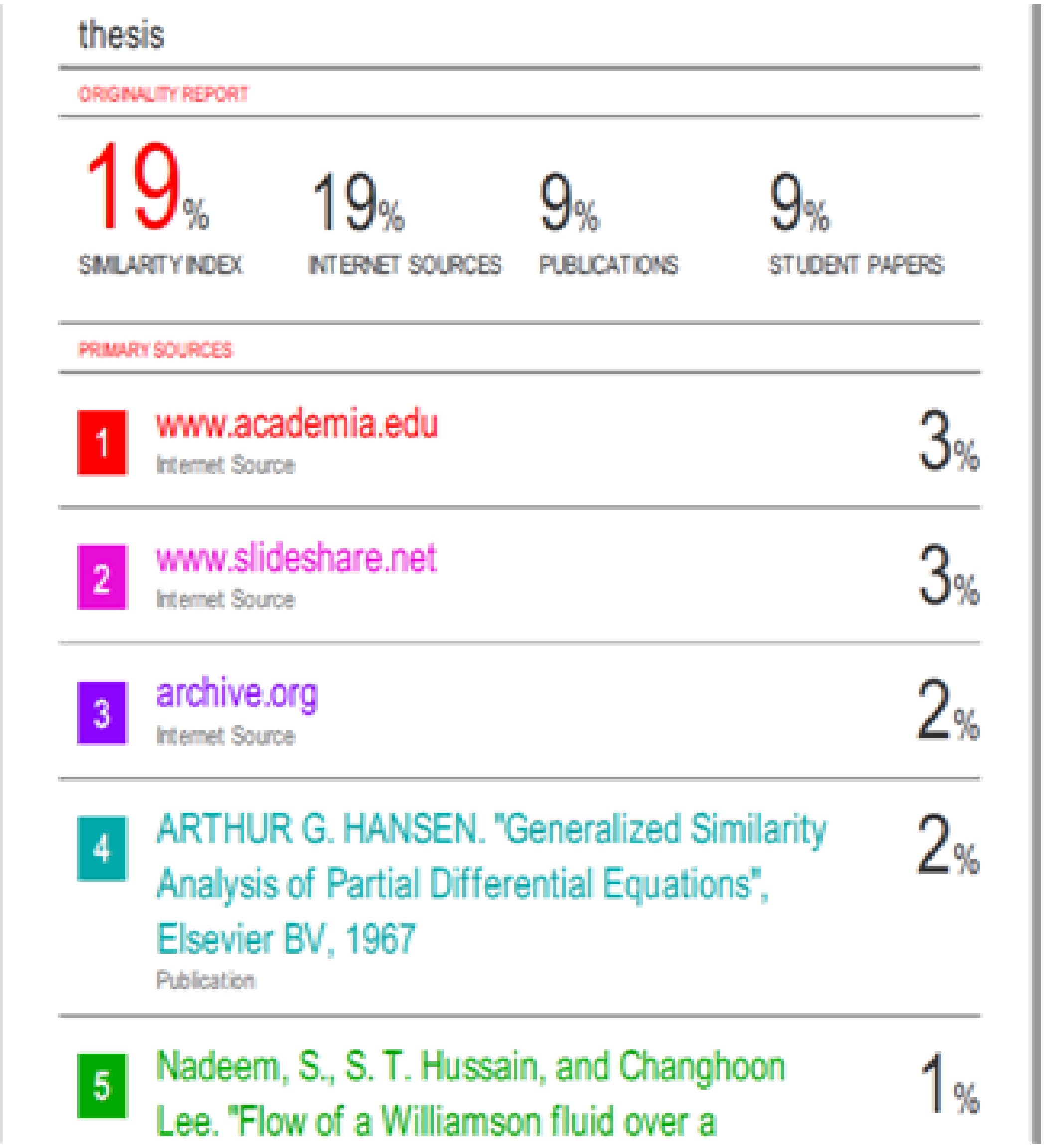

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ABSTRACT

The complex rehology of biological fluids has motivated investigations involving different non-Newtonian fluids. In recent years, non-Newtonian fluids have become more and more important industrially. Academic curiosity and practical applications have generated considerable interest in finding the solutions of differential equations governing the motion of non-Newtonian fluids. The governing equations of such non-Newtonian fluids are highly non linear partial differential equations. These fluids flow problems present some interesting challenges to researchers in engineering, applied mathematics and computer science.

Two dimensional Magneto Hydro Dynamic (MHD) and non-MHD laminar boundary layer flow of non-Newtonian fluid is analyzed in the present work. Three non-Newtonian fluid models for the stress-strain relationship Sisko fluid model, Prandtl fluid model and Williamson fluid model are considered for the different flow geometry (semi-infinite flat plate & semi infinite moving plate). Governing non linear partial differential equations (PDE) are transformed into non linear ordinary differential equations (ODE) using group theoretical methods. The quasi linearization method is then applied to convert the non linear ODE into linear ODE. The numerical solutions of the resulting linear ordinary differential equations are obtained using Finite difference method (method carried out with Microsoft excel and MATLAB coding). The graphical presentation of the velocity profile is given for each problem.

Calculations on non-Newtonian media present a new challenge in flow analysis. Simulating these types of flows in order to calculate pipe and pump sizes presents a significant challenge to the engineer. The stress-strain relationship for different type of viscous-inelastic fluids and similarity equations using these relationships will be helpful to many researchers and engineers for further research.

Our aim is to modify similarity techniques and apply it to find similarity solutions of MHD and non-MHD boundary layer flows of non-Newtonian fluids. Applications of these

techniques are useful for the treatment of engineering boundary value problems. Our aim is to propose quite a new idea in the theory of similarity analysis and that is approximate similarity technique. We hope that our proposed approximate similarity technique will be useful to treat the non-linear partial differential equations.

Scopes of the new proposed technique are wide. It will be useful to solve non-similarity equations and also hope to derive some closed form solutions. It may be used for finding the exact solutions for defined fluid models.

Acknowledgement

I am grateful to receive valuable help and encouragement from several persons in course of this research.

Foremost, I would like to express my sincere thanks to my research supervisor Prof. (Dr.) Manisha P. Patel, Assistant Professor in Mathematics at Sarvajanik College of Engineering and Technology (SCET), Surat, for her continuous support during my research work. Her deep insight and ever-growing knowledge have contributed a great deal in my approach to the subject. Her guidance, motivation and enthusiasm have striven to make this research work possible. I feel short of words to express my gratitude towards her.

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Words cannot express how grateful to my mother, father, mother-in-law, father-in-law, brother and sisters for all of the sacrifices that you’ve made on my behalf. Your prayer for me has sustained me thus far. I am very much thankful to my husband, Dr. Vipul Patel for his inspiration and confidence in my ability. Without his patience and support, this work would not have been possible. I feel short of words to express my gratitude towards him. I would also like to thank my beloved daughter Dhani for her moral and emotional support and for always cheering me up.

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JAYSHRI PATEL

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List of Abbreviations

FDM :Finite Difference Method

ODE :Ordinary Differential Equation

PDE :Partial Differential Equation

MHD :Magneto Hydro Dynamic

QLM :Quasi Linearization Method

GNF : Generalized Newtonian Fluids

List of Symbols

u : Velocity components in X-direction

v : Velocity components in Y-direction

w : Velocity components in Z-direction

U : Main stream velocity in X-direction

$\bar{\bar{\tau}}$: Shear Stress component

$\bar{\bar{\Delta}}$: Strain rate component

e_{lm} : Strain rate component

n : Flow behavior index

η : Similarity variable

f, g : Similarity functions

$A, \alpha_i's, \beta_i's$: Real constants/parameters

G : Group notation

B_0 : Imposed magnetic induction

ψ : Stream function

L : Characteristic length

K_n : Knudsen number

F : Force

σ_i & S : Cauchy stress tensors

α : Deformation angle

ρ : Density

W : Specific wight

S : Specific gravity

μ : Dynamic Viscosity

υ : Kinematic Viscosity

s, a, b, A, B : Fluid parameter

t : Time

M_a : Mach number

R_e : Reynold’s number

Pr : Prandtl number

Cp : Specific heat

K : Thermal conductivity

τ_{ij} : stress tensor in the direction of j-axis perpendicular to i-axis

F_i : Inertia force

F_v : Viscous force

F_p : Pressure force

F_g : Gravity force

h : width of interval

B_0 : Imposed magnetic induction

σ : Electrical conductivity

M_0 : magnetic field strengths/ Magnetic constant

P : Pressure

I : Identity vector

μ_0 : Limiting viscosities at zero

μ_∞ : Limiting viscosities at infinite

A_1 : The first Rivlin-Ericksen tensor

Γ : Time constant

U_∞ : Velocity at the wall

B : Stretching parameter along X-axis

λ : Dimensionless Williamson Parameter

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CHAPTER-1

FLUID MECHANICS AND APPLICATIONS

1.1 INTRODUCTION

It is fact that our earth is covered with 75 % of water and 100 % with air. This air and the water of seas and rivers are always in motion. Also the flows of gases, water and sewage in pipes, an irrigation canals, air flow around rockets, motion of express trains, aircraft, automobiles and boats etc… act as the resistance on such flows. Moreover, the movement of passengers on the platform of a railway station can be considered as forms of flow. In this way, we have too closed relationship for fluid flows that is the fluid mechanics. The study of flow is really a very popular thing to all of us.

The extension of fluid mechanics is too vast and has unlimited practical applications ranging from microscopic biological systems to automobiles, airplanes and spacecraft propulsion. For example fans, turbines, pumps, artillery, studies of breathing, blood flow, oceanography, hydrology, energy generation etc…Fluid mechanics is very exciting and interesting subject with historically one of the most challenging subjects for researchers.

1.2 DEFINITION OF FLUID MECHANICS

Fluid is a set of particles that are continually deforms under applied tangential or shear stress however it may be very small. This deformation due to application of shear stress constitutes a flow. Mechanics is an oldest branch of physical sciences which deals with the both moving and stationary under the effects of forces. Together both the terms fluids and mechanics then fluid mechanics is that branch of physics which deals with the motion and forces on fluid particles that are continually flows under an applied shear stress. In other words, we can also say that the fluid mechanics is an oldest branch of physics in which we deals with the mechanics of liquids, gases and plasmas at rest as well as when it is in motion.

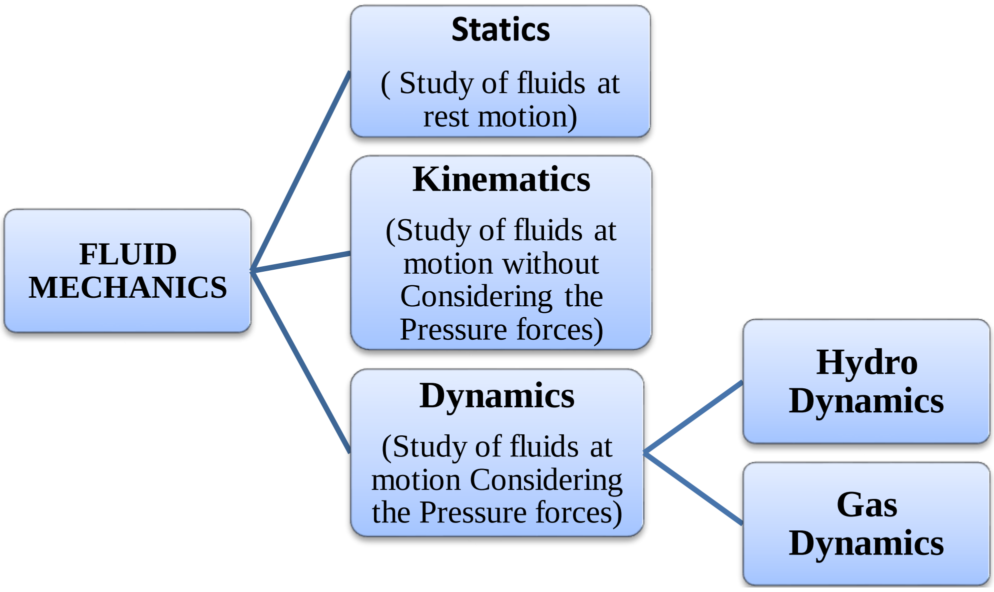


Figure 1.1 Classification fluid mechanics

Fluid mechanics divided into three main branches **(Figure 1.1)** Fluid Statics, Fluid Dynamics and Fluid Kinematics. In fluid statics, the study of fluids at rest; while in fluid kinematics, the study of the effect of forces on fluid motion where pressure forces are not considered. Fluid dynamics is the study of the effect of forces when fluid is in motion. Fluids which are not compressible (like liquids and gases at low speeds) are generally referred as hydrodynamics. Liquid flows in open channels and pipes are included in hydraulics, a subcategory of hydrodynamics. The flows of fluids that undergo significant density changes (compressible), such as the flow of gases through nozzles at high speeds are included in gas dynamics. Aerodynamics branch deals with the flow of gases (especially air) over bodies such as rockets, aircraft and automobiles at high speed as well as low speed. Few other specialized categories are oceanography, hydrology and meteorology deal with flows which occur naturally.

Fluid Mechanics is also referred to as fluid dynamics by considering fluid at rest as a special case of motion with zero velocity. Fluid Dynamics is a branch of continuum mechanics and an active field of research with many problems that are partly or wholly unsolved. The solution of a fluid mechanic problem normally involves calculating for various properties of the fluid, such as velocity, pressure, density and temperature as a function of space and time. Mathematically, it is too complex so it can best be solved by numerical method.

1.3 HISTORY OF FLUID MECHANICS

One of the most engineering problems faced by humankind with the development of cities was the supply of water for farming and domestic uses. Our modern lifestyles can be retained only with great amount of water, and it is clear from archaeology that every successful civilization of prehistory invested in the construction as well as maintenance of water systems. The most spectacular engineering from a technical viewpoint was done at the Hellenistic town of Pergamum in present-day Turkey. There, from 283 to 133 BC, they construct a series of pressurized lead and clay pipelines up to 45 km long shown in **Figure 1.2** that operated at pressures exceeding. Archimedes had given the earliest recognized contribution to the theory of fluid mechanics. He first developed the buoyancy principle and applied them to floating and submerged bodies for deriving a form of the differential calculus as part of the analysis in history. Then Leonardo da Vinci (1452–1519) gives the equation of conservation of mass in one-dimensional time independent flow.

Figure 1.2 Segment of Pergamon clay pipe line

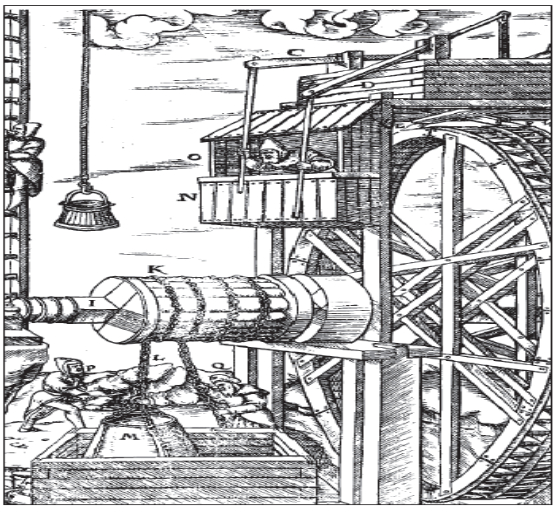


Figure1.3 A mine hoist powered by a reversible water wheel

During the middle ages the application of fluid machinery step by step expanded and elegant piston pumps were developed for dewatering mines, and the watermill and windmill were perfected to grind grain, forge metal for different purpose. For the first time in recorded human history vital work was done without the power supplied by a person or animal **(Figure 1.3)** and these inventions are credited with enabling the later industrial revolution. Then after the scientific method was perfected and adopted by Galileo (1564–1642), Simon Stevin (1548–1617), Edme Mariotte (1620–1684) and Evangelista Torricelli

(1608–1647) were among the first to apply hydrostatic pressure distributions and vacuums to fluids. Blaise Pascal (1623–1662) had integrated and refined their work .The Italian monk, Benedetto Castelli (1577–1644) was the first to publish a statement of the continuity principle for fluids. Sir Isaac Newton (1643–1727) applied his laws of motion and the law of viscosity of the linear fluids which is now called Newtonian fluids. The development of fluid mechanics theory up through the end of the 18th century had little impact on engineering since fluid properties and parameters were not properly quantified, and most theories were abstractions that could not be quantified for design purposes. That was to alter with the development of the French school of engineering led by Riche de Prony (1755–1839). Antonie Chezy (1718–1798), Louis Navier (1785–1836), H. Darcy (1803–1858) and many other contributors to fluid theory were students / instructors at the schools. By the mid of 19th century fundamental advances were coming on many fronts. Flow in capillary tubes for multiple fluids was accurately measured by the physician Jean Poiseuille (1799–1869), whereas the difference between laminar and turbulent flow in pipes had been given by German scientist Gotthilf Hagen (1797–1884). The classic pipe experiment in 1883, showed the importance of the dimensionless number was published by Lord Osborne Reynolds (1842–1912). The dimensionless number give name after him Reynolds number. Similarly, in parallel to the early work of Navier (1785-1836) and Stokes (1819–1903) gives the general equations of fluid motion with Newtonian viscous terms that take their names.

The Irish and English scientists and engineers, including in addition to Reynolds and Stokes, William Thomson, Lord Kelvin (1824–1907), Lord Rayleigh (1842–1919) and Sir Horace Lamb (1849–1934) has been given notable contribution in the expansion of fluid theory in the late 19th century. They have investigated numerous problems such as dimensional analysis, irrotational flow, cavitations, vortex motion and waves. Their work also explored the connection between fluid mechanics, thermodynamics and heat transfer. Their primary invention was completed and included all the major characteristics of modern craft. The Navier–Stokes equations were nearly impossible to solve to this time therefore it is of little use. In a pioneering paper published in 1904, by the German Ludwig Prandtl (1875–1953), showed that fluid flows with very small viscosity can be divided into a boundary layer, near solid surfaces and interfaces, patched onto a nearly inviscid outer layer, where the Euler and Bernoulli equations are applied such as air flows and water flows. The theory of Boundary-layer has established to be a very important tool in advanced flow analysis. Layer closed to the walls, the boundary layer, where the friction effects are important and an outer layer where such effects are negligible and the simplified Euler and Bernoulli equations are applicable.

In the mid of 20th century the theories, properties and parameters of fluid were very well defined so this time could be considered as a golden time for fluid mechanics. Due to this, a large expansion could be possible in the sector of industrial, aeronautical, chemical and water resources. And hence fluid mechanics pushed in the new directions.

1.4 APPLICATIONS OF FLUID MECHANICS

Like most scientific disciplines, Fluid mechanics is vast and touches nearly each human endeavour. Fluid mechanics has a wide range of applications, including civil engineering, mechanical engineering, chemical engineering, geophysics, biomedical engineering, astrophysics, and biology. It is applicable in our daily activities as well as in the design of almost all advanced engineering systems from vacuum cleaners to supersonic aircraft. So it is very important to us as a basic knowledge and understanding of the fundamentals fluid mechanics.

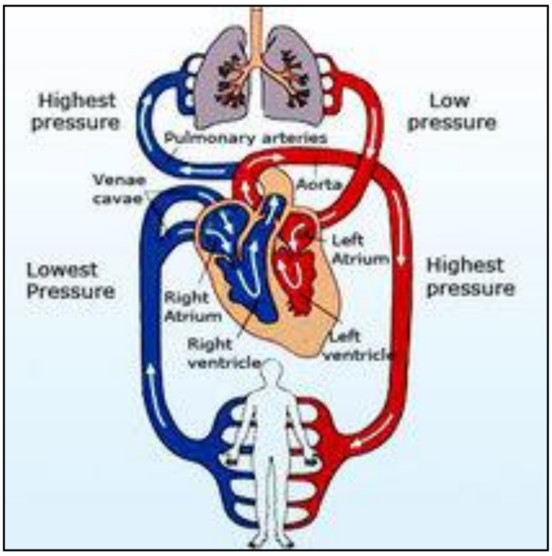


Figure 1.4 Blood circulation in human body

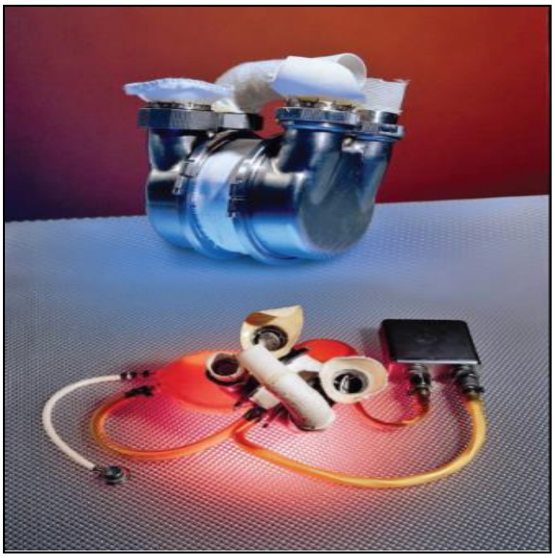


Figure-1.5 Artificial Heart

Let’s begin with some functions in human body follow some rules of fluid mechanics. The heart constantly pumping blood to every parts of a body through the veins and arteries and lungs are the medium of airflow in alternating directions shown in **Figure 1.4.** Fluid dynamics is also extensively applied in the technique for artificial hearts **(Figure 1.5)**, dialysis systems and breathing machines. The piping systems for cold water, natural gas, and sewage for an individual home **(Figure 1.6)** etc… is designed basically on the fundamental of fluid mechanics. So we can say that our ordinary houses are an exhibition hall full with the applications of fluid mechanics. The same can be said for the piping and ducting network of air-conditioning and heating systems which are used in industries **(Figure 1.7)**.

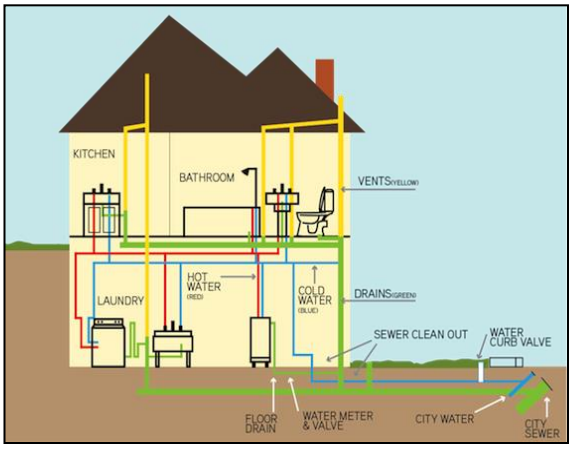


Figure 1.6 Plumbing System in house

Figure 1.7 Industrial Application

All transportation issues involve fluid motion with well-developed specialise in aerodynamics of aircraft and rockets, in naval hydrodynamics of ships and submarines. Most probably all our electric energy is developed either from water flow or from steam flow through turbine generators. All these components are designed by using fluid mechanics. We can also see various applications of fluid mechanics in an automobile. All components related to the transportation of the fuel from the fuel tank to the cylinders the fuel pump, fuel line, fuel injectors or carburettors as well as the mixture of the fuel and the air in the cylinders, the purging of combustion gases in exhaust pipes are analyzed using fluid mechanics. Fluid mechanics is also utilised in the design of the hydraulic brakes, automatic transmission, the power steering and lubrication systems, the cooling system of the engine block including the radiator and the water pump, and even the tires. The sleek streamlined shape of recent model cars is the result of efforts to minimize the drag using extensive analysis of flow past a surface. Hence, fluid mechanics plays a significant role in the design and analysis of aircraft, rockets, wind turbines, boats, submarines, jet engines, biomedical devices, the cooling of electronic components, and the transportation of water, crude oil, and natural gas. It is also applicable in the design of buildings, bridges, and even billboards to make sure that the structures can face up to wind loading.

Figure 1.8 Natural flows

Various natural phenomena such as the rain cycle, weather patterns and increase of ground water to the top of trees, ocean waves and currents in large water bodies are also governed by fluid mechanics principles **Figure 1.8.**

1.5 FLUID AS A CONTINUUM

The continuum is a property of a system which is considered as a continuous function of space and time. It is well-known that, for fluids (gases and liquids) the molecules are widely spaced in gases on other sides, closely spaced for a liquid. The distance between these molecules is quite larger than the diameter of molecular. Also note that the molecules are not fixed in a lattice but move about freely relative to every alternative. Statistically the average distance the molecules travelled between two successive collisions is called a mean free path (λ)and characteristic length of the medium is denoted by L (characteristic dimension is depend upon the problem. for example: In case of pipe flow it will be the diameter of the pipe) then the dimension less parameter λ/L is known as Knudsen number denoted by K_n.

Mathematically written as, $K_n = \frac{\lambda}{L}$

When, $K_n < 0.01$: Continuum is valid

$K_n > 0.01$: The concept of continuum does not hold good

$0.01 < K_n < 0.1$: Continuum is slip flow

$0.1 < K_n < 10$: Continuum is transition flow

$K_n > 10$: Free medium flow

Here, we considered the flow regimes is always less than 0.01 and it is say that the fluid is a continuum.

1.6 SHEAR STRESS AND NORMAL STRESS

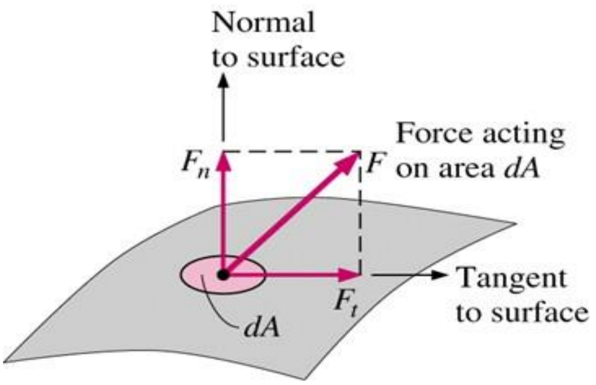


Figure 1.9 Shear stress and Normal stress on the surface of a fluid element

In a **Figure 1.9** small area dA considered on the surface of fluid element. The force component F_t acting along the surface direction and the force component F_n acting normal to the surface of the force F acting on the unity of surface are called Tangential Stress (τ- Shear Stress) and Normal Stress(σ) respectively.

Mathematically it is given by,

The Normal Stress $\sigma = \lim_{dA\to 0} \frac{\mathrm{Fn}}{dA}$

And, Shear Stress $\tau = \lim_{dA\to 0} \frac{\mathrm{Ft}}{dA}$

If a body in rest motion, the normal stress is called a **Pressure**.

1.7 FLUID DEFORMATIOM

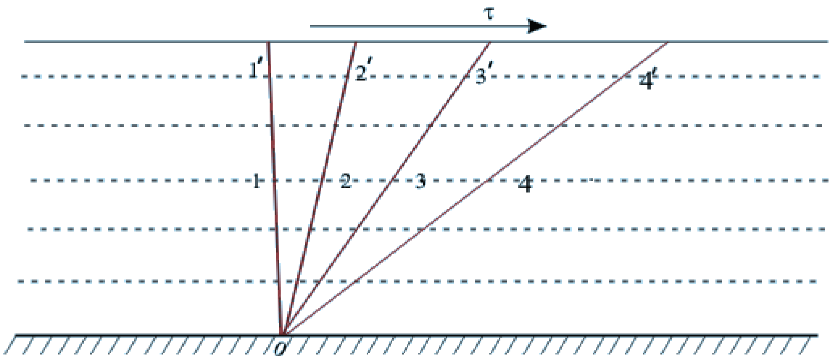


Figure 1.10 Deformation of a fluid body

Assume that a fluid with two parallel plates placed in a big container of water, the fluid layer in contact with the upper plate. When shear stress τ applied on any location in a fluid, the plate continuously is moving at the velocity of the plate no matter how it is small. From the **Figure 1.10** initially the fluid element at the position 0-1-1', which is at rest, then it will be followed by 0-2-2' then to 0-3-3' then to 0-4-4' and so on. The tangential stress in a fluid body depends on deformation of velocity and it will be decreases with the depth fluid and reaching zero at the lower plate. It is done because of friction between each and every fluid layers.

1.8 SOLID DEFORMATION

Any object that is consisting of the same substance throughout is called a solid. In other words, there is no hollow between any two substance throughout the object is called a solid.

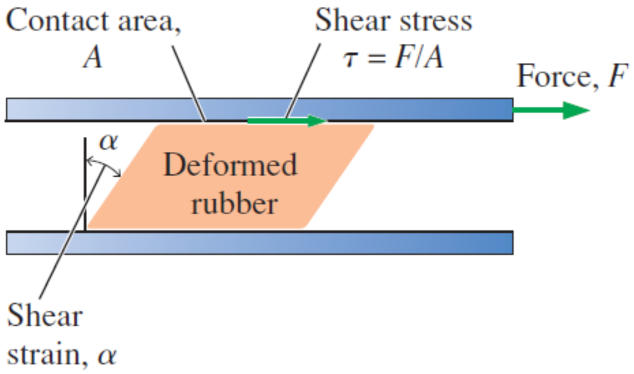


Figure 1.11 Deformation of a solid body

Let's assume a rubber block of rectangular shape placed tightly between two parallel plates. The rubber block deforms with the deformation angle α, it is known as shear strain. In **Figure 1.11** the lower plate as fixed and the upper plate is pulled with some force F. The shear strain increased in proportional to the applied force F. No slip condition is assumed for the rubber and plates; when the upper surface of the rubber is displaced by an amount equal to the displacement of the upper plate while the lower surface remains stationary. In equilibrium, the net force acting on the plate in the horizontal direction should be zero. Therefore a force F acting on the plate in opposite direction and it is equal to F. This opposing force is develops at the plate and rubber interface due to friction which is written by the formula $F = \tau A$, here the stress tensor is denoted by τ and A is the contact area between the rubber block and the upper plate. The rubber returns to its original position if the force is removed.

1.9 DEFORMATION DIFFERENCE BETWEEN SOLID AND FLUID

We divided any substance into two parts a solid and a fluid (gas, liquid) on the basis of their ability to resist an applied shear (or tangential) stress that tends to change its shape. Under the applied shear stress a solid is deforming, other side a fluid deforms continuously under the influence of shear stress, no matter how small. So, from above experiments we can says that,

(a) The elasticity of solid is shown in tension, compression and shearing stress wherever fluids have only compressibility.

(b) Under constant shear force, in a solids stress is proportional to strain, but in a fluids stress is proportional to strain rate.

(c) A solid is stops deforming at some fixed strain angle, but a fluid never stops deforming and it approaches a certain rate of strain.

1.10 PROPERTIES OF FLUID

Any characteristic of a system is called a property. Some familiar properties are pressure *P*, temperature *T*, volume *V*, and mass *m*. The list can be extended to more specific that viscosity, density, Specific weight, etc… Now any properties can be divided into two parts, intensive property or extensive property. Properties are independent of the mass of a system is called intensive property. For example, Temperature, Pressure, Density, etc...Properties are dependent of the mass of a system is called extensive property.

For example, total mass, total volume, total momentum, etc... We define some extensive and intensive properties as below.

1.10.1 DENSITY / MASS DENSITY [ρ]

Density or Mass Density defined as the ratio between the mass of a fluid to its unit volume. In other words, the mass per unit volume of a fluid is called density.

Density [ρ] = $\frac{\boldsymbol{Mass\ of\ fluid}}{\boldsymbol{Volume\ of\ fluid}}$

Unit of density is $\frac{Kg}{m^3}$ or $\frac{gm}{m^3}$

At normal temperature density of water = 1 $\frac{gm}{cm^3}$. It is maximum at 4^0c.

At normal temperature density of air = 0.0013 $\frac{gm}{cm^3}$

1.10.2 SPECIFIC WEIGHT/WEIGHT DENSITY [W]

Weight Density is defined as the ratio between the weights of fluid to its volume.

In other words, weight per unit volume of a fluid is called weight density.

$$W = \frac{\text{weight of fluid}}{\text{Volume of fluid}}$$

Unit of specific weight (in SI) is N/m^3

1.10.3 SPECIFIC VOLUME $\left[\frac{1}{\rho}\right]$

Specific volume is defined as the volume of a fluid occupied by a unit mass or volume per unit mass.

$$\frac{1}{\rho} = \frac{\text{Volume of fluid}}{\text{Mass of fluid}}$$

Unit of specific volume is m^3/Kg or m^3/gm

1.10.4 SPECIFIC GRAVITY [S]

Specific volume is defined as the ratio of the weight density of a fluid to the weight density of a standard fluid.

For a liquid, standard fluid is taken as water

For a gas, the standard fluid is taken air.

$$S \text{ (for liquid)} = \frac{\text{weight density of liquid}}{\text{weight density of water}}$$

$$S \text{ (for gases)} = \frac{\text{weight density of gas}}{\text{weight density of air}}$$

1.10.5 VISCOSITY [μ]

Viscosity is one of the most important fluid properties. Its impact we understood only in case of the fluid is moving. When fluid is in motion the fluid elements move with different velocities due to the friction of fluid within the elements. Hence due to this reason shear stress occurs between the fluid elements. Sir Isaac Newton gives the relationship between the shear stress and the velocity gradient.

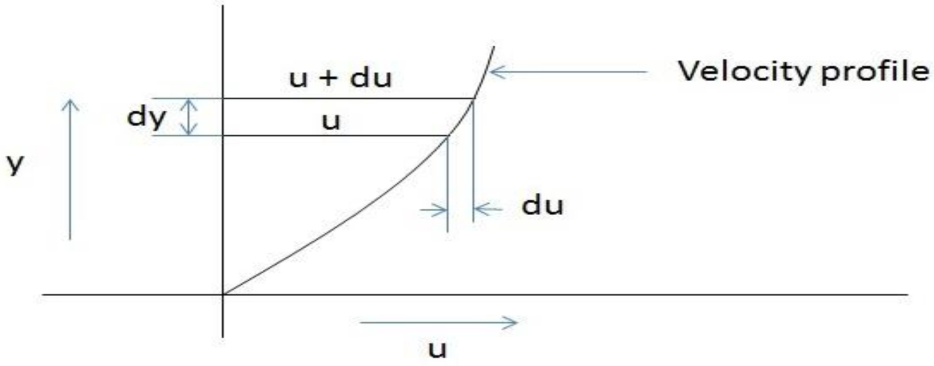


Figure 1.12 Velocity near solid boundary

Consider two layers of a fluid, the movement of one layer of fluid over another adjacent layer of the fluid in same direction. When two layers of a fluid with distance '*dy*' apart, moving the top layer faster and tries to draw the lower slowly moving layer their different velocities , say u and $u + du$ respectively as show in **Figure 1.12.** The viscosity together with relative velocity causes a shear stress acting between the fluid layers. By Newton's third law, the top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer. Newton postulated that this shear stress is proportional to the rate of change of velocity with respect to y. It is denoted by symbol τ (Tau).

$$\tau \propto \frac{du}{dy}$$

$$\therefore \tau = \mu \frac{du}{dy} \tag{1.1}$$

The proportionality relation in equation (1.1) between the shear stress and the rate of shear strain is known as the constitutive equation or **Newton's law of viscosity**. Common Fluids such as water, air, mercury etc… satisfies Newton's law of viscosity and are known as Newtonian fluids.

On the other hand, toothpaste, blood, paints etc… do not obey this linear relationship between τ and $\frac{du}{dy}$ so are called Non-Newtonian fluids.

Where, μ is the proportionality constant and it is called the coefficient of viscosity or simply it is called the viscosity. It depends only on the nature of fluid (not on *du* or *dy*).

Sign of τ depends upon the sign of $\frac{du}{dy}$.

The relation between viscosity [μ] and temperature [*t*] is given by,

For liquid, $\mu = \mu_0 \left(\frac{1}{1 + \alpha t + \beta t^2} \right)$

For gas,

$$\mu = \mu_0 + \alpha t - \beta t^2$$

From the above relation we conclude that:

For liquids, the viscosity μ is nearly independent of pressure and decreases rapidly with increasing temperature. When we increase temperature in that case bonds between molecules are becomes weak and since these bonds contribute to viscosity, the coefficient of viscosity is decreased.

While in the case of gases, the viscosity is independent of pressure but it increase with increasing temperature. Here we increase temperature intermolecular forces in gases are not as an important factor in viscosity as collisions between the molecules and thus increasing the coefficient of viscosity.

A striking result of the kinetic theory of gases is that the viscosity of a gas is independent of the density of a gas. Viscosity is the principal factor resisting motion in laminar flow. However, when the velocity has increased to the point at which the flow becomes turbulent, pressure differences resulting from eddy currents rather than viscosity provide the major resistance to motion. The value of viscosity is small for thin fluids, such as water or air, but it takes large value in case of highly viscous fluid such as oil or glycerin.

Units of Viscosity

- MKS Units: $\frac{Kg}{m\,.\,sec}$
- **S**.I. Units: Poise (the name after French Physician Jean Louis Marie Poiseuille, 1799-1869) or N.s/m^2
- C.G.S Unit of viscosity is Poise= dune-sec/cm^2
- 1 Poise (P) = 0.1 $\frac{Kg}{m\,.\,sec}$ = 0.01 Pascal second (Pa.s)
- 1 Poise (P) = 100 Centipoises (Cp)

- 1Cp =1 milliPascal second (mPa.s)

Table 1.1 Viscosity values for common fluids at room temperature

Substance	**Viscosity** $(\mu - mPa.s)$
Air	0.01
Water	1
Castor oil	6×10^2
Olive oil	10^2
Honey	10^4
Corn syrup	10^5
Mercury	1.5
Ethyl alcohol	1.2
Bitumen	1.08×10^5

1.10.6 KINEMATIC VISCOSITY [ν]

The effect of viscosity on the motion of a fluid is determined by the ratio of dynamic viscosity μ to the density ρ rather than by μ alone. This ratio is known as 'kinemtic viscosity' denoted by ν.

In other words, Viscosity per unit density is called Kinematic Viscosity.

Mathematically it is written as, $\nu = \frac{Dynemic\ viscosity}{Density} = \frac{\mu}{\rho}$

Units of kinematic viscosity

- S.I. Unit: m^2/s
- C.G.S. unit: stoke = cm^2/s , The named after George Gabriel Stokes
- 1 stoke (St)=10^{-4} m^2/s
- 1 centistokes (cst) = 10^{-6} m^2/s

1.10.7 APPARENT VISCOSITY

Apparent viscosity is the slope of the sharing stress versus rate of strain which varies with the rate of strain for non-Newtonian fluids. For Newtonian fluids, the apparent viscosity is the same as the dynamic viscosity (or simply viscosity).

Mathematically written it is as, $\eta = \frac{\tau}{du/dy} = k\left(\frac{du}{dy}\right)^{n-1}$

1.10.8 NO-SLIP CONDITION

Through experimental observations it is established that, the relative velocity between the solid surface and the adjacent fluid particles is becomes zero. In other words, we can say that the fluid elements in contact with a stationary surface attain the velocity of the surface. It is known as the “no-slip” condition. Also it is note that this behaviour of no-slip at the solid surface should not be confused with the wetting of surfaces by the fluids. For example, mercury flowing in a stationary glass tube will not wet the surface, but will have zero velocity at the wall of the tube. The wetting property results from surface tension, whereas the no-slip condition is a consequence of fluid viscosity.

The velocity and the temperature of fluid at a point of contact with solid surface is same as the velocity and temperature of that solid surface,

$(V)_{fluid} = (V)_{wall}$ and $(T)_{fluid} = (T)_{wall}$

These conditions are called the no-slip condition or no temperature-jump condition. These conditions are used as boundary conditions in the analysis of fluid flow past a solid surface.

1.10.9 BOUNDARY LAYERS

The boundary layer concept was first introduced by Ludwig Prandtl, a German aerodynamicist, in 1904. Let us now follow the impact as a flow approaches a solid body, to make it simple. Consider a flat plate shown in **Figure 1.13**.

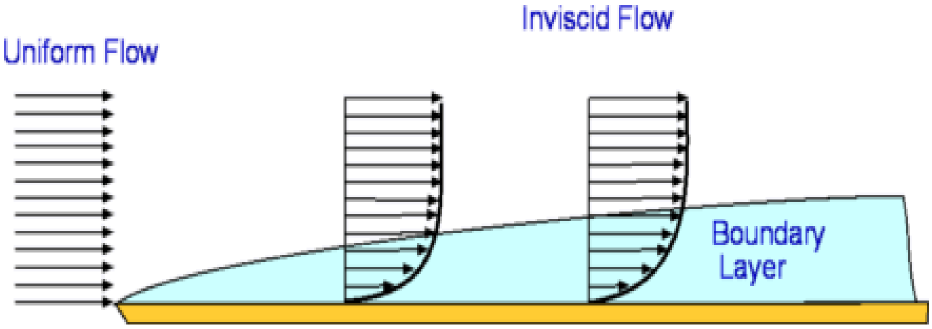


Figure 1.13 Formation of a Boundary Layer

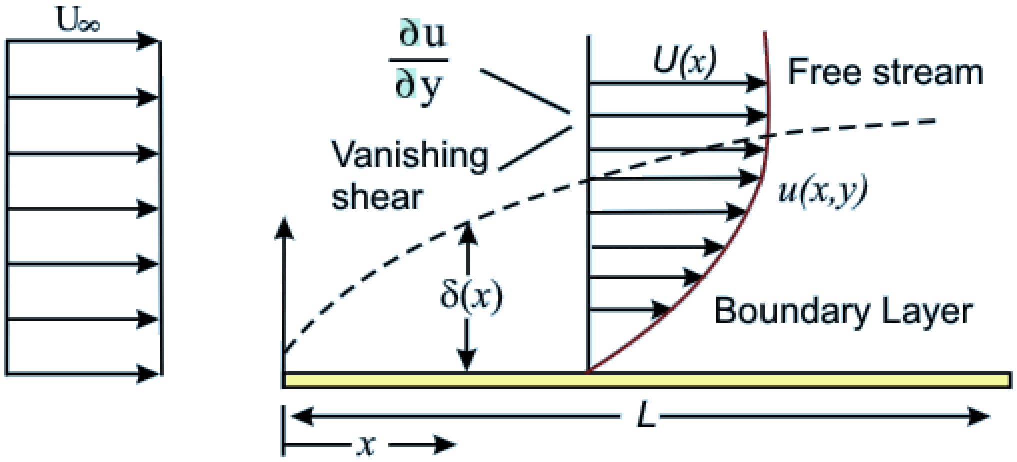


Figure 1.14 Free stream layer and boundary layer for flow past a flat plate

A uniform flow (inviscid flow) with velocity U_∞ was applied in the front of a flat plate. As soon as the flow 'hits' the plate No Slip Conditions gets into action. As a result, the velocity on the body becomes zero. Since the effect of viscosity is to resist fluid motion, the velocity closed to the solid surface continuously decreases towards downstream. But away from the flat plate the speed is equal to the free stream value U_∞ **(Figure 1.14)**. Consequently a velocity gradient is set up in the fluid in a direction of normal to the flow direction. Also the component of velocity gradient in a direction normal to the surface is large as compared to the velocity gradient in the stream wise direction. However, outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer (since the velocity gradient $\frac{\partial u}{\partial y}$ is negligible).

In the normal direction, within this thin layer, the gradient $\frac{\partial u}{\partial y}$ is very large compared to the gradient in the flow direction.

1.11 TYPES OF FLUID

Fluid is basically classified into following five categories:

1.11.1 IDEAL FLUID

A fluid, which is not compressible and inviscid is known as an ideal fluid. Since all real fluid has some viscosity, Ideal fluid (perfect fluid) is only an imaginary fluid.

1.11.2 REAL FLUID

A viscous fluid is known as Real fluid (Practical fluid). All the existing fluids are real fluids.

1.11.3 NEWTONIAN FLUID

A real fluid, in which the shear stress is directly proportional to the rate of shear strain (velocity gradient), known as a Newtonian fluid. i.e. Fluids which obey the Newtonian law of viscosity are known as Newtonian fluids. For example: Water, air, Kerosene, metals, salts, etc…

Graph of all type fluids shown in **Figure 1.15.**

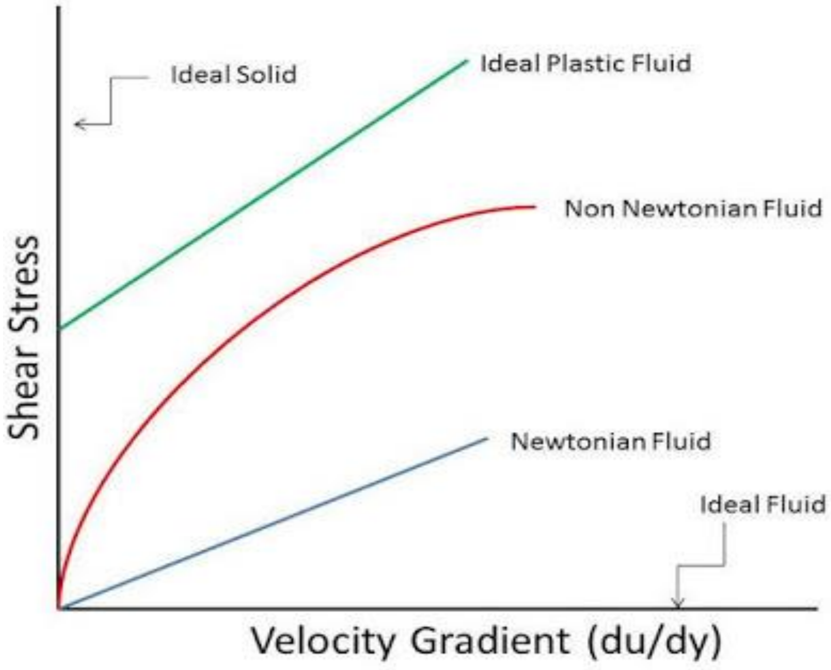


Figure 1.15 Types of fluid

1.11.4 NON- NEWTONIAN FLUIDS

A real fluid, in which the shear stress is directly not proportional to the rate of shear strain (velocity gradient), known as Non-Newtonian fluid. i.e. Fluids which do not obey the Newtonian law of viscosity are called Non- Newtonian Fluids.

For Example: Printer ink, blood, mud, slurries, polymer solutions, etc….

1.11.5 IDEAL PLASTIC FLUID

A fluid, in which the shear stress is more than the yield value and then it is proportional to the rate of shear strain (velocity gradient) such fluid known as an Ideal plastic fluid.

1.12 DETAILED FOR NON-NEWTONIAN FLUID

Any fluids are classified in two different ways; either according to their response to the externally applied pressure or according to the effects produced under the action of a shear stress.

According to the first scheme of classification fluids are called ‘compressible’ and ‘incompressible’ fluids, depending upon whether or not the volume of an element of fluid is dependent on its pressure. A gas possesses compressibility influence the flow characteristics, while a liquid possesses an incompressible influence the flow characteristics. Compared to the first scheme the effects of shearing which is of greater importance.

Due to most common fluid property viscosity, when the fluid elements move with different velocities, each element will feel some resistance due to fluid friction within the elements. Therefore shear stress can be identified between the fluid elements with different velocities. The relationship between the shear stress and the velocity gradient was given by Sir Isaac Newton. It is known as Newton’s law of viscosity and expressed as in equation (1.1).

In general, the viscosity of a fluid depends on both temperature and pressure, although the dependence on pressure is rather weak. For liquids, both the dynamic and kinematic viscosities are practically independent of pressure, and any small variation with pressure is usually disregarded, except at extremely high pressures.

In a Newtonian fluid, the relation between the shear stress and the shear rate is linear as shown in equation (1.1). In a non-Newtonian fluid, the relation between the shear stress and the shear rate is not linear and can even be time-dependent. This general relationship can be expressed as equation (1.2), Therefore a constant coefficient of viscosity cannot be defined.

$$\tau = A\left(\frac{du}{dy}\right)^n + B(t) \tag{1.2}$$

Therefore, the apparent viscosity (shear stress/shear rate) is not constant at a given pressure and temperature. It depends on flow geometry, shear rate, etc. and sometimes even on the kinematic history of the fluid particles under consideration.Such materials can be divided into the below three general classes:

(a) Fluids for which the shear rate at a point is decided only by the value of the shear stress at that point. These fluids are known as unsteady or purely viscous or inelastic or generalized Newtonian Fluids (GNF).

(b) Fluids for which the relation between shear stress and shear rate depends, in addition, upon the shearing duration and their kinematic history; they are known as time-dependent fluids.

(c) System exhibiting characteristics of both ideal fluids and elastic solids and showing partial elastic recovery, after deformation; these are categorized as visco-elastic or elastic-viscous fluids.

Again further classification of purely viscous and visco-elastice fluids are shown in **Table 1.2** and there graph for shear stress versus shear rate is shown in **Figure 1.16.**

Table: 1.2 Classification of Non Newtonian fluids

Non- Newtonian Fluid $\left[\tau = A\left(\frac{du}{dy}\right)^n + B(t)\right]$		
Purely Viscous Fluids		**visco-elastice Fluids**
Time- Independent	**Time- Dependent**	
1. Pseudo Plastic $\tau = A\left(\frac{du}{dy}\right)^n \quad ,n<1$ **Example: Blood , Milk** **2. Dilatant Fluids** $\tau = A\left(\frac{du}{dy}\right)^n \quad ,n>1$ **Example: Butter** **3. Bingham or Ideal Plastic Fluids** $\tau = A\left(\frac{du}{dy}\right)^n + \tau_0$ **Example: Water suspensions of clay and flyash**	**1. Thixotropic fluids** $\tau = A\left(\frac{du}{dy}\right)^n + B(t)$ Where, B(t) is decreasing **Example: Printer Ink, Crude oil** **2. Rheopectic Fluids** $\tau = A\left(\frac{du}{dy}\right)^n + B(t)$ Where, B(t) is increasing **Example: Rare liquid solid suspension**	**Visco - elastice fluids** $\tau = A\frac{du}{dy} + B$ **Example: Liquid and solid combinations in pipe**

The graph of the shear stress versus shear rate will not remain constant by changing the shear rate. Fluids are known as Pseudo plastic fluid (shear-thinning) for which the viscosity decreases with the increase in shear rate. While in the dilatants fluids (shear-thickening) viscosity will be increase with the increase of shear rate.

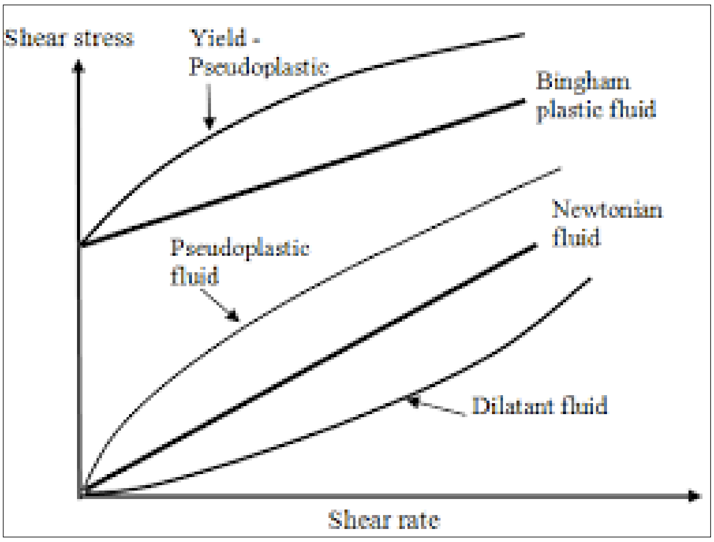
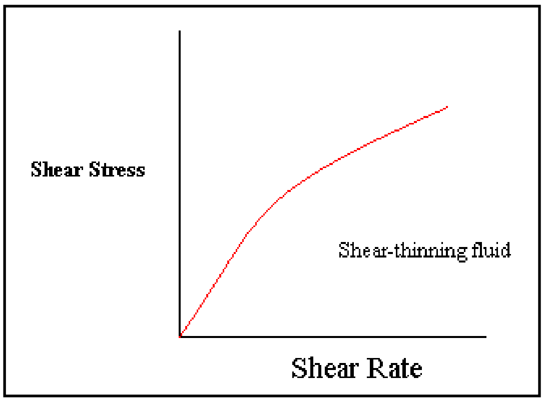


Figure 1.16 Shear stress vs Shear rate

Shear-thinning behaviour is more common than the shear-thickening. A graph of shear thinning fluid and shear thickening fluid for shear stress versus shear rate is given by **Figure 1.17** and **Figure 1.18** respectively.



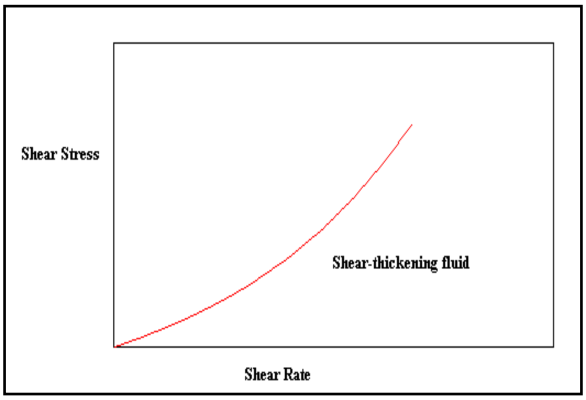


Figure 1.17 Graph for Shear-thinning fluid

Figure 1.18 Graph for Shear-thickening fluid

Examples of shear-thinning fluids are polymer melts such as molten polystyrene, polymer solutions such as polyethylene oxide in water and some paints. Paint flows comfortably when it is sheared with a brush. But if the shear stress is removed, it cannot flow easily because of the increase in viscosity. Of course the solvent evaporates soon and then the point sticks to the surface. The behaviour of paint is a bit more complex than this, because at a given shear rate, the viscosity changes with time.

Visco-elastice behaviour is another important class of non-Newtonian fluids. Fluid falls in this category behave both of solid (elastic) and fluids (viscous). These fluids will not flow on applying a small shear stress. In other words it behave as a solid still yield stress τ_0, the shear stress must exceed τ_0 the fluid is flow.

Consider the simple example of toothpaste. It would be good if the paste does not flow at the slightest amount of shear stress. We need to apply an adequate force before the toothpaste will start flowing. So, visco-plastic fluids behave like solids when the applied shear stress is less than the yield stress. Once it exceeds the yield stress, the visco-plastic fluid will flow just like a fluid. Bingham plastics are a special class of visco-plastic fluids that exhibit a linear behavior of shear stress against shear rate. Typical visco-plastic behaviors are illustrated in **Figure 1.19**.

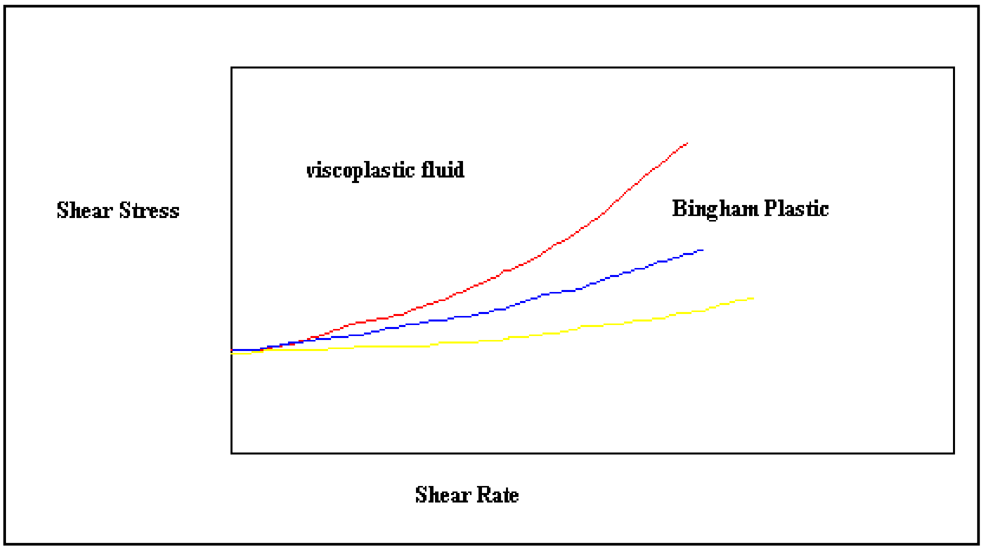


Figure 1.19 Visco-plastic behaviours

More examples of Visco-plastic fluids are drilling mud, nuclear fuel slurries, egg-white, mayonnaise, blood. Also, some paints exhibit a yield stress. Several industrially important polymers melt and solutions are viscoelastic.

Well this is not an adequate discussion for the behaviour of non-Newtonian fluids. Some fluids of this category are having time dependent behaviour. That is the viscosity of fluid may change with time under a given constant shear rate. In the case of thyrotrophic fluid, the viscosity will decrease with time even under a constant applied shear stress. However, when the stress is removed, the viscosity will gradually recover with time as well. For the rheoplastic fluid the viscosity increases with time when a constant shear stress is applied.

1.13 TYPES OF FLUID FLOW

1.13.1 VISCOUS AND INVISCID FLOW

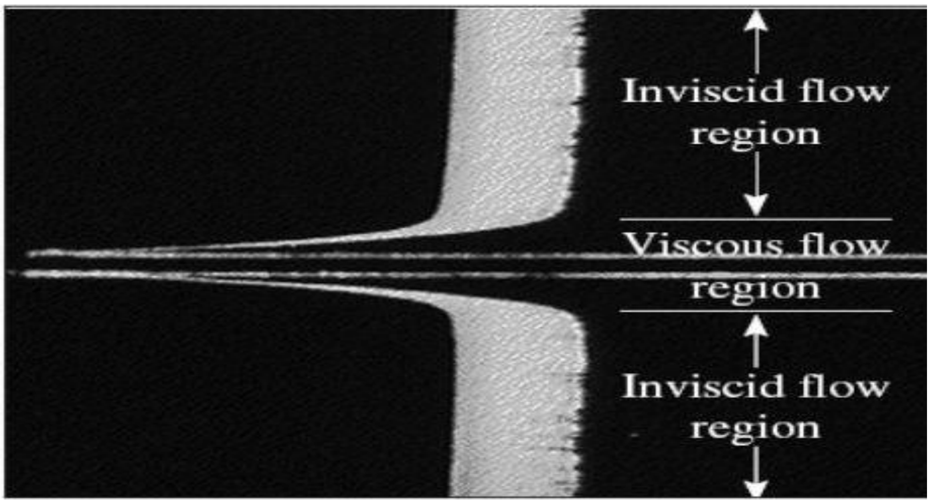


Figure-1.20 Viscous and inviscid flow region

When two fluid layers move relative to each other, a friction force will be develop between them and the slower layer tries to slow down the faster layer. This internal resistance to flow is quantified by the fluid property viscosity, which is a measure of internal stickiness of the fluid. Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases. There is no fluid with zero viscosity, and thus all fluid flows involve viscous effects to some degree. Flows in which the frictional effects are significant are called viscous flows however, in many flows in practical interest, there are regions (typically regions not close to solid surfaces) where viscous forces are negligibly small compared to inertial or pressure forces. Neglecting the viscous terms in such inviscid flow regions greatly simplifies the analysis without much loss in accuracy. The development of viscous and inviscid regions of flow as a result of inserting a flat plate parallel into a fluid stream of uniform velocity is shown in **Figure 1.20**. The fluid sticks to the plate on both sides because of the no-slip condition, and the thin boundary layer in which the viscous effects are significant near the plate surface is the viscous flow region. The region of flow on both sides away from the plate and unaffected by the presence of the plate is the inviscid flow region.

1.13.2 INTERNAL AND EXTERNAL FLOW

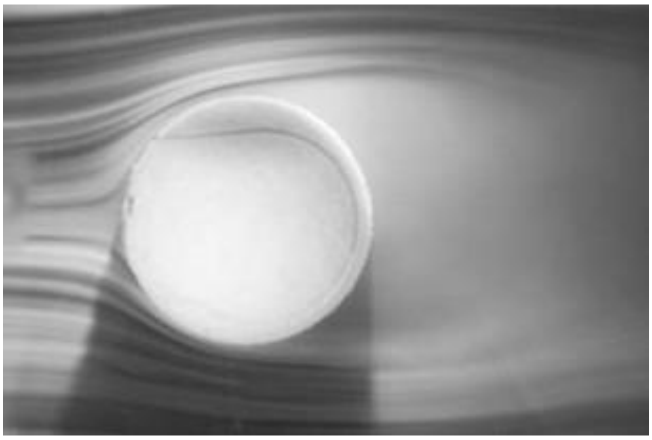


Figure-1.21 External flow over tennis ball and gives the turbulent flow behind

A fluid flow is classified as being internal or external, depending on whether the fluid is forced to flow in a confined channel or over a surface. The flow of an unbounded fluid over a surface such as a plate, a wire or a pipe is called external flow. The flows of water in rivers and irrigation ditches are examples of type of flow. The flow in a pipe or duct completely bounded by solid surfaces is called internal flow. In external flows the viscous effects are limited to boundary layers near solid surfaces and to wake regions downstream of bodies. In **Figure 1.21** shows that external flow over tennis ball and gives the turbulent flow behind.

1.13.3 COMPRESSIBLE AND INCOMPRESSIBLE FLOW

A fluid is classified into compressible or incompressible depending upon the level of variation of density during the fluid flow. A flow is said to be an incompressible if the density remains constant throughout fluid flow. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion. The density of liquid is constant thus the flow of liquids is usually referred to as incompressible substances at pressure of 210 atm. On the other hand, gases are highly compressible fluid. When analyzing rockets, spacecraft, and other systems that involve high-speed gas flows, the flow speed is often expressed in terms of the dimensionless Mach number defined in 1.14.3.

When Liquid flows are incompressible to a high level of accuracy, but the level of variation in density in gas flows and the consequent level of approximation made when modelling gas flows as incompressible depends on the Mach number. Gas flows can often be approximated as incompressible if the density changes are under about 5 percent, which is usually the case when Ma < 0.3. Therefore, the compressibility effects of air can be neglected at speeds under about 100 m/s. Note that the flow of a gas is not necessarily a compressible flow.Compressibility effects are very important in the design of modern high-speed aircraft and missiles, power plants, fans and compressors.

1.13.4 LAMINAR AND TURBULENT FLOW

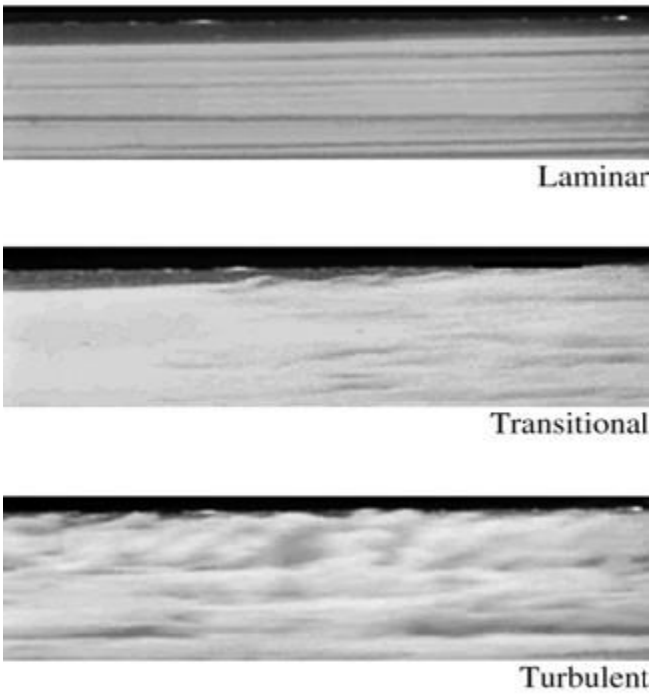


Figure-1.22 Laminar, Transitional and Turbulent Flows

Some flows are smooth and orderly while others are rather chaotic. The highly ordered fluid motion characterized by smooth layers of fluid is called laminar. The word laminar comes from the movement of adjacent fluid particles together in “laminates.” The flow of high-viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluid motion that typically occurs at high velocities and is characterized by velocity fluctuations is called turbulent **(Figure 1.22)**. The flow of low-viscosity fluids such as air at high velocities is typically turbulent. The flow regime greatly influences the required power for pumping. A flow that alternates between being laminar and turbulent is called transitional The experiments conducted by Osborn Reynolds in the 1880s resulted in the establishment of the dimensionless Reynold's number (Re) as the key parameter for the determination of the flow regime in pipes.

When, $R_e \leq 2000$; the pipe flow is laminar.

$R_e > 2000$; the pipe flow is turbulent.

1.13.5 NATURAL (OR UNFORCED) AND FORCED FLOW

A fluid flow is said to be natural or forced, depending on how the fluid motion is initiated. In force flow a fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan. In natural flows any fluid motion is due to natural means such as the buoyancy effect, which manifests itself as the rise of the warmer (and thus lighter) fluid and the fall of cooler (and thus denser) fluid. In solar hot-water systems, for example, the thermo siphoning effect is commonly used to replace pumps by placing the water tank sufficiently above the solar collectors.

1.13.6 STEADY AND UNSTEADY FLOW

The terms steady and unsteady are used frequently in engineering, thus it is important to have a clear understanding of their meanings. The term steady implies no change at a point with time. The opposite of steady is unsteady.

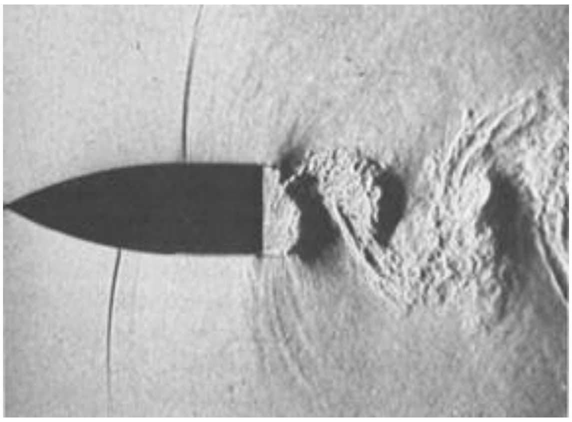


Figure-1.23(a) Unsteady flow

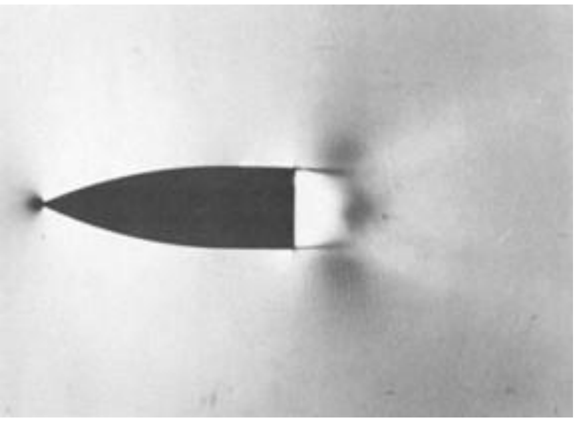


Figure-1.23(b) Steady flow

Many devices such as turbines, compressors, boilers, condensers, and heat exchangers operate for long periods of time under the same conditions, and they are classified as steady-flow devices. During steady flow, the fluid properties can change from point to point within a device, but at any fixed point they remain constant. Therefore, the volume, the mass, and the total energy content of a steady-flow device or flow section remain constant in steady operation. Steady-flow conditions can be closely approximated by devices that are intended for continuous operation such as turbines, pumps, boilers, condensers, and heat exchangers of power plants or refrigeration systems. Some cyclic devices, such as reciprocating engines or compressors, do not satisfy the steady-flow conditions since the flow at the inlets and the exits is pulsating and not steady. However, the fluid properties vary with time in a periodic manner, and the flow through these devices can still be analyzed as a steady-flow process by using time-averaged values for the properties. Some fascinating visualizations of fluid flow are provided. A nice illustration of an unsteady-flow field is shown in **Figure 1.23(a)**, taken from Van Dyke's book. **Figure 1.23(b)** is an instantaneous snapshot from a high-speed motion picture; it reveals large, alternating, swirling, turbulent eddies that are shed into the periodically oscillating wake from the blunt base of the object. The eddies produce shock waves that move upstream alternately over the top and bottom surfaces of the airfoil in an unsteady fashion. The resulting time-averaged flow field appears "steady" since the details of the unsteady oscillations have been lost in the long exposure.

1.14 SOME FLUID NUMBERS

In the governing equations of fluid flow non-dimensionalization takes an important role in the both theoretical and computational reasons. Non-dimensional scaling provides a physical insight into the importance of various terms in the system of governing equations. Computationally, dimensionless forms have the added benefit of providing numerical scaling of the system discrete equations, thus providing a physically linked technique for improving the ill-conditioning of the system of equations. Moreover, dimensionless forms also allow us to present the solution in a compact way. Some of the important dimensionless numbers used in fluid mechanics, gas dynamics and heat transfer are given below.

1.14.1 PRANDTL NUMBER (P_r)

The dimensionless number Prandtl Number (the named after the German physicist Ludwig Prandtl) is defined as the ratio of momentum diffusivity and thermal diffusivity.

In other words, the thermal boundary layer thickness is similarly the distance from the body at which the temperature is 99% of the temperature found from an inviscid solution. The ratio of the two thicknesses (velocity boundary layer and thermal boundary layer) is governed by the Prandtl number.

Mathematically written as, $$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{\mu \backslash \rho}{k \backslash (Cp\ \rho)} = \frac{Cp\,\mu}{k}$$

Where, α = Thermal diffusivity (SI units: m^2/s).

Cp = Specific heat (SI units: J/kg-K).

μ = Dynamic Viscosity (SI units: Pa s = N s/m^2).

K = Thermal conductivity (SI units: W/m-K).

ρ = Density (SI units: kg/m^3).

When, $P_r = 1$: the two boundary layers are the same thickness.

$P_r > 1$: the thermal boundary layer is thinner than the velocity boundary layer.

$P_r < 1$: the thermal boundary layer is thicker than the velocity boundary layer (which is the case for air at standard conditions).

$Pr << 1$: the thermal diffusivity (heat) dominates the behavior.

$Pr >> 1$: the momentum diffusivity (velocity) dominates the behavior.

It is used in heat transfer and free and forced convection calculations and which is depends only on fluid and its properties.

In heat transfer problems, the Prandtl number controls the relative thickness of the momentum and thermal boundary layers. When Pr is small, it means that the heat diffuses quickly compared to the velocity (momentum). This means that for liquid metals the thickness of the thermal boundary layer is much bigger than the velocity boundary layer.

For example, liquid mercury indicates that the heat conduction is more significant compared to convection so thermal diffusivity is dominant. for engine oil, convection is very effective in transferring energy from an area in comparison to pure conduction, so momentum diffusivity is dominant.

Table 1.3 Some typical values for Pr

Substance	Prendtl Number (Pr)
Air	0.7-0.8
Oxygen	0.63
Water	7
Engine Oil	100-40000
Glycerol	1000
Polymer melts	10000

1.14.2 REYNOLDS NUMBER (R_e)

The dimension less number, the ratio of inertial forces to viscous forces is known as the Reynolds number.

The concept of **Reynolds Number** was introduced by Sir George Stokes in 1851 but the Reynolds number was named by Arnold Sommerfeld in 1908 after Osborne Reynolds (1842-1912), who popularized its use in 1883.

Mathematically written as, $$R_e = \frac{Inertail\ force}{Viscous\ \ force} = \frac{\rho V^2 / D}{\mu V / D^2} = \frac{\rho VD}{\mu} = \frac{VD}{\nu}$$

Where, V = the flow velocity (m/s).

D = characteristic linear dimension (m).

ρ = fluid density (kg/m^3).

μ = dynamic viscosity (Pa.s).

ν = kinematic viscosity (m^2/s).

It is a convenient parameter for predicting if a flow condition will be laminar or turbulent (flow pattern). It can be interpreted that when the viscous forces are dominant (slow flow, low Re, Re < 2000) they are sufficient enough to keep all the fluid particles in line, and then the flow is laminar. Even very low Re indicates viscous creeping motion, where inertia effects are negligible. When the inertial forces dominate over the viscous forces (when the fluid is flowing faster and Re >4000) then the flow is turbulent and 2000<Re <4000 then flow is transition.

$0 < Re < 1$: highly viscous laminar "creeping" motion.
$1 < Re < 100$: Laminar, strong Reynolds number dependence.
$100 < Re < 10^3$: Laminar boundary Layer theory useful.
$10^3 < Re < 10^4$: transition or turbulence.
$10^4 < Re < 10^6$: turbulent, moderate Reynolds number dependence.
$10^6 < Re < \infty$: turbulent, slight Reynolds number dependence.

1.14.3 MACH NUMBER (M_a)

The flow speed is often expressed in terms of the dimensionless number is known as Mach in honour of Ernst Mach, a late 19th century physicist who studied gas dynamics. The Mach number M_a allows us to define flight regimes in which compressibility effects vary. Mach number defined as,

$$Ma = \frac{V}{C} = \frac{Speed\ of\ Flow}{Speed\ of\ sound}$$

Where, c = The speed of sound

= 346 m/s in air at room temperature at sea level.

When $Ma<0.8$: the flow is Subsonic
$0.8<Ma<1.2$: the flow is transonic
$Ma =1$: the flow is sonic
$1.2<Ma <5$: the flow is supersonic
$5<Ma <10$: the flow is hypersonic
$Ma >10$: the flow in hypervelocity

The trigonometric sine of the cone angle b is equal to the inverse of the Mach number M and the angle is therefore called the Mach angle.

$\text{Sin}(b) = 1 / M_a$

1.15 TIME LINE, PATH LINE, STREAK LINE AND STREAMLINES

In the analysis of problems in fluid mechanics, frequently it is advantageous to obtain a visual representation of a flow field. Such representation is provided by timelines, path lines, streak lines and streamlines.If a number of adjacent fluid particle in a flow field are marked at a given instant, they form a line in the fluid at that instant; this line is called timeline see **Figure 1.24.**

A path line is the path or trajectory traced out by a moving fluid particle see **Figure 1.25.** To make a path line visible, we might identity a fluid particle at a given instant e.g. by the use of dye, and then take a long exposure photograph of its subsequent motion. The line traced out by the particle is a path line.

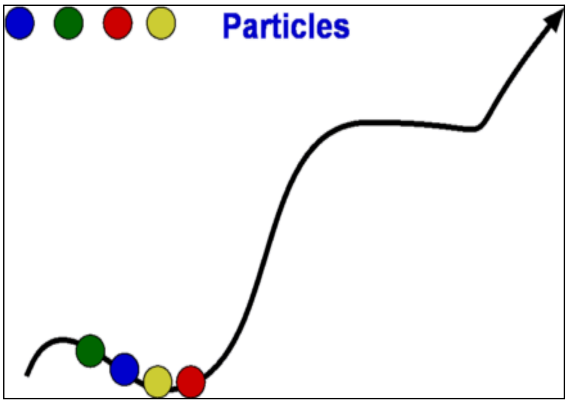


Figure1.24 Timeline

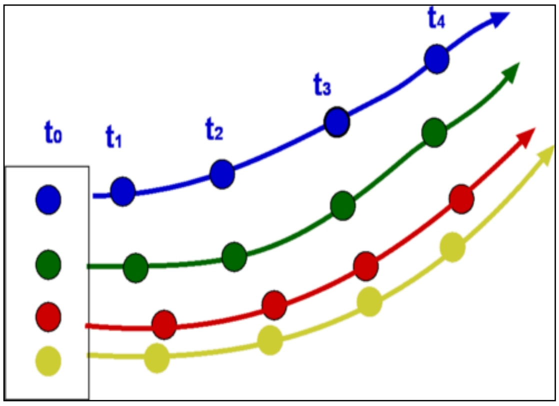


Figure 1. 25 Path lines

On the other hand, we might choose to focus our attention on a fixed location in space and identify again by the use of dye, all fluid particles passing through this point. After a short period of time we would have a number of identifiable fluids particles in the flow, all of which had, at some time, passes through one fixed location in space. The line joining these fluid particles is defined as a streak lines see **Figure 1.26.**

Streamlines are lines drawn in the flow field so that a given instant they are tangent to the direction of flow at every point in the flow field. Since the streamlines are tangent to the velocity vector at every point in the flow field, there can be no flow across a streamline see **Figure 1.27**.

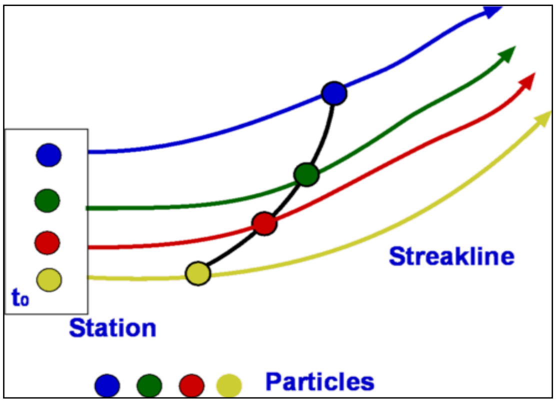


Figure 1.26 Streaklines

Figure 1.27 Streamlines

Thus, streamline define at a single instant in a flow do not intersect. In steady flow, the velocity at each point in the flow field remains constant with time and consequently, the streamlines do not vary from one instant to the next. This implies that a particle located on a given streamline will remain on the same streamline. Furthermore, consecutive particles passing through a fixed point in space will be on the same streamline and, subsequently will, remain on this streamline. Thus in a steady slow, path line, streakiness and streamlines are identical lines in the flow field.

The shape of the streamlines may vary from instant if the flow in unsteady. In the case of unsteady flow, path lines streamlines and streak lines do not coincide.

1.16 STREAM FUNCTION :

A continuous function $\psi(x, y, t)$ called the stream function, that replace the two velocity components *u (x, y, t)* and *v (x, y, t)* along the x and y direction respectively.

$$u = \frac{\partial \psi}{\partial y} \quad and \quad v = -\frac{\partial \psi}{\partial x}$$

The stream function is a device which allows us to wipe out the contenting equation and solve the momentum equation directly for the single variable ψ .

CHAPTER-2

GOVERNING EQUATIONS AND FLUID MODELS

2.1 SYSTEM

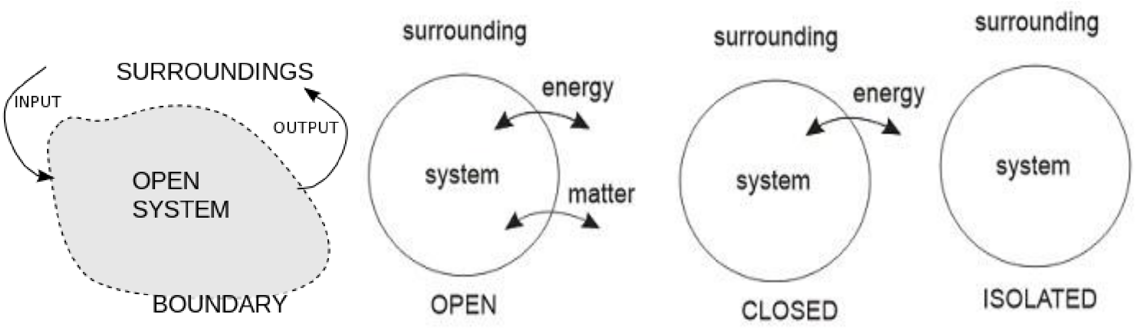


Figure 2.1 System types

In the analysis of problem the quantity of matter in the space is known as a system. The external part of the system is called the surrounding part and system is separated from surrounding part is called boundary of the system **Figure 2.1**. This boundary may be solid or imaginary and may be moving or fixed.

Any System is divided into three types as follows:

2.1.1 CONTROL VOLUME SYSTEM (OPEN SYSTEM)

There is no any change in the volume of the system mass and energy crosses the boundary of the system is called control volume system. In general, Most of the engineering devices are open system.

2.1.2 CONTROL MASS SYSTEM (CLOSED SYSTEM)

The system having fixed mass with fixed identity. In other wards we can say that there is no mass transfer in the system is called control mass system.

2.1.3 ISOLATED SYSTEM

The system in which neither energy nor mass interact between the systems and the surrounding is called isolated system. In other words, it is a fixed mass and fixed energy with same identity.

2.2 THE DIFFERENTIAL EQUATION OF CONSERVATION OF MASS (CONTINUITY EQUATION)

There is no accumulation of mass. In other words, Mass is neither created nor destroyed in the fluid element is known as the conservation of Mass law. The equation based on the principle of the conservation of mass is known as equation of continuity. For finding all basic differential equations of motion either an elemental system or an elemental control volume is considered.

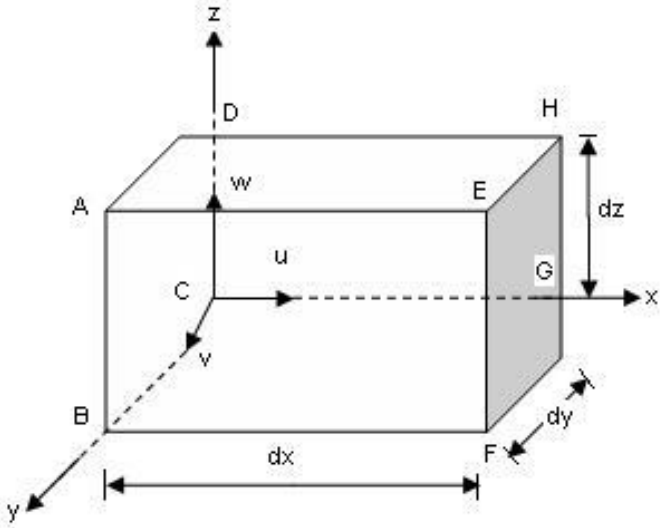


Figure 2.2 Elemental fixed control volumes in Cartesian coordinate system

Let us consider one fluid element having length *dx, dy* and *dz* in the direction of *x, y* and *z* respectively and *u, v* and *w* are the in late velocity component along *x, y* and *z* direction respectively.

Mass entering in the phase ABCD per second = ρAV

$$= \rho \times Area\ of\ \ ABCD\ \times Velocity\ in\, x - direction$$

$$= \rho\, dy\, dz\, u$$

Mass living in the phase EFGH per second $= \rho\, dy\, dz\, u + \frac{\partial}{\partial x}\left(\rho\, dy\, dz\, u\ \right) dx$

$$\text{Therefore, Rate of increase in mass x-direction} = \begin{pmatrix}\text{Mass entering in}\\ \text{the phase ABCD}\\ \text{per second}\end{pmatrix} - \begin{pmatrix}\text{Mass living in}\\ \text{the phase EFGH}\\ \text{per second}\end{pmatrix}$$

$$= (\rho\, dy\, dz\, u) - \left(\rho\, dy\, dz\, u + \frac{\partial}{\partial x}(\rho\, dy\, dz\, u)\, dx\right)$$

$$= (\rho\, dy\, dz\, u) - \left(\rho\, dy\, dz\, u + \frac{\partial}{\partial x}(\rho\, dy\, dz\, u)\, dx\right)$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dxdy\, dz$$

Similarly,

Rate of increase in mass y-direction $= -\frac{\partial}{\partial y}(\rho v)\, dxdy\, dz$

Rate of increase in mass z-direction $= -\frac{\partial}{\partial z}(\rho w)\, dxdy\, dz$

Total rate of increase in mass $= -\frac{\partial}{\partial x}(\rho u)\, dxdy\, dz - \frac{\partial}{\partial y}(\rho v)\, dxdy\, dz - \frac{\partial}{\partial z}(\rho w)\, dxdy\, dz$

Now by conservation of mass law, mass is neither created nor destroyed in the fluid element. So, net increase of mass per unit time in the fluid element must equal to the total rate of increased mass in the fluid element.

$$\frac{\partial \rho}{\partial t} dxdy\, dz = -\frac{\partial}{\partial x}(\rho u)\, dxdy\, dz - \frac{\partial}{\partial y}(\rho v)\, dxdy\, dz - \frac{\partial}{\partial z}(\rho w)\, dxdy\, dz$$

The volume element term (*dx dy dz*) cancelled out on both the sides. We get partial differential equation having the partial derivatives of velocity components and density of fluid.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2.1}$$

The equation (2.1) called **The Equation of Continuity**. It is valid for an infinitesimal control volume system. Because it do not required any assumptions except that the density and velocity are continuum functions. In other words we can say that this equation satisfies for the flow of types steady, unsteady, viscous, frictionless, compressible Or incompressible.

Also, we know from the definition of divergence of a vector *V*.

$$\frac{\partial}{\partial u}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = \nabla \cdot (\rho V) \tag{2.2}$$

Substitute the value of equation (2.2) in equation (2.1) we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \tag{2.3}$$

Cylindrical Polar Coordinates:

In cylindrical polar coordinates continuity equation (2.1) written as,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{2.4}$$

Compressible and Steady Flow:

For steady flow, $\frac{\partial \rho}{\partial t} = 0$

So, equation (2.1) and (2.4) reduce as,

Cartesian coordinate:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2.5}$$

Cylindrical coordinate:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{2.6}$$

Incompressible flow:

For incompressible flow the density changes are negligible with respect to time so equations (2.5) and (2.6) are reduced as,

Cartesian coordinate: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.7}$$

Cylindrical coordinate:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0 \tag{2.8}$$

2.3 FORCES ACTING ON THE SYSTEM

2.3.1 INERTIA FORCE (*F_i*)

The product of the mass of fluid element and its acceleration is defined as inertia force.

Mass of the Fluid element $= \rho l^3$ where, l = the characteristic dimension

Acceleration of the Fluid element $=\frac{v}{t}=\frac{v^2}{l}$ $\left(\because t=\frac{l}{v}\right)$

So, the magnitude of $F_i = ma = \rho l^2 v^2$

2.3.2 VISCOUS FORCE (*F_v*)

Due to shear stress the viscous force arising in flow of fluid.

Therefore, Shear stress = rate of shear strain x viscosity

= velocity gradient x viscosity

$$=\frac{v}{l}\times\mu$$

Magnitude of F_v = the surface area at which the shear stress applied shear stress

$$=\mu\ \frac{v}{l}l^2\ =\mu v l$$

2.3.3 PRESSURE FORCE (F_p)

Due to the difference of pressure, the pressure force arising in a flow field.

Mathematically it is written as, $F_p = \nabla p\ l^2$

2.3.4 GRAVITY FORCE (F_g)

Due to the weight (gravitational attraction) of the fluid element the gravity force arising in a flow field.

Mathematically it is written as, $F_g = \rho\ l^3\ g$

2.3.5 CAPILLARY FORCE (SURFACE TENSION FORCE) (F_c)

The capillary force arises due to the existence of an interface between two fluids. The surface tension force acting in the direction of tangent to a surface is the product of the length (*l*) of a linear element on the surface perpendicular to which the force acts and the coefficient of surface tension (σ).

Mathematically it is written as, $F_c = l\ \sigma$

2.3.6 COMPRESSIBILITY FORCE (ELASTIC FORCE) (F_e)

Due to the consideration of compressibility of the fluid force elastic force comes into its flow. When we compressed the fluid (a decrease in volume), the increase in pressure is directly proportional to the bulk modulus of elasticity and gives rise to a such force known as the elastic force.

Mathematically it is written as, $F_e = E\,l^2$

2.4 DYNAMIC SIMILARITY

In general, the flow of a fluid does not involve all the forces simultaneously. Therefore the dimensionless parameter dynamic similarity is defined. It is defined as the ration of dominant forces causing on the flow. Some of the examples are given in the following **Table-2.1.**

Table 2.1 Dynamic similarity

Ratio of the forces	Pertinent dimensionless term as the criterion of dynamic situations of fluid flow	Name	Recommended symbol
$\frac{Inertia\ force}{Viscous\ force}$	$\frac{\rho\, l\, V}{\mu}$	Reynolds number	*Re*
$\frac{Pressure\ force}{Inertia\ force}$	$\frac{\Delta p}{\rho\, V^2}$	Euler number	*Eu*
$\frac{Inertia\ force}{Gravity\ force}$	$\frac{V}{(\sqrt{lg})}$	Froude number	*Fr*
$\frac{Surface\ tension\ force}{Inertia\ force}$	$\frac{\sigma}{\rho\, V^2 l}$	Weber number	*Wb*
$\frac{Inertia\ force}{Elastic\ force}$	$\frac{V}{\sqrt{E_s/\rho}}$	Mach number	*Ma*

2.5 NAVIER-STOKES DIFFERENTIAL EQUATION

Generalized equations of motion of a real flow are named after its inventors (Navier and Stokes) known as Navier-Stokes equations. These equations are derived using the

Newton's second law of motion. Which states that, the external forces acting on a body parts is equals to the product of mass and acceleration.

Now, there are two types of external forces acting on the body:

1. Force acting throughout the mass of a body. This includes gravitational force and electromagnetic force.
2. Force acting on the boundary. This includes pressure force and frictional force.

Let us consider per unit mass body force is,

$$\vec{f}_b = \hat{i}\ f_x + \hat{j}\ f_y + \hat{k}\ f_z \tag{2.9}$$

And per unit volume surface force is,

$$\vec{f}_s = \hat{i}\ f_{sx} + \hat{j}\ f_{sy} + \hat{k}\ f_{sz} \tag{2.10}$$

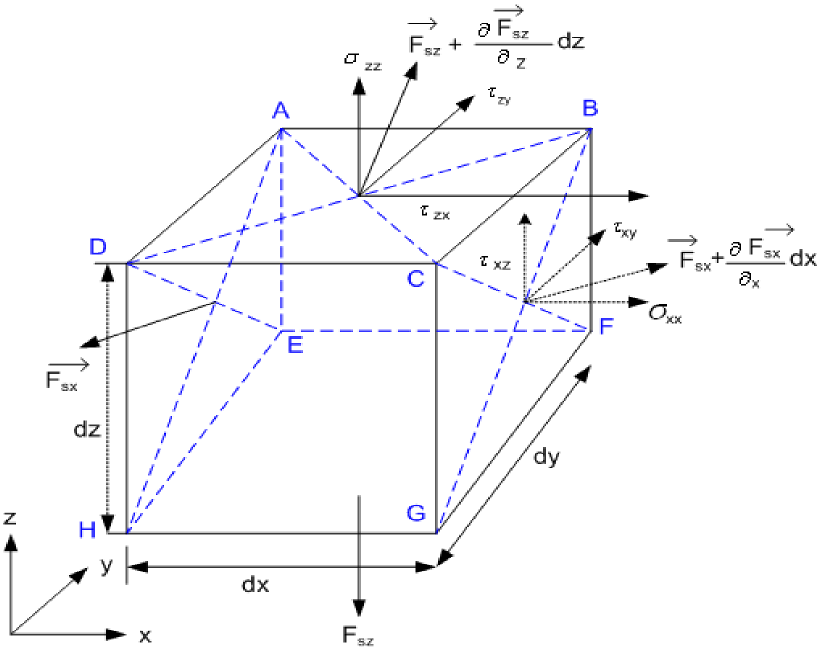


Figure 2.3 Stress components and their location in a fluid element

From the flow of fluid consider a differential fluid element shown in **Figure 2.3.** f_{sx} denotes the surface force acting on the face ABCD per unit area and defined as,

$$\vec{f}_{sx} = \hat{i}\,\sigma_x + \hat{j}\,\tau_{xy} + \hat{k}\,\tau_{xz}$$

And the surface force acting on the face BFGC per unit area is given by,

$$\vec{f}_{sx} + \frac{\partial \vec{f}_{sx}}{\partial x} dx$$

Due to imbalance of surface forces the resultant force acting on the body is given by,

$$\frac{\partial \vec{f}_{sx}}{\partial x}\, dx\, dy\, dz$$

Similarly, the resultant total surface forces acting on all six surfaces is,

$$\left(\frac{\partial \vec{f}_{sx}}{\partial x}+\frac{\partial \vec{f}_{sy}}{\partial y}+\frac{\partial \vec{f}_{sz}}{\partial z}\right) dx\, dy\, dz$$

Therefore, total net surface force *df* per unit volume is

$$d\vec{f} = \frac{\partial \vec{f}_{sx}}{\partial x}+\frac{\partial \vec{f}_{sy}}{\partial y}+\frac{\partial \vec{f}_{sz}}{\partial z} \tag{2.11}$$

Now the surface force acting on any surface is given by two components, τ-shearing stresses and σ- normal stresses. Given by the equation,

$$\left.\begin{aligned} \vec{f}_{sx} &= \hat{i}\,\sigma_x + \hat{j}\,\tau_{xy} + \hat{k}\,\tau_{xz} \\ \vec{f}_{sy} &= \hat{i}\,\tau_{yx} + \hat{j}\,\sigma_y + \hat{k}\,\tau_{yz} \\ \vec{f}_{sz} &= \hat{i}\,\tau_{zx} + \hat{j}\,\tau_{zy} + \hat{k}\,\sigma_z \end{aligned}\right\} \tag{2.12}$$

These three surface force components contain nine scalar quantities and its form a stress tensor. The set of this nine stress tensor component can be written as in matrix notation,

$$A = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \tag{2.13}$$

Here stress tensor matrix A is symmetric.

Take the rotation of the fluid element about y-axis then,

$$\begin{aligned} \dot{\omega}_y dI_y &= (\tau_{xz} dydz)dx - (\tau_{zx} dxdy)dz \\ &= (\tau_{xz} - \tau_{xz})dx\, dy\, dz \end{aligned} \tag{2.14}$$

Where, $\dot{\omega}_y$ = angular acceleration about y-axis

dI_y = moment of inertia about y-axis

Now in left hand side of above equation (2.14) contains fifth power of linear dimensions, and right side third power of linear dimensions which is contradict the element to a point. So it is true only in case of $\tau_{xz} = \tau_{zx}$. Similarly by considering remaining two axis we get $\tau_{xy} = \tau_{yx}$ and $\tau_{zy} = \tau_{yz}$.

Combining equation (2.11) and equation (2.12), the total net surface force acting per unit volume is given by,

$$d\vec{f}_s = \hat{i}\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)+\hat{j}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right) + \hat{k}\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right) \tag{2.15}$$

For velocity field,

$$\frac{D\vec{V}}{Dt} = \hat{i}\frac{Du}{Dt}+\hat{j}\frac{Dv}{Dt}+\hat{k}\frac{Dw}{Dt} \tag{2.16}$$

By Newton’s second laws of motion applied to the differential element we get,

$$\rho(dx\,dy\,dz)\frac{D\vec{V}}{Dt} = (\mathrm{d}\vec{\mathrm{f}}_s)(dx\,dy\,dz)+\rho\vec{f}_b(dx\,dy\,dz)$$

Or $$\rho\frac{D\vec{V}}{Dt} = d\,\vec{\mathrm{f}}_s+\rho\vec{f}_b \tag{2.17}$$

Substitute equation (2.16), (2.15) and (2.9) in equation (2.17) we obtain

$$\rho\frac{Du}{Dt} = \rho f_x + \left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right) \tag{2.18a}$$

$$\rho\frac{Dv}{Dt} = \rho f_y + \left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right) \tag{2.18b}$$

$$\rho\frac{Dw}{Dt} = \rho f_z + \left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right) \tag{2.18c}$$

According to Stokes’ laws of viscosity, shear stress is directly proportional to normal stresses and rate of shear strain which is given by

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \tag{2.19a}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \tag{2.19b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \tag{2.19c}$$

$$\sigma_x = -p+2\mu\frac{\partial u}{\partial x}+\mu'\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \tag{2.19d}$$

$$\sigma_y = -p + 2\mu\frac{\partial v}{\partial y} + \mu'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad (2.19e)$$

$$\sigma_z = -p + 2\mu\frac{\partial w}{\partial z} + \mu'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad (2.19f)$$

Substitute the value of equation (2.19a)-(2.19f) in equation (2.18a)-(2.18c),

$$\begin{aligned}\rho\frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\circ\vec{V}\right) + \frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\\ + \frac{\partial}{\partial z}\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\end{aligned} \qquad (2.20a)$$

$$\begin{aligned}\rho\frac{Dv}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\circ\vec{V}\right) + \frac{\partial}{\partial z}\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\\ + \frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\end{aligned} \qquad (2.20b)$$

$$\begin{aligned}\rho\frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\circ\vec{V}\right) + \frac{\partial}{\partial x}\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\\ + \frac{\partial}{\partial y}\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\end{aligned} \qquad (2.20c)$$

These differential equations (2.20a)-(2.20c) considered to be an accurate presentation of the fluid motion is known as Navier-Stokes equations.

2.6 DIFFERENTIAL EQUATION OF LAMINAR BOUNDARY LAYER

If the Reynolds number is very large and the inertial forces will be predominant in such a case the effect of viscosity can be considered as confined in a thin layer known as velocity boundary layer; adjacent to a solid boundary. Any disturbance created in the laminar flow in boundary layer is ultimately damped. This is known as Laminar Boundary layer. The boundary layer concept was first introduced by Ludwig Prandtl, a German aerodynamicist, in 1904 and he derived the equations for boundary layer flow from the correct reduction of Navier-Stokes equations. He assumed that for fluids having relatively small viscosity, the effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies over which the fluid flows. Thus, close to the body is the boundary layer where shear stresses exert an increasingly larger effect on the fluid as one moves from free stream towards the solid boundary. However, outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer.

Now we derived equation for two dimensional, laminar, steady and incompressible flows.

For Steady flow, $\frac{\partial \rho}{\partial t} = 0$

For incompressible flow, $\rho = \text{constant}$

And flow is considered in two dimensional spaces.

The continuity equation (2.1) reduced as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.21}$$

The general form of continuity equation (2.21) for three dimensional is simplified in case of incompressible flow by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad or \quad \nabla \cdot V = 0 \tag{2.22}$$

Substitute equation (2.22) into Navier–Stokes equations (2.20*a*), (2.20*b*) and (2.20*c*), for a two-dimensional steady incompressible flow in Cartesian coordinates are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad , \; x-direction$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad , y-direction$$

Where, $\nu = \frac{\mu}{\rho}$ =Kinematic viscosity

Now to convert equations in non-dimensional forms, Let us consider length L and velocity scales U_∞. The non-dimensional variables are:

$$u' = \frac{u}{U_\infty}, \; v' = \frac{v}{U_\infty}, \; p' = \frac{p}{\rho U^2_\infty}, \; x' = \frac{x}{L}, \; y' = \frac{y}{L}$$

The pressure is non-dimensional by twice the dynamic pressure

$$p_d = \frac{1}{2}\rho U^2_\infty$$

So, Momentum equations are converted as,

$$u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=-\frac{1}{\rho}\frac{\partial p'}{\partial x'}+\frac{1}{R_e}\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right) \tag{2.23}$$

$$u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}=-\frac{1}{\rho}\frac{\partial p'}{\partial x'}+\frac{1}{R_e}\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right) \tag{2.24}$$

Where, $\mathrm{R}_e=\frac{\rho U_\infty L}{\mu}$

Now the effect of the shear stresses outside the boundary layer of the flow is very small compared to values inside the boundary layer.

So, the velocity gradient $\frac{\partial u}{\partial y}$ is negligible in the normal direction of the flow within this thin layer and it is very large compared to the velocity gradient $\frac{\partial u}{\partial x}$ in the direction of flow.

Along the *x*-direction, momentum equation (2.23) is reduced as,

$$u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=-\frac{1}{\rho}\frac{\partial p'}{\partial x'}+\frac{1}{R_e}\frac{\partial^2 u'}{\partial y'^2} \tag{2.25}$$

For the momentum equation in the y- direction,

All the terms of equation (2.24) are smaller in magnitude then momentum equation (2.23) in *x*-direction. Therefore, equation (2.24) can only balanced if $\frac{\partial p'}{\partial x'}$ is of the same order of magnitude as other terms.

Thus, momentum equation (2.24) in *y*- direction is reduced as,

$$\frac{\partial p'}{\partial x'}=0 \tag{2.26}$$

Form the momentum equation (2.26) we can say that the pressure across the boundary layer does not change. The pressure is impressed on the boundary layer, and its value is determined by hydrodynamic considerations.

Also we can say that the pressure p is only a function of *x* variable and it is easily determined by the inviscid flow outside the boundary layer.

$$p'=\frac{1}{2}\rho U_\infty^2 \tag{2.27}$$

Combined the equation (2.27) and equation (2.25) the momentum equation in *x*-direction we get,

$$u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=-U_\infty\frac{dU_\infty}{dx'}+\frac{1}{R_e}\frac{\partial^2 u'}{\partial y'^2} \tag{2.28}$$

Finally in order to magnitude analysis, the Navier-Stokes equations (2.23) and (2.24) are simplified into equations given below.

$$u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=-U_\infty\frac{dU_\infty}{dx'}+\frac{1}{R_e}\frac{\partial^2 u'}{\partial y'^2} \tag{2.29}$$

$$\frac{\partial p}{\partial y}=0 \tag{2.30}$$

These equations (2.29) and (2.30) are known as Prandtl' s boundary layer equations. The available boundary conditions are:

$$at \quad y=0, u=v=0$$

$$at \quad y=\delta\, or\, \infty, u=U(x)$$

We solve the Prandtl boundary layer equations for velocity components $u'(x,y)$ and $v'(x,y)$ in *x* and *y* direction respectively and stream velocity *U* obtained from the outer in viscid fluid flow analysis. The equations are solved by commencing at the leading edge of the body and moving to downstream at the desired location.

2.7 STRESS–STRAIN RELATIONSHIP OF SOME VISCO INELASTIC FLUID MODELS:

We know that the Navier-Stokes equations cannot describe the behavior of fluids having high molecular weights. Due to the variety of such fluids it is very difficult to suggest a single constitutive equation which can describe the properties of all non-Newtonian fluids. Therefore many models of non-Newtonian fluids have been proposed and usually it is classified as: (i) Fluids for which shear stress depends only on the rate of shear (ii) Fluids for which relation between shear stress and rate of strain depends on time (iii) The visco inelastic fluids which possess both elastic and viscous properties. A common feature of visco inelastic non-Newtonian class of fluids is that when at rest they are isotropic and homogenous and when they are subjected to a shear, the resultant stress depends only on the rate of shear. However, this sub-class shows diverse behavior in response to applied stress. A number of rheological models have been proposed to explain such a diverse behavior. Some of these models which have attracted researchers. Due to the growing use of these non-Newtonian materials in various manufacturing and processing industries, considerable efforts have been directed towards understanding their flow characteristics. Many of the inelastic non-Newtonian fluids, encountered in chemical engineering processes are known to follow the so-called "power-law model" in which the shear stress varies according to a power function of the strain rate (Metzner *et al.* 1956). The well-known Ostwald-de-Waele model (or power-law fluid model) is purely phenomenological; however, it is useful in that approximately describes a great number of real non-Newtonian fluids. This model behaves properly under tensor deformation. Use of this model alone assumes that the fluid is purely viscous.

The formulation of empirical relations for different non-Newtonian fluid and its evaluation in terms of known variables is indeed very difficult task. Here we shall develop the empirical relationship for two-dimensional boundary layer equations of non-Newtonian fluids. A number of industrially important fluids such as molten plastics, polymers, pulps and foods exhibit non-Newtonian fluid behavior.

Mathematically it can be represented in the form,

$$\bar{\bar{\tau}} = -\left\{ m \left| \frac{1}{2} \bar{\bar{\Delta}} : \bar{\bar{\Delta}} \right|^{\frac{n-1}{2}} \right\} \bar{\bar{\Delta}} \tag{2.31}$$

Where, m and n are physical constants. They differ for different fluids and can be determined experimentally.

$$\bar{\bar{\Delta}}\,\bar{\bar{\Delta}} = \sum_{l=1}^{3} \sum_{m=1}^{3} e_{l\,m}\ e_{m\,l}$$

Now,

$$\bar{\bar{\Delta}} = \mathrm{e}_{i\,j} \text{ and } \bar{\bar{\tau}} = \tau_{i\,j} \tag{2.31a}$$

Hence,

$$\bar{\bar{\Delta}} : \bar{\bar{\Delta}} = (e_{11})^2 + (e_{22})^2 + (e_{33})^2 + 2(e_{12})^2 + 2(e_{23})^2 + 2(e_{13})^2 \tag{2.32}$$

Where,

$$e_{11} = \frac{\partial u'}{\partial x'} + \frac{\partial u'}{\partial x'};\ e_{22} = \frac{\partial v'}{\partial y'} + \frac{\partial v'}{\partial y'};\ e_{33} = \frac{\partial w'}{\partial z'} + \frac{\partial w'}{\partial z'};\ e_{12} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = e_{21} \ldots\text{etc} \tag{2.32a}$$

$$\begin{aligned} \text{and} \quad \tau_{i\,j} &\Rightarrow \tau'_{x'x'},\quad \tau'_{y'y'},\quad \tau'_{z'z'},\ \tau'_{x'y'} \ldots\ldots\text{etc} \\ eij &\Rightarrow e_{x'x'},\quad e_{y'y'},\quad e_{z'z'},\ e_{x'y'} \ldots\ldots\text{etc} \end{aligned} \tag{2.32b}$$

Hence we get,

δ^m [1 1 1 $\frac{1}{\delta^2}$ $\frac{1}{\delta^2}$

$$\tau'_{x'x'} = -m\left\{\left(\frac{1}{2}\right)^{\frac{n-1}{2}}\left[4\left(\frac{\partial u'}{\partial x'}\right)^2+4\left(\frac{\partial v'}{\partial y'}\right)^2+4\left(\frac{\partial w'}{\partial z'}\right)^2+2\left(\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right)^2+2\left(\frac{\partial v'}{\partial z'}+\frac{\partial w'}{\partial y'}\right)^2\right.\right. \tag{2.33}$$

1 1]

$$\left.\left.+2\left(\frac{\partial v'}{\partial z'}+\frac{\partial w'}{\partial x'}\right)^2\right]^{\frac{n-1}{2}} 2\frac{\partial u'}{\partial x'}\right\}$$

δ_m [1 1 1 $\frac{1}{\delta^2}$ $\frac{1}{\delta^2}$ 1

$$\tau'_{y'x'} = -m\left\{\left(\frac{1}{2}\right)^{\frac{n-1}{2}}\left[4\left(\frac{\partial u'}{\partial x'}\right)^2+4\left(\frac{\partial v'}{\partial y'}\right)^2+4\left(\frac{\partial w'}{\partial z'}\right)^2+2\left(\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right)^2+2\left(\frac{\partial v'}{\partial z'}+\frac{\partial w'}{\partial y'}\right)^2+2\left(\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial x'}\right)^2\right]^{\frac{n-1}{2}}\right. \tag{2.34}$$

$\frac{1}{\delta}\delta]$

$$\left.\left(\frac{\partial u'}{\partial y'}+2\frac{\partial v'}{\partial x'}\right)\right\}$$

δ_m [1 1 1 $\frac{1}{\delta^2}$ $\frac{1}{\delta^2}$ 1

$$\tau'_{y'z'} = -m\left\{\left(\frac{1}{2}\right)^{\frac{n-1}{2}}\left[\begin{array}{l}4\left(\frac{\partial u'}{\partial x'}\right)^2+4\left(\frac{\partial v'}{\partial y'}\right)^2+4\left(\frac{\partial w'}{\partial z'}\right)^2+2\left(\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right)^2+2\left(\frac{\partial v'}{\partial z'}+\frac{\partial w'}{\partial y'}\right)^2\\ +2\left(\frac{\partial u'}{\partial y'}+\frac{\partial w'}{\partial x'}\right)^2\end{array}\right]^{\frac{n-1}{2}}\right. \tag{2.35}$$

$\frac{1}{\delta^2}$ δ]

$$\left.\left(\frac{\partial w'}{\partial y'}+\frac{\partial v'}{\partial z'}\right)\right\}$$

And similarly other components of τ_{ij} can be derived.

We assume as before $k = o(\delta^m)$, where m is to be determined writing only the order of various terms we get the following order relation

$$0(1) = 0\left(\delta^{m-n-1}\right) + 0\left(\delta^{m-n+1}\right) \tag{2.36}$$

Taking m = n + 1 we get the following non-vanishing component

$$\tau'_{y'x'} = -m\left\{\left(\frac{1}{2}\right)^{\frac{n-1}{2}}\left[2\left(\frac{\partial u'}{\partial y'}\right)^2 + 2\left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u'}{\partial y'}\right\}$$

Or

$$\tau'_{y'x'} = -m\left\{\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u'}{\partial y'}\right\} \tag{2.37}$$

Similarly, other non-vanishing components will be

$$\tau'_{y'z'} = -m\left\{\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w'}{\partial y'}\right\} \tag{2.38}$$

Remaining shear-stress components vanish under the boundary-layer assumption. In a similar way the stress-strain relationship for various visco-inelastic fluid models can be derived.

Few fluid models which we have used are discussed below.

2.7.1 SISKO FLUID MODEL

So many real fluids are available which is follows the Sisko model of non-Newtonian fluids proposed by Sisko. Polymeric suspensions such as waterborne coatings are known to be non-Newtonian in nature and are follow the Sisko model. The viscosity of such coatings depends on the shear rate and the strain history. An example that lends itself to such types of coating is metallic automotive basecoat. Of course the most well know Sisko fluids are lubricating greases. In fact most psueodplastic fluids, drilling fluids and cement slurries without yield stress follow the Sisko model. For the Sisko fluid model shearing stress v/s rate of strain is given by the relations (for 3-D flows).

$$\tau'_{y'x'} = \left\{a + b\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\right\}\frac{\partial u'}{\partial y'}$$

$$\tau'_{y'z'} = \left\{a + b\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\right\}\frac{\partial w'}{\partial y'}$$

2.7.2 PRANDTL FLUID MODEL

Ludwig Prandtl (1875-1953)

Born in Germany, Prandtl taught at Hanover Engineering College and then Gottingen University. He successfully observed, by using the floating tracer method, that the surface of bodies is covered with a thin layer having a large velocity gradient, and so advocated the theory of the boundary layer. He is called the creator of modern fluid dynamics. Furthermore, he taught such famous scholars as Blasius and Karman.

Peristaltic transport widely occurs in many biological systems for instance, food swallowing through the esophagus, intra-urine fluid motion, circulation of blood in small blood vessels and the flows of many other glandular ducts. Several theoretical and experimental studies have been undertaken to understand peristalsis through abrupt changes in geometry and realistic assumptions. A review of much of the early literature is presented in an article by Jaffrin and Shapiro(1987). The peristaltic flow of a power-law fluid in an asymmetric channel was investigated by Subba Reddy et al. (2007). Nagendra et al. (2008) have studied the peristaltic flow of a Jeffrey fluid in a tube. Recently, Akbar et al. (2012) have discussed the peristaltic flow of a Prandtl fluid in an asymmetric channel. Noreen Sher Akbar (2014) have discussed that the blood flow analysis of Prandtl fluid in tapered stenos arteries.

Shearing Stress v/s rate of Strain is given by (for 3-D flow),

$$\tau_{y'x'} = \frac{A\sin^{-1}\left\{\frac{1}{C}\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}\right\}}{\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}} \frac{\partial u'}{\partial y'}$$

$$\tau_{y'z'} = \frac{A\sin^{-1}\left\{\frac{1}{C}\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}\right\}}{\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}} \frac{\partial w'}{\partial y'}$$

2.7.3 PRANDTL – EYRING

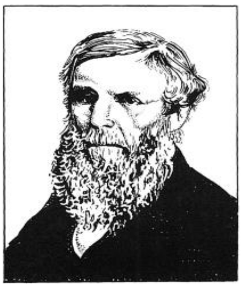
Prandtl-Eyring model is an approximation of perfectly plastic fluids. Features of this model are the behavior of lubricant.

Shearing Stress v/s Rate of Strain is given by(for 3-D flow),

$$\tau_{y'x'} = \frac{A\sinh^{-1}\left\{\frac{1}{B}\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}\right\}}{\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}}\frac{\partial u'}{\partial y'}$$

$$\tau_{y'z'} = \frac{A\sinh^{-1}\left\{\frac{1}{B}\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}\right\}}{\left[\left(\frac{\partial u'}{\partial y'}\right)^2 + \left(\frac{\partial w'}{\partial y'}\right)^2\right]^{1/2}}\frac{\partial w'}{\partial y'}$$

2.7.4 WILLIAMSON FLUID MODEL



William Froude (1810-79)

He born in England and engaged in shipbuilding. In his sixties started the study of ship resistance, building a boat testing pool (approximately *75*m long) near his home. After his death, this study was continued by his son, Robert Edmund Froude (1846-1924). For similarity under conditions of inertial and gravitational forces, the non dimensional number used carries his name.

Williamson fluid is characterized as a non-Newtonian fluid with shear thinning property i.e., viscosity decreases with increasing rate of shear stress. Chyme in small intestine is assumed to behave like Williamson fluid. Peristaltic motion is one of the most characteristics fluid transport mechanism in many biological systems. It pumps the fluids against pressure rise. Williamson (1929) discussed the flow of pseudo plastic materials and proposed a model equation to describe the flow of pseudo plastic fluids and experimentally verified the results. Lyubimov and Perminov (2002) discussed the flow of a thin layer of a Williamson fluid over an inclined surface in the presence of a gravitational field. Cramer *et al.* (1968) showed that this model fits the experimental data of polymer solutions and particle suspensions better than other models. In the Williamson fluid model, both the maximum viscosities (μ∞) and minimum viscosities (μ0) are considered. So, for pseudo plastic fluids (for which the apparent viscosity does not go to zero at infinity), it will give better results.

Shearing Stress v/s rate of Strain is given by(for 3-D flow),

$$\tau_{y'x'} = \frac{A\,\dfrac{\partial u'}{\partial y'}}{B + \left[\left(\dfrac{\partial u'}{\partial y'}\right)^2 + \left(\dfrac{\partial w'}{\partial y'}\right)^2\right]^{1/2}} + \mu_\infty \frac{\partial u'}{\partial y'}$$

$$\tau_{y'z'} = \frac{A\,\dfrac{\partial w'}{\partial y'}}{B + \left[\left(\dfrac{\partial u'}{\partial y'}\right)^2 + \left(\dfrac{\partial w'}{\partial y'}\right)^2\right]^{1/2}} + \mu_\infty \frac{\partial w'}{\partial y'}$$

CHAPTER-3

SIMILARITY ANALISIS AND NUMERICAL METHODS

3.1 SIMILARITY ANALYSIS

An exigent problem faced by engineers and applied mathematicians are working in a field of study is to find exact solutions of the basic equations arising in that field while employing a minimum number of simplifying assumptions. In number of cases, the basic equations of any fluid expressing the physical laws are partial differential equations. Ideally, one hopes to find exact solution of these equations. In certain instances standard methods of solution (separation of variables, Laplace transforms etc…) are of value and solutions can be found. Nevertheless, there are numbers of problems available for which the solution can't found by the usual classical methods. This is particularly true if the equations encountered are nonlinear. An excellent example of a nonlinear system of equations which presents formidable obstacles to classical analysis is the system of equations of motion for a viscous fluid flow.

We are interested to find the solution of such sets of partial differential equations for applied fields. But for the solution of such partial differential equation common classical methods failed to yield result. So we have to apply transformations that are reduced the system of partial differential equations into system of ordinary differential equations. These solutions are known as similarity solutions.

In similarity analysis method, we find the transformation between independent and dependent variables appeared in the given fluid flow equations in such a way that the numbers of independent variables are at least one less than in the transformed differential equation compared to the original partial differential equation. For instance if a given partial differential equation contains two independent variables then after applying similarity transformation it is transformed into an ordinary differential equation (i.e. equation having one independent variable). Due to this fact, the major application of

similarity transformations has been used to reduction of certain classes of non-linear partial differential equations in to ordinary differential equations. Blasius (1908) gives model example for similarity analysis method. He converted the system of nonlinear partial differential equations for flow over a flat plate into only one ordinary differential equation.
In recent times it is more demanded and increase interest to find the necessary and sufficient conditions for the existence of similarity solutions. Also find the general solution method for similarity solutions of various classes of non linear partial differential equations. From the literature review there are four similarity analysis methods. Which are as follows :

(a) Group theoretic
(b) Separation of variables
(c) Free Parameter
(d) Dimensional Analysis

We have applied group theoretic method for our problems which is discussed below.

3.1.1 GROUP THEORETIC METHOD

Mathematically, Group theory method is the most difficult approach for finding similarity transformation. This concept will eventually lead to more general developments of similarity analysis. For better understanding of this concept we will consider one simple application.

According to this method we can reduce a system of partial differential equations having two independent variables into a system of ordinary differential equations having one independent variable. For that we can find a relation between new independent variable η and old independent variables *x* and *y* in such way that which is absolutely invariant under a taken group transformation. As well as we can find a relation between the dependent variables that is also an absolute invariant under a taken group transformation.

Let’s consider the basic equations for laminar boundary–layer flow of Newtonian fluid over a flat plate (M. Patel, 2007).

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{3.1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (3.2)$$

Where, ν = Kinematic viscosity

With the boundary conditions are:

$$u=0 \ and \ v=0 \ at \ y=0 \qquad (3.2a)$$

$$u=U_0 \quad at \quad y\to\infty \qquad (3.2b)$$

Take one-parameter linear group transformation of variables *x, y, u* and *v* as

$$G:\begin{cases}\bar{x}=a^{n}x\\ \bar{y}=a^{m}y\\ \bar{u}=a^{p}u\\ \bar{v}=a^{q}v\end{cases} \qquad (3.3)$$

Where, *a* = one parameter group transformation constant.

We need to determine the values of constants *n*, *m*, *p* and *q* in such way that given partial differential equations are invariant under taken group transformation. For this we substitute expressions from equation (3.3) in equations (3.1) and (3.2) we have,

$$a^{-2m+k}\ \bar{u}\ \frac{\partial \bar{u}}{\partial \bar{x}}+a^{-n-m+l}\ \bar{v}\ \frac{\partial \bar{u}}{\partial \bar{y}}=\nu a^{-m+2l}\ \frac{\partial^{2}\bar{u}}{\partial \bar{y}^{2}} \qquad (3.4a)$$

$$a^{-m+k}\frac{\partial \bar{u}}{\partial \bar{x}}+a^{-n+l}\frac{\partial \bar{v}}{\partial \bar{y}}=0 \qquad (3.4b)$$

For finding the invariance it is necessary to choose the power of "*a*" in such manner that the transformed equations look exactly the same in terms of the new variables as they did with the old. We need only to equate these exponents and cancel out the common powers of *a*.

Equating powers of *a* in equations (3.4*a*) and (3.4*b*) we get the following system of equations,

$$-2m+k=-n-m+l=-m+2l \qquad (3.5)$$

$$-m+k=-n+l \qquad (3.6)$$

From the set of equations (3.5) and (3.6) we get relation between constants,

$$n=-l \quad and \quad m=k-2l \qquad (3.7)$$

Now we define new independent variable η using previous independent variables *x* and *y* which is absolute invariant under the taken group transformation.

$$\eta = y\,x^{s}$$

(3.8)

The invariance of η is satisfied by

$$\eta = yx^{s} = \bar{y}\,\bar{x}^{s} = a^{m}y\left(a^{n}x\right)^{s} \tag{3.9}$$

We get,

$$s = -{}^{m}\!/_{n}$$

(3.10)

Also, we define new dependent variables using previous variables in such a way that they are absolute invariant under the taken group transformation.

$$f_1 = u\,x^{r} \tag{3.11}$$

$$f_2 = v\,x^{t} \tag{3.12}$$

The invariance dependent variables f_1 and f_2 satisfies,

$$u\,x^{r} = \bar{u}\,\bar{x}^{r} = (a^{k-2l}u)(a^{k}x)^{r} \tag{3.13}$$

$$v\,x^{t} = \bar{v}\,\bar{x}^{t} = (a^{-l}\ \mathrm{v})(a^{k}x)^{t} \tag{3.14}$$

Equating powers of constant a on both the sides of equations (3.13) and (3.14) we get,

$$k - 2l + kr = 0 \tag{3.15}$$

$$-l + kt = 0 \tag{3.16}$$

Finally, from the equation (3.15) the value of r and from the equation (3.16) the value of t is substitute in equation (3.11) and (3.12) respectively.

$$f_1(\eta) = ux^{r} = ux^{-1+2l/k} \qquad or \qquad u = x^{-1+2l/k}f_1(\eta) \tag{3.17}$$

$$f_2(\eta) = vx^{t} = vx^{l/k} \qquad or \qquad v = x^{-l/k}f_2(\eta)$$

(3.18)

It is now possible to substitute these expressions for u and v in the original set of equations (3.1) and (3.2) we have ordinary differential equations in new dependent variables f_1 and f_2.

The continuity equation (3.2) is reduced as,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\left(1-\frac{2l}{k}\right)x^{-2l/k}f_1+x^{1-(2l/k)}f_1{}'\frac{\partial\eta}{\partial x}+x^{-l/k}f_2{}'\frac{\partial\eta}{\partial y}=0 \tag{3.19}$$

Now,

$$\frac{\partial\eta}{\partial x}=\frac{\partial}{\partial x}\;y\;x^{-l/k}=-\frac{l}{k}y\,x^{-(l/k)-1}=-\frac{l}{k}\;\frac{\eta}{x} \tag{3.20}$$

$$\frac{\partial\eta}{\partial y}=x^{-l/k} \tag{3.21}$$

$\therefore$ Equation (3.19) becomes,

$$\left(1-\frac{2l}{k}\right)x^{-2l/k}f_1+x^{-2l/k}f_1{}'\left(-\frac{l}{k}\right)\eta+x^{-2l/k}f_2{}'=0 \tag{3.22}$$

Variable $x\neq 0$ so we have,

$$\left(1-\frac{2l}{k}\right)f_1+f_1{}'\left(-\frac{l}{k}\right)\eta+f_2{}'=0 \tag{3.23}$$

Now let's choose $l/k=1/2$ to simplify our expression in equation (3.23) we obtain,

$$f_2{}'=\frac{f_1{}'\eta}{2} \tag{3.24}$$

We substitute our expressions for u and v from equation (3.17) and (3.18) respectively into the equation (3.2) with the choosing $m/n=1/2$ we get,

$$-\frac{f_1f_1{}'}{2}\;\frac{\eta}{x}+x^{-1/2}\;f_2\;f_1{}'x^{-1/2}=\nu\,f_1{}''\;x^{-1} \tag{3.25}$$

Again for $x\neq 0$ we have

$$f_2\;f_1{}'+\frac{f_1f_1{}'}{2}\eta=\nu\,f_1{}'' \tag{3.26}$$

Thus we have two ordinary differential equations (3.24) and (3.26) for the functions f_1 and f_2. The results can be put in more conventional form by letting $f_1=F'$

Then the form of equation (3.24)

$$\begin{aligned} f_2&=\int\frac{F''\eta}{2}d\eta+const.\\ &=\frac{1}{2}\left(\eta F'-F\right)+const. \end{aligned} \tag{3.27}$$

Taking integration constant zero and substitute the value of f_2 from equation (3.27) in equation (3.26) we get

$$\frac{1}{2}(\eta F'-F)F''-\frac{F'F''}{2}\eta=\nu F'''$$

It is also written as,

$$\nu F'''+\frac{FF''}{2}=0 \tag{3.28}$$

Finally, equation (3.28) is similarity equation for defined flow problem. The major approaches to finding similarity transformations have been outlined up to this point. In this method we combined two methods Group transformation method for reduction of independent variables and separation of variables method for combined system of equations into a single equation. So together both the methods are called “dimensional analysis method” can be established as being a special group theoretical method.

3.1.2 FREE PARAMETER METHOD

The free parameter method is also one of the types of similarity analysis method. Using this method we reduce the number of independent variables as well as we find proper transformation of the boundary values. The choice of transformation is determined by the boundary values of the original problem and takes transformation of dependent variables in such way that the boundary values of defined flow problem is transformed into meaningful boundary values in terms of the new set of dependent variables.

The particular procedure is perhaps best illustrated by example. In this example we also point out one of the basic weakness in the use of the free parameter type of analysis. Consider a classical flow over a flat plate. We denote u by the velocity of the flow within the boundary layer in the direction of flow (along x-axis) and v is the velocity in the boundary layer normal to the flow(along y-axis). The velocity of the fluid is equal to zero on the surface of the plate and varies from zero to the “mainstream” velocity U_0 in a very thin region near the surface of the plate called the boundary layer. The flow equations for defined flow problem are given by equations (3.1) and (3.2).

First we eliminate one of the dependent variables u or v from the equation (3.2).
From equation (3.1) velocity component v given by,

$$v=-\int_0^y\left(\frac{\partial u}{\partial x}\right)dy \tag{3.29}$$

Substitute value of equation (3.29) in equation (3.2),

$$u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\int_0^y\left(\frac{\partial u}{\partial x}\right)dy=\nu\frac{\partial^2 u}{\partial y^2} \tag{3.30}$$

Now our goal is to convert equation (3.30) into single independent variable η. Also we express *u* as a function of η in such a way that the boundary conditions equations (3.2*a*) and (3.2*b*) which have been specified for the original equation carry over into meaningful boundary conditions in the transformed equation.

For this take $\frac{u}{U_0}$ be a function of independent variable η as,

$$\frac{u}{U_0}=f'(\eta) \tag{3.31}$$

With Boundary conditions;

$$f'=\frac{u}{U_0}=0\ \ at\ \ \eta=\eta_0 \tag{3.32}$$

$$f'=\frac{u}{U_0}=1\ \ at\ \eta=\eta_\infty \tag{3.33}$$

Where,

$$\eta_0=\eta(0,y),\quad \eta_\infty=\eta(x,\infty)$$

Substitute the value of equation (3.31) in equation (3.30) we get,

$$F'F''\frac{\partial\eta}{\partial x}-\frac{\partial}{\partial x}\int_0^y F'dy\ F''\frac{\partial\eta}{\partial y}\ =\frac{\nu}{U_0}\left[F'''\left(\frac{\partial\eta}{\partial y}\right)^2+F''\frac{\partial^2\eta}{\partial y^2}\right] \tag{3.34}$$

Now assume that

$$\frac{\partial\eta}{\partial y}=g(x) \tag{3.35}$$

So equation (3.34) will reduce as

$$F'''-\left(\frac{U_0}{\nu g^3}\ \frac{dg}{dx}\right)F\,F''=0 \tag{3.36}$$

The problem of reducing equation (3.36) to an ordinary differential equation in η reduces to merely choosing.

$$\frac{U_0}{\nu\, g^3}\ \frac{dg}{dy}=cons\tan t=-\frac{1}{2} \tag{3.37}$$

Together equation (3.36) and (3.37) we get,

$$F''' + \frac{1}{2} FF'' = 0 \tag{3.38}$$

With boundary conditions:

$$f(0) = f'(0) = 0 \quad and \quad \lim_{\eta \to \infty} f' = 1 \tag{3.39}$$

Equation (3.38) called the Blasius equation.

In this method we specifies an acceptable form for the new dependent variables but it does not give any conditions on the transformed independent variables which is the chief weakness of this method. The proper form for the independent, however, ultimately is chosen. Group theory method is better to apply.

3.2 QUASILINEARIZATION METHOD

The quasi linearization approach is a generalized Newton-Raphson technique for functional equations. It is converges quadratically to the exact solution if there is convergence and it has monotone convergence.

Let us consider the nonlinear second order differential equation as follows:

$$y''(x) = \phi(y'(x), y(x), x) \tag{3.40}$$

With the boundary conditions

$$y(a) = a_0 \quad and \quad y(b) = b_0, \ a \le x \le b \tag{3.41}$$

The function ϕ can now be expanded about $y_0(x)$ by using the Taylor's series expansion. After neglecting second and higher order terms we have,

$$\phi(y'(\mathrm{x}), y(x), x) = \phi(y_0'(x), y_0(x), x) + (y(x) - y_0(x))\left(\frac{\partial \phi}{\partial y}\right)_{(y_0', y_0, x)} + (y'(x) - y_0'(x))\left(\frac{\partial \phi}{\partial y'}\right)_{(y_0', y_0, x)} \tag{3.42}$$

From equation (3.40) and (3.42) we have,

$$y''(x) \approx \phi(y_0'(x), y_0(x), x) + (y(x) - y_0(x))\left(\frac{\partial \phi}{\partial y}\right)_{(y_0', y_0, x)} + (y'(x) - y_0'(x))\left(\frac{\partial \phi}{\partial y'}\right)_{(y_0', y_0, x)} \tag{3.43}$$

Solving equation (3.43) for $y(x)$ and say it $y_1(x)$.

Therefore, equation (3.43) can be written as,

$$y_1''(x) = \phi(y_0'(x), y_0(x), x) + (y_1(x) - y_0(x))\left(\frac{\partial \phi}{\partial y}\right)_{(y_0', y_{0,} x)} + (y_1'(x) - y_0'(x))\left(\frac{\partial \phi}{\partial y'}\right)_{(y_0', y_{0,} x)}$$

Similarly, above equation expanded about $y_1(x)$ we have,

$$y_2''(x) \approx \phi(y_1'(x), y_1(x), x) + (y_2(x) - y_1(x))\left(\frac{\partial \phi}{\partial y_1}\right)_{(y_1', y_{1,} x)} + (y_2'(x) - y_1'(x))\left(\frac{\partial \phi}{\partial y_1'}\right)_{(y_1', y_{1,} x)}$$

Continue the procedure up to desired accuracy. We have recurrence relation as,

$$y_{r+1}''(x) \approx \phi(y_r'(x), y_r(x), x) + (y_{r+1}(x) - y_r(x))\left(\frac{\partial \phi}{\partial y_r}\right)_{(y_r', y_{r,} x)}$$

$$+ (y_{r+1}'(x) - y_r'(x))\left(\frac{\partial \phi}{\partial y_r'}\right)_{(y_r', y_{r,} x)}$$

(3.44)

Where, r = 0, 1, 2, 3…

Where, $y_r(x)$ is known and using it we can obtain $y_{r+1}(x)$. Equation (3.44) is always a linear differential equation and boundary conditions are:

$$y_{r+1}(a) = a_0 \quad and \quad y_{r+1}(b) = b_0, \;\; a \le x \le b$$

Similarly, one can follow the same procedure for higher order nonlinear differential equations to obtain the recurrence relation

$$L^n y_{r+1}(x) = \phi(y_r^{n-1}(x), ..., \mathrm{y}_r'(x), y_r(x), x)$$

$$+ \sum_{i=0}^{n-1} (y_{r+1}^i(x) - y_r^i(x))\phi_{y^i}(y_r^{n-1}(x), ..., \mathrm{y}_r'(x), y_r(x), x) \qquad (3.45)$$

Where, n is the order of the differential equation. Equation (3.45) is always a linear differential equation and can be solved recursively.

3.2 NUMERICAL METHOD: FINITE DIFFERENCE METHOD

The solution of Ordinary Differential Equations (ODEs) is found by two ways first is discrete method and second is numerical method. Discrete methods are one of the oldest and most successful areas of numerical calculation. One of the examples to need for such calculation is the problem of predicting the motions of planets or other bodies in the space. The governing differential equations for the sun and the planet are known and Newton showed that how to solve it exactly. In the last three centuries no one has ever found its exact solution in the case of more than two bodies. So, the numerical calculation of such orbits is effortless by modern standards.

The most popular families of numerical methods for solution of ODEs are:

- Finite Difference Method
- Finite Element Method
- Range -Kutta Method

For finding the solution of two ending point boundary value problem using finite difference method, we replace the derivatives occurring in the differential equation as well as in the boundary conditions by means of their finite difference approximations. By Substituting finite difference we obtain sets of linear equation and it is solve by any standard procedure.

Now procedure to find the appropriate finite difference (central difference) approximation of the derivatives as follows.

The Taylor's series expansion of $f(\mathrm{x+h})$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\mathrm{x}) + \frac{h^3}{6} f'''(\mathrm{x}) + \ldots\ldots$$

For which we obtain,

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(x) - \frac{h^2}{6} f'''(\mathrm{x}) + \ldots\ldots$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h) \quad , \qquad (3.46)$$

Equation (3.46) is called the forward difference approximation formula for $f'(x)$

Similarly, by taking Taylor's series expansion for $f(\mathrm{x-h})$,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \ldots\ldots$$

Neglecting second and higher order terms we obtain,

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \quad , \qquad (3.47)$$

Equation (3.47) is called the backward difference approximation formula for $f'(x)$.

A central difference approximation formula for $f'(x)$ is given by taking addition of equations (3.46) and (3.47) we have,

$$f'(x)=\frac{f(x+h)-f(x-h)}{2h}+O(h) \tag{3.48}$$

It is clear that it gives better approximation for $f'(x)$ than the forward and backward difference approximation.

Again adding for $f''(x),\ f'''(x)$ we have,

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}+O(h^2) \tag{3.49}$$

$$f'''(x)=\frac{f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)}{2h^3}+O(h^3) \tag{3.50}$$

In similar manner, it is possible to derive the central finite difference approximations formula for higher order derivatives to solve boundary value problems.

We divide the range [$x_{0,}\,x_{\mathrm{n}}$] into *n* equal subintervals of width *h*.

So that the value of *x* at i$^{\text{th}}$ node is given by,

$$x_i=x_0+ih, \qquad i=1,2,3,.....,\mathrm{n}$$

And their corresponding values of $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ at i$^{\text{th}}$ node point is obtained by,

$$f(x_i)=f_i=f(x_0+ih), \qquad i=1,2,3,.....,\mathrm{n}$$

The values of at the i$^{\text{th}}$ node can now be written as,

$$f_i'(x)=\frac{f_{i+1}-f_{i-1}}{2h}+O(h) \tag{3.51a}$$

$$f_i''(x)=\frac{f_{i-1}-2f_i+f_{i+1}}{h^2}+O(h^2) \tag{3.51b}$$

$$f_i'''(x)=\frac{f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2}}{2h^3}+O(h^3) \tag{3.51c}$$

Equations (3.51*a*)-(3.51c) central difference formulas for derivatives are substituting in ODE we have system of linear equations then it is solve by Finite Difference method with boundary conditions we have solution. If we increase the number of subintervals we find the solution up to desired accuracy.

CHAPTER-04

NUMERICAL SOLUTION OF BOUNDARY LAYER EQUATION OF PRANDTL FLUID FLOW

4.1 INTRODUCTION

The complex rheology of biological fluids has motivated investigations involving different non-Newtonian fluids. In recent years, non-Newtonian fluids have become more and more important industrially. Polymer solutions, polymer melts, blood, paints and slurries, shampoo, toothpaste, clay coating and suspensions, grease, cosmetic products, custard, are the most common examples of non-Newtonian fluids. Academic curiosity and practical applications have generated considerable interest in finding the solutions of differential equations governing the motion of non-Newtonian fluids. The property of these fluids is that the stress tensor is related to the rate of deformation tensor by some non-linear relationship. These fluids present some interesting challenges to researchers in engineering, applied mathematics and computer science. Thus wide usages of these fluids have prompted modern researchers to explore extensively; the fields of non-Newtonian fluids.

The similarity solutions play an important role in investigating the non-Newtonian effects especially in the boundary layer flows. Under the requirements of both academic as well as practical application to find the similarity solution of differential equation of fluid (Non-Newtonian) motion. In addition to the fact that it is the only class of exact solutions for the governing differential equations it serves also as a reference to check the approximate solutions. However, these solutions have limitations in cases of boundary layer flows as it exists for the limit case of velocities at the edge of the similarity analysis.

4.2 HISTORICAL REVIEW

In 1969, Moran et al. (1968) have explained similarity analysis in very systematic way to convert given system of partial differential equations in to a system of ordinary differential equations.

Abd-el-Malek et al. (2002), Parmar and Timol (2011), Adnan et al. (2011), Darji and Timol (2013) and Jain and Timol (2015) have applied deductive group transformation method for various types of fluid flow problems.

Very rare information is available in literature about the Prandtl fluid model. Recently, many researchers discussed about the peristaltic transport of Prandtl fluid.Akbar et al. (2012) examined the Prandtl fluid model in an asymmetric channel. Sucharitha et al. (2012) and Navaneeswara et al. (2012) discussed the conducting Prandtl fluid in a porous channel. Jothia et al. (2012) further explained the peristaltic transport of same model under the effects of magnetic field. The flow equations for the Prandtl fluid model are developed in both a fixed and moving frame of references by Nadeem et al. (2014).N Akbar et al. (2014) Have been discussed the blood flow analysis of Prandtl fluid model in tapered stenosed arteries. The governing equations for considered model are presented in cylindrical coordinates. Perturbation solutions are constructed for the velocity, impedance resistance, wall shear stress and shearing stress at the stenosis throat. The effects of magnetohydrodynamic (MHD) on peristaltic transport of Prandtl fluid in a symmetric channel have been studied under the assumptions of long wave length and low-Reynolds number by A. Alsaedia et al. (2013). S. Jothia et al.(2012) have been studied the MHD peristaltic flow of a Prandtl fluid in a uniform channel under the assumptions of long wavelength and low Reynolds number. Series solutions of axial velocity and pressure gradient are given by using regular perturbation technique when Prandtl number is small. P. Gangavathi et al. (2014) have been discussed the effect of magnetic field on the peristaltic flow of a Prandtl fluid in an asymmetric channel under the assumptions of long wavelength and low Reynolds number.

Mathematically, the empirical relation for the Prandtl fluids Model is written M. Patel et al (2013).

$$\tau = A\sin^{-1}\left(\frac{e}{C}\right)$$

Where, *A* and *C* are the material constants of the fluid. A possible generalization is

$$\tau_{ij} = \frac{A\sin^{-1}\left[\left(\sum_{l=1}^{3}\sum_{m=1}^{3}\frac{e_{lm}\,e_{ml}}{2C^2}\right)^{\frac{1}{2}}\right]}{\left(\sum_{l=1}^{3}\sum_{m=1}^{3}\frac{e_{lm}\,e_{ml}}{2}\right)^{\frac{1}{2}}}\,e_{ij}$$

Where, τ and e are the stress tensor and the rate of deformation tensor respectively; *A* and *C* are the Prandtl fluid parameter; *m* and *l* are fluid index.

4.3 PRESENT INVESTIGATION

This chapter develops to find the numerical solution of two-dimensional boundary layer equations for Prandtl fluid model which is non-Newtonian fluid model and characterized by the property that their components of deviatory stress is related to the rate of strain tensor component by an arbitrary continuous function. Flow past on flat plate as well as moving plate is discussed.

The similarity analysis is made of two-dimensional boundary layer equations for the flows of purely viscous, steady and non-Newtonian fluids along with boundary conditions flat plate as well as moving plat for similarity solutions. This flow equation is highly non-linear PDE so it is very difficult to find its exact solution. Hence we try to find its numerical solution in this chapter. The governing non-linear partial differential equations of the laminar boundary layer flow of non-Newtonian fluid are transformed into non-linear ordinary differential equations using one parameter group transformation method. Again given ODE is also highly non-linear so we converted into linear ODE by quasi-linearization method and find its numerical solution by Finite difference method & graphical presentation using MATLAB solver.

4.4 GOVERNING EQUATIONS:(FOR THE FLOW PAST FLAT PLATE)

The two dimensional laminar boundary layer flows past a semi-infinite flat surface is considered. The geometry of present flow problem is shown in **Figure 4.1.** The governing equations of continuity and momentum of laminar boundary layer flow of Prandtl fluid past a semi-infinite flat surface are:

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4.1}$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial}{\partial y}\tau_{yx}, \tag{4.2}$$

With boundary conditions are:

$$y=0;\ u=0,v=0 \tag{4.3}$$

$$y\to\infty\ ;u=U(x) \tag{4.4}$$

The Prandtl Fluid Model for 2-D flow problems is: (from 2.7.2)

$$\tau_{yx}=\left\{A\sin^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)\right\} \tag{4.5}$$

We can expand τ_{yx} by Taylor's series expansion,

$$\tau_{yx}=\left\{A\sin^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)\right\}=\frac{A}{C}\frac{\partial u}{\partial y}+\frac{A}{6C^3}\left(\frac{\partial u}{\partial y}\right)^3+\ldots$$

Here, $\frac{\partial u}{\partial y}$ is small so we take first two terms and higher order terms are neglected.

$$\tau_{yx}=\left\{A\sin^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)\right\}=\frac{A}{C}\frac{\partial u}{\partial y}+\frac{A}{6C^3}\left(\frac{\partial u}{\partial y}\right)^3 \tag{4.6}$$

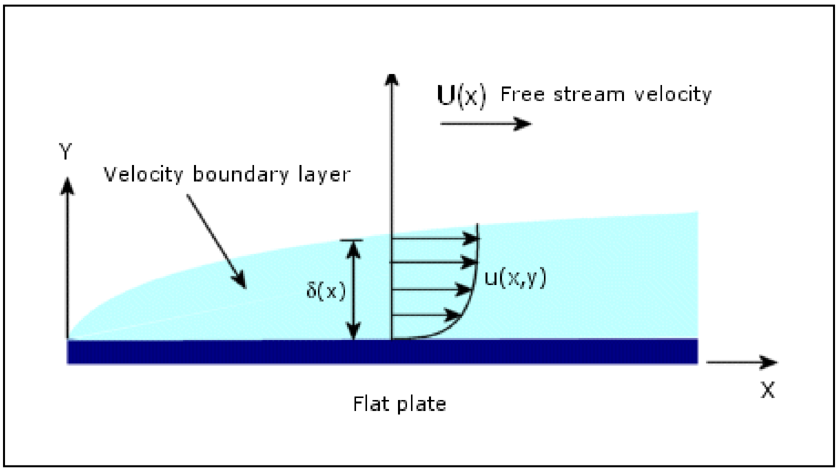


Figure 4.1 Flow Geometry

Now, introducing the stream function ψ (x, y)

$$u=\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{\partial\psi}{\partial x} \tag{4.7}$$

Stream function in equation (4.7) satisfies the continuity equation (4.1). Substitute the value of equation (4.7) and equation (4.6) in the momentum equation (4.2) and in the boundary conditions.

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\frac{A}{C}\frac{\partial^3\psi}{\partial y^3}+\frac{A}{2C^3}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^3\psi}{\partial y^3} \tag{4.8}$$

Boundary conditions are:

$$\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial x}=0 \quad ;y=0 \tag{4.9}$$

$$\frac{\partial\psi}{\partial y}=U(x) \qquad ;y\to\infty \tag{4.10}$$

4.5 SIMILARITY SOLUTION : USING ONE PARAMETER DEDUCTIVEGROUP THEORY METHOD

To obtain the similarity deductive solutions of the equations (4.8) under the boundary condition (4.9) to (4.10), we apply one-parameter group method. After applied this method, two independent variables *x, y* transforms into one independent variable η.

The group *G* of a class of one-parameter is taken of the form,

$$G:\bar{S}=h^s(a)s+k^s(a) \tag{4.11}$$

Where, *S* stands for $x,y,\psi,U,h^s \; and \; k^s$ are real valued and differentiable in their real argument *a*.

$$G:\begin{cases}\bar{x}=h^x(a)x+k^x(a)\\ \bar{y}=h^y(a)y+k^y(a)\\ \bar{\psi}=h^\psi(a)\psi+k^\psi(a)\\ \bar{U}=h^U(a)U+k^U(a)\end{cases} \tag{4.12}$$

$$\bar{S}_i=\left[\frac{h^s}{h^i}\right]S_i \;\; ; \;\; \bar{S}_{ij}=\left[\frac{h^s}{h^ih^j}\right]S_{ij} \;\; ;i=y,x;\; j=y,x \tag{4.13}$$

Equation (4.8) is said to be invariantly transformed for some function *M* (*a*), whenever

$$\begin{gathered}\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\,\partial\bar{y}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}-\bar{U}\frac{d\bar{U}}{d\bar{x}}-\frac{A}{C}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}-\frac{A}{2C^3}\left(\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right)^2\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}= \\ M(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-U\frac{dU}{dx}-\frac{A}{C}\frac{\partial^3\psi}{\partial y^3}-\frac{A}{2C^3}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^3\psi}{\partial y^3}\right]\end{gathered} \tag{4.14}$$

Substituting (4.12) and (4.13) in (4.14).We have,

$$\begin{gathered}\frac{h^{2\psi}}{h^{2y}h^{x}}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{h^{2\psi}}{h^{2y}h^{x}}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{h^{U}}{h^{x}}(h^{U}U+k^{U})\frac{dU}{dx}-\frac{A}{C}\frac{h^{\psi}}{h^{3y}}\frac{\partial^3\psi}{\partial y^3} \\ -\frac{A}{2C^3}\frac{h^{3\psi}}{h^{7y}}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^3\psi}{\partial y^3}= \\ M(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-U\frac{dU}{dx}-\frac{A}{C}\frac{\partial^3\psi}{\partial y^3}-\frac{A}{2C^3}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^3\psi}{\partial y^3}\right]\end{gathered} \tag{4.15}$$

From equation (4.15).We have the relation,

$$\frac{h^{2\psi}}{h^{2y}h^{x}}=\frac{h^{2\psi}}{h^{2y}h^{x}}=\frac{h^{2U}}{h^{x}}=\frac{h^{\psi}}{h^{3y}}=\frac{h^{3\psi}}{h^{7y}}=M(a)\quad,\quad\frac{k^{U}h^{U}}{h^{x}}=0 \tag{4.16}$$

In a similar manner, the boundary conditions in equation (4.9) and (4.10) also being invariant so from the equation (4.9) and (4.10) we have relations,

$$K^{y}=K^{U}=0,\ h^{U}=\frac{h^{\psi}}{h^{y}} \tag{4.17}$$

Solving equation (4.16) and (4.17),

$$h^{\psi}=h^{2y},\ h^{x}=h^{3y},\ h^{U}=h^{y} \tag{4.18}$$

Which is convert the group *G* of the form

$$G:\begin{cases}\bar{x}=h^{3y}x+k^{x}\\ \bar{y}=h^{y}y\\ \bar{\psi}=h^{2y}\psi+k^{\psi}\\ \bar{U}=h^{y}U\end{cases} \tag{4.19}$$

The above group *G* transforms the equations (4.8)-(4.10) invariantly.

If η = η(*x*,*y*) is the absolute invariant of the independent variables, then the absolute invariant of the dependent variables is

$$g_{j}(x,y;\psi,U)=\phi_{j}[\eta(x,y)];\qquad j=1,2. \tag{4.20}$$

The basic theorem in group theory states that: a function g(x,y,ψ,U) is an absolute invariant of a one parameter if it satisfies the following first-order linear differential equation.

$$\sum_{i=1}^{4}(\alpha_i S_i + \beta_i)\frac{\partial g}{\partial S_i} = 0; \qquad S_i = x, y, \psi, U \tag{4.21}$$

Where,

$$\alpha_i = \frac{\partial h^{s_i}}{\partial a}(a^0),\ \beta_i = \frac{\partial k^{s_i}}{\partial a}(a^0); \qquad i = 1, 2, 3, 4 \tag{4.22}$$

And a^0 is the identity element of the group *G*.

Using equations (4.16) and (4.17) and equation (4.22), we have the following values of α's and $\beta's$,

$$\frac{\alpha_1}{3} = \alpha_2 = \frac{\alpha_3}{2} = \alpha_4;\ \beta_2 = \beta_4 = 0 \tag{4.23}$$

From the equation (4.21),

$$(\alpha_1 x + \beta_1)\frac{\partial g}{\partial x} + (\alpha_2 y + \beta_2)\frac{\partial g}{\partial y} + (\alpha_3 \psi + \beta_3)\frac{\partial g}{\partial \psi} + (\alpha_4 U + \beta_4)\frac{\partial g}{\partial U} = 0 \tag{4.24}$$

The absolute invariants of the independent variable η and dependent variables *U* and ψ are determined using equation. (4.24) and (4.23) as,

$$\eta = \frac{y}{(\alpha_1 x + \beta_1)^{1/3}}$$

$$\psi = (\alpha_1 x + \beta_1)^{2/3} f(\eta) - \frac{\beta_3}{2\alpha_1}$$

$$U = (\alpha_1 x + \beta_1)^{1/3} \tag{4.25}$$

Using the absolute invariants given in equation (4.25) in equations (4.8)-(4.10) we obtained the non linear ordinary differential equations along with boundary conditions as below:

$$f'^2 - 2 f f'' - \frac{3A}{C} f''' - \frac{3A}{2C^3} f''^2 f''' - 1 = 0 \tag{4.26}$$

With Boundary Conditions:

$$f'(0) = f(0) = 0 \tag{4.27}$$

$$f'(\infty) = 1 \tag{4.28}$$

Now for the solution of equation (4.26) under the boundary conditions (4.27)-(4.28) first we convert quasi linearization method into linear ordinary differential equation, as

$$
\begin{aligned}
& f_n'^2(\eta) - 2 f_n(\eta) f_n''(\eta) - \frac{3A}{C} f_n'''(\eta) - \frac{3A}{2C^3}\left(f_n''(\eta)\right)^2 f_n'''(\eta) - 1 \\
& \qquad - 2 f_n''(\eta)[f_{n+1}(\eta) - f_n(\eta)] + 2 f_n'(\eta)[f_{n+1}'(\eta) - f_n'(\eta)] \\
& -2 f_n(\eta)[f_{n+1}''(\eta) - f_n''(\eta)] - \frac{3A}{2C^3} f_n'''(\eta) 2 f_n''(\eta)[f_{n+1}''(\eta) - f_n''(\eta)] \\
& -\frac{3A}{C}[f_{n+1}'''(\eta) - f_n'''(\eta)] - \frac{3A}{2C^3}\left(f_n''(\eta)\right)^2 [f_{n+1}'''(\eta) - f_n'''(\eta)] = 0
\end{aligned}
$$

Simplifying above equation we have,

$$
\begin{aligned}
& -f_n'^2 - 1 - 2 f_n''(\eta) f_{n+1}(\eta) + 2 f_n'(\eta) f_{n+1}'(\eta) - 2 f_n(\eta) f_{n+1}''(\eta) \\
& +2 f_n(\eta) f_n''(\eta) - \frac{3A}{C^3} f_n'''(\eta) f_n''(\eta) f_{n+1}''(\eta) - \frac{3A}{C} f_{n+1}'''(\eta) = 0
\end{aligned}
\tag{4.29}
$$

To fit the curve, consider the solution

$$f_n = A_0\eta^2 + A_1\eta + A_2 \tag{4.30}$$

Under the boundary conditions in (4.27) and (4.28) we get the values of constants,

$$A_0 = 0.5,\ A_1 = 0,\ A_2 = 0$$

So equation (4.30) can be written as,

$$f_n = 0.5\eta^2 \tag{4.31a}$$

$$\therefore\ \ f_n' = \eta \tag{4.31b}$$

$$\therefore\ \ f_n'' = 1 \tag{4.31c}$$

$$\therefore\ \ f_n''' = 0$$

(4.31*d*)

Substitute the values of equations (4.31a)–(4.31c) in equation (4.29) we get,

$$\frac{3A}{C} f_{n+1}''' + \eta^2\ f_{n+1}'' - 2\eta \mathrm{f}_{n+1}' + 2 f_{n+1} = -1 \tag{4.32}$$

4.6 NUMERICAL SOLUTION : FINITE DIFFERENCE METHOD

Here the equation (4.32) is linear ordinary differential equation now we will find its numerical solution using Finite difference method, Substitute the values of equations (3.51*a*)–(3.51*c*) in equation (4.32). So at the ith node we get, the system of linear equations,

$$\frac{3A}{C}f_{i+2}+[-\frac{6A}{C}+2h\eta^{2}-2h^{2}\eta]f_{i+1}+[\frac{6A}{C}+2h\eta^{2}+2h^{2}\eta]f_{i-1} \\ +[-4h\eta^{2}+4h^{3}]f_{i}-\frac{3A}{C}f_{i-2}=-2h^{3} \qquad (4.33)$$

This system of linear equations we solve it using MATLAB solver by dividing the interval [0,1] into 1000 subintervals having equal length of each subintervals *h*=0.001. Given numerical solution is presented graphically as below.

4.7 GRAPHICAL PRESENTATION

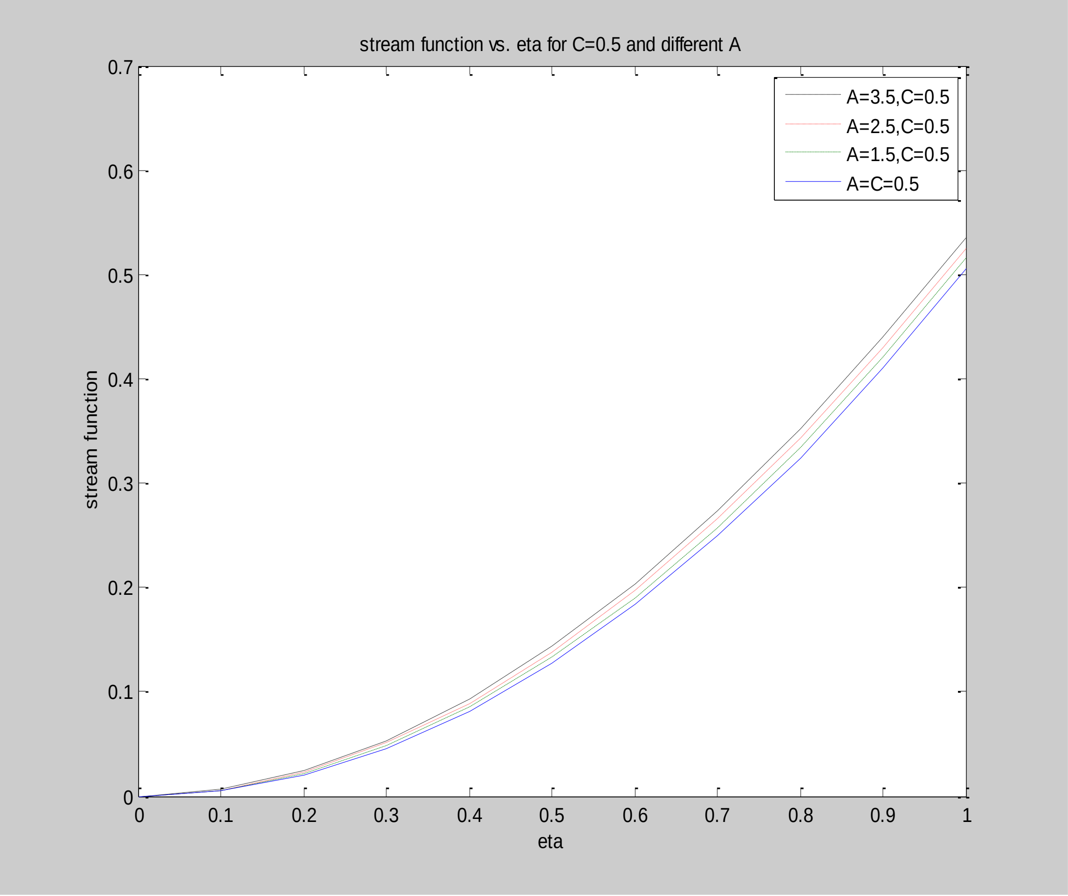


Figure 4.2 Stream function vs. Eta for different values of A

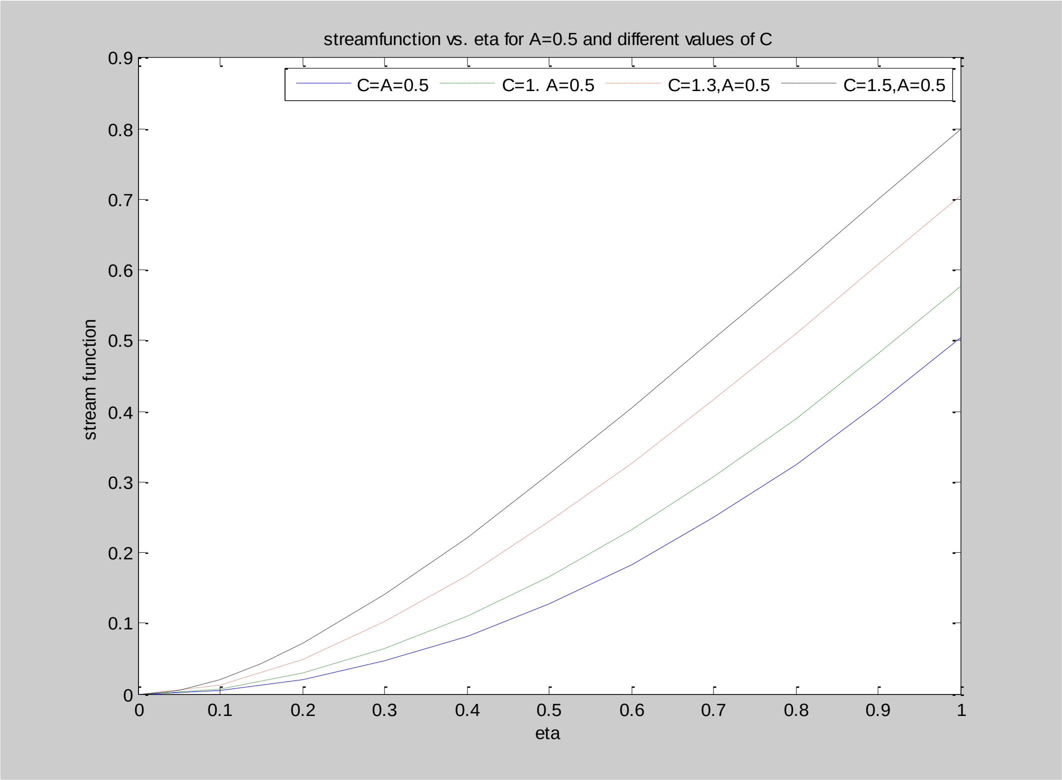


Figure 4.3 Stream function vs. Eta for different values of C

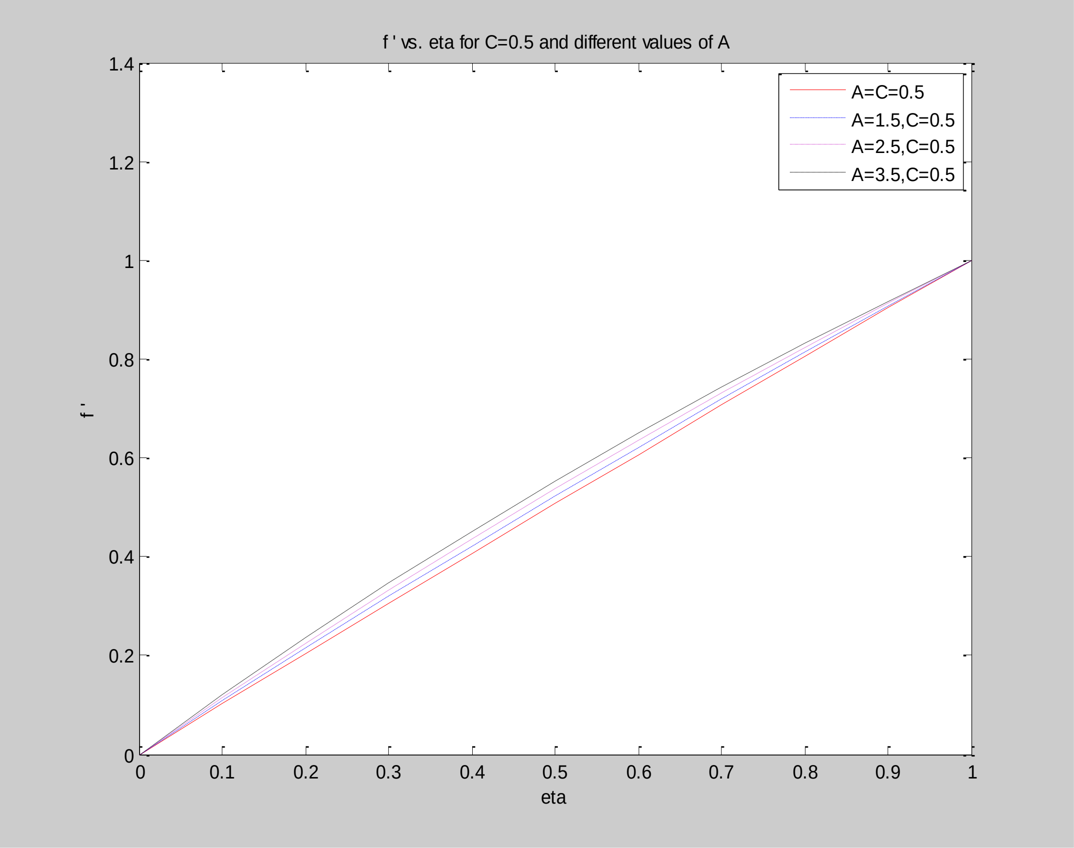


Figure 4.4 Velocity profile vs. Eta for different values of A

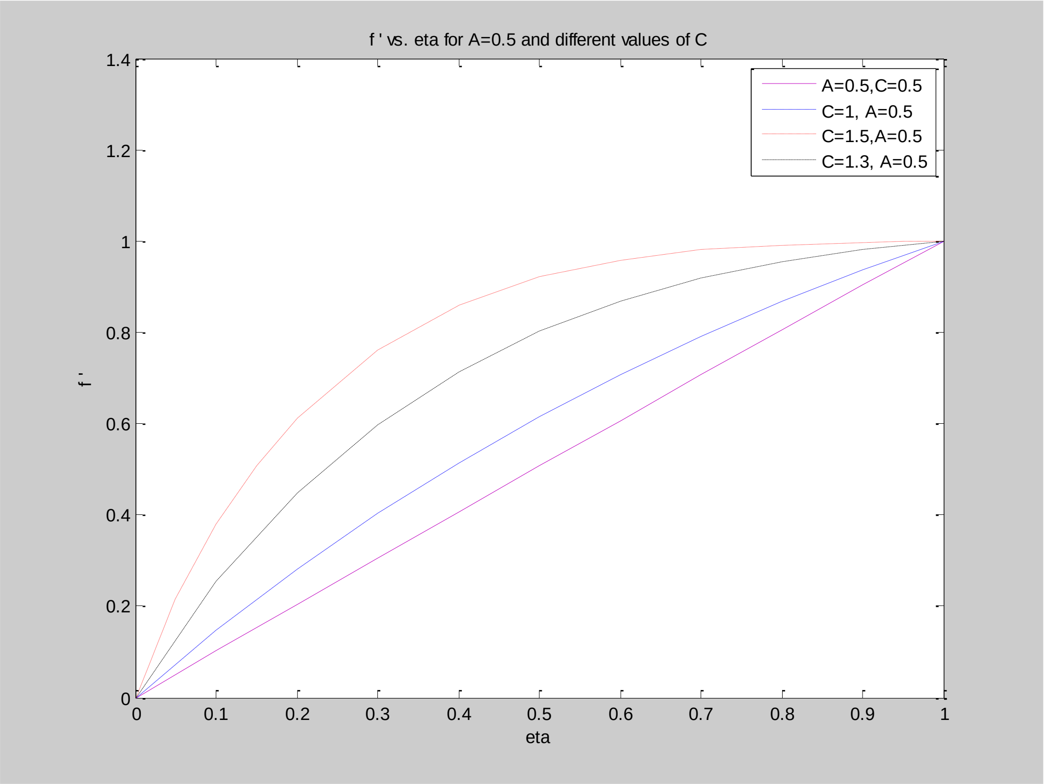


Figure 4.5 Velocity profile vs. Eta for different values of C

4.8 RESULT AND DISCUSSION

Using one parameter deductive group method, the PDE of flow problems are transformed into non linear ODE. Then non linear ODE is converted into linear ODE by quasi-linearization method and then solved it using numerically finite difference method and presented graphically by MATLAB solver. **Figure 4.2** and **Figure 4.3** gives the flow pattern for various values of the Prandtl fluid parameter A and C respectively. **Figure 4.4** and **Figure 4.5** present the velocity profiles. **Figure 4.4** represents the velocity profiles vs. eta for different values of A and fix value of C=0.5. **Figure 4.5** represents the velocity profiles vs. eta for different values of C and fix value of A=0.5. For variable values of C, the difference in increase in velocity is seems more clearly. Therefore from **Figure 4.4** and **Figure 4.5**, it conclude that the velocity profile increase more rapidly for different values of C.

4.9 GOVERNING EQUATIONS:(FOR THE FLOW PAST MOVING PLATE)

The two dimensional laminar boundary layer flows past a semi-infinite moving surface is considered. The governing equations of continuity and momentum of laminar boundary layer flow of Prandtl fluid past a semi-infinite moving surface are:

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial}{\partial y}\tau_{yx},$$

With boundary conditions:

$$y=0: u=1, v=1 \tag{4.34}$$

$$y\to\infty\ ; u=0 \tag{4.35}$$

For the Pradtl Fluid model stress tensor τ is taken from equation (4.5) and using one parameter group theory method find it similarity equation we get momentum equation is same as equation (4.26) and the boundary conditions as in equation (4.34) and (4.35) are reduced as,

$$f'(0)=f(0)=1 \tag{4.36}$$

$$f'(\infty)\to 0 \tag{4.37}$$

Applying quasi linearization method to convert non linear differential equation (4.26) into linear ordinary differential equation under the boundary condition equations (4.36)-(4.37) as,

To fit the curve, consider the solution

$$f_n=A_0\eta^2+A_1\eta+A_2 \tag{4.38}$$

Using boundary conditions in equations (4.36)-(4.37) we have constants,

A_0 = - 0.499995, A_1=1, A_2 =1

So, equation (4.38) can be written as,

$$f_n=-0.499995\eta^2+\eta+1 \tag{4.39a}$$

$$\therefore\quad f_n^{'}=-0.99999\eta+1 \tag{4.39b}$$

$$\therefore\quad f_n^{''}=-0.99999 \tag{4.39c}$$

$$\therefore \quad f_n^{'''} = 0 \tag{4.39d}$$

Substitute the values of equations (4.39*a*) - (4.39*d*) in the equation (4.29) we get the linear differential equation as,

$$\begin{aligned} -\frac{3A}{C} f_{n+1}^{'''} + [0.99999\eta^2 - 2\eta - 2]\, f_{n+1}^{''} + [2 - 1.99998\eta]\, f_{n+1}^{'} + 1.499998\, f_{n+1} \\ = 3.99998 - 0.99999\,\eta - 0.99998\eta^2 \end{aligned} \tag{4.40}$$

Here the equation (4.40) is linear ordinary differential equation now we will find its numerical solution using finite difference method. Substitute the values of equations (3.51*a*)–(3.52*c*) in equation (4.40). So at the ith node we get the system of linear equations,

$$\begin{aligned} -\frac{3A}{C} f_{i+2} + [\frac{6A}{C} + 1.99998h\,\eta^2 - 4h\eta - 4h + 2h^2 - 1.99998h^2\eta]\, f_{i+1} \\ + [-\frac{6A}{C} + 1.99998h\,\eta^2 - 4h\eta - 4h - 2h^2 + 1.99998h^2\eta]\, f_{i-1} \\ + [-3.99996h\,\eta^2 + 8h\,\eta + 8h + 3.99996h^3]\, f_i + \frac{3A}{C} f_{i-2} \\ = [3.99998 - 0.99999\eta - 0.99998\eta^2\,]2h^3 \end{aligned} \tag{4.41}$$

This system of linear equations we solve it using MATLAB solver by dividing the interval [0,1] into 1000 subintervals having equal length of each subintervals h=0.001. Given numerical solution is presented graphically as below.

4.10 GRAPHICAL PRESENTATION

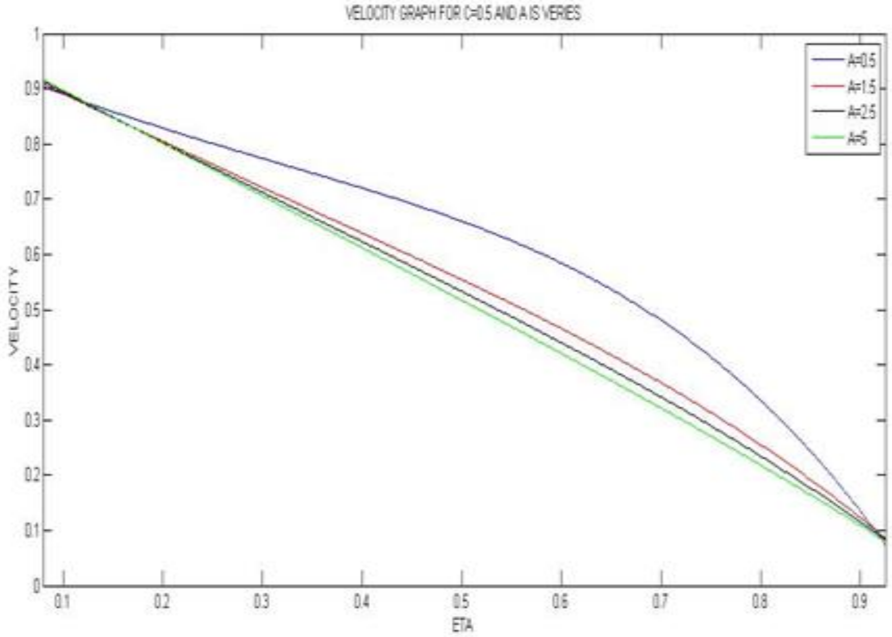


Figure-4.6 Velocity profile vs. Eta for different values of A

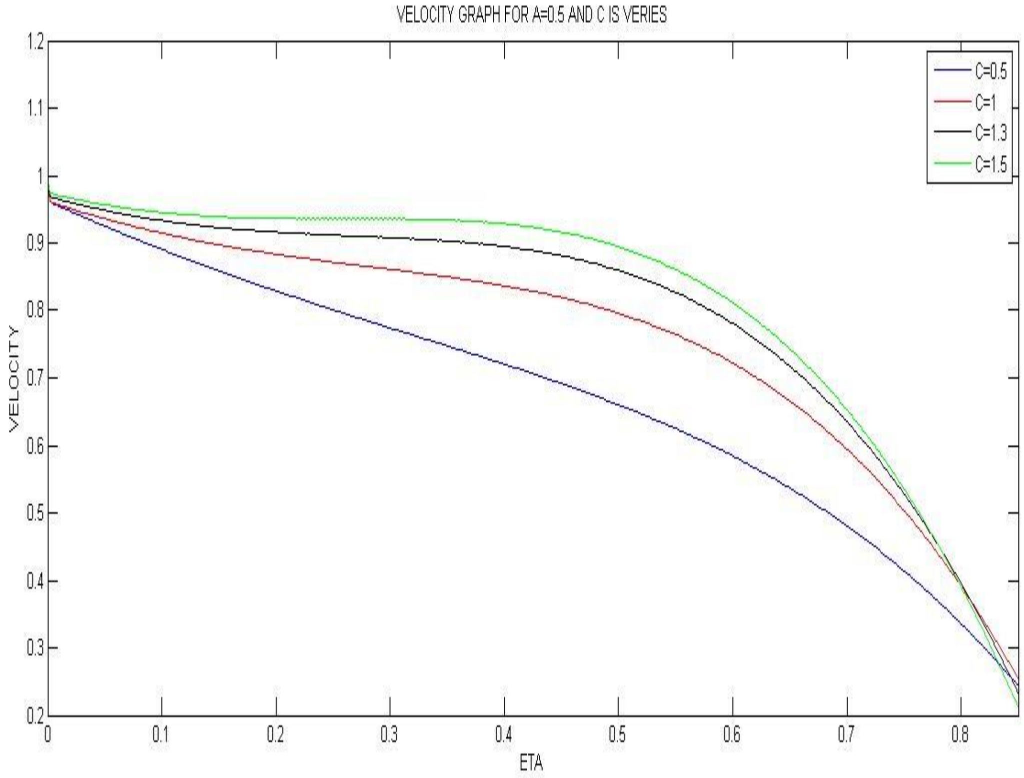


Figure 4.7 velocity profile vs. Eta for different values of C

4.11 RESULT AND DISCUSSION

Using one parameter group method, the PDE of flow problems are transformed into ODE. The ODE are then solved numerically and presented graphically by MATLAB solver in the present work. **Figure 4.6** and **Figure 4.7** present the velocity profiles. **Figure 4.6** represents the velocity profiles vs. eta for different values of A and fix value of C=0.5. **Figure 4.7** represents the velocity profiles vs. eta for different values of C and fix value of A=0.5. For variable values of C, the difference in increase in velocity is seems more clearly. Therefore from **Figure 4.6** and **Figure 4.7**, it conclude that the velocity profile increase more rapidly for different values of C.

4.12 CONCLUSION

The methods for obtaining similarity transformations are devided into two categories (i) Direct methods and (ii) group thoretic methods. The direct methods such as, seperation of variables do not invoke group invarience. On the other hand, group theoretic methods are more elegant mathematically. The main concept of invariance under a group of transformation is invoked always . In the present paper, the governing non-linear partial differential equations of the laminar boundary layer flow of non-newtonian fluid are tranformed into non-linear ordinary differential equations using one parameter deductive method. The Prandtl fluid model of non-Newtonian fluids is considered for the stress-strain relationship.

CHAPTER-05

NUMERICL SOLUTION OF MHD BOUNDARY LAYER EQUATIONS OF SISKO FLUID FLOW

5.1 INTRODUCTION

In practical applications and academic viewpoints it is most considerable interest to find the solutions of governing partial differential equations of non-Newtonian fluids flow. In non-Newtonian fluids the stress tensor and the rate of deformation tensor are always related by some non-linear relationship. Some examples of non-Newtonian fluids are shampoo, toothpaste, paint, clay coating and suspensions, grease, cosmetic products, custard, blood and slurries etc… which obey this property of fluids. Due to great commercial importance of such all the fluids it has been most considerable interest in non-Newtonian fluids.

It is too difficult to exhibits all properties of non-Newtonian fluids by a single model and also it cannot be described as simple like Newtonian fluids. Moreover, there has been much confusion in the further classification of non-Newtonian fluids. The non-Newtonian fluids are usually classified in three categories such as: (i) Fluids for which shear stress depends only on the rate of shear (ii) Fluids for which relation between rate of strain and shear stress depends on time (iii) Fluids for which the visco-inelastic fluids which possess both viscous and elastic properties. Therefore the rate of shear and the shearing stress are always very important for the mathematical structure of non-Newtonian fluids. But deriving the mathematical formulation of such fluid model is really very difficult task.

5.2 HISTORICAL REVIEW

In past few decades, several researchers have analysed the problems of non-Newtonian boundary layer flow of fluids past a different geometries. Bird et al. (1960), Hansen et al. (1968), Kapur et al. (1982), Lee et al.(1966), Manisha et al. (2005,2008,2009,2010), Timol et al. (1986,2004), Wells (1964)). Many fluids in the real world are non-Newtonian by nature. The study of such fluids is very much important due to their vast applications in the field of engineering sciences and industries. Several fluid models of non-Newtonian fluid have been investigated. A brief instruction and classification on various fluid models of non-Newtonian fluid is given in detailed by Manisha et al.(2010,2013). Plenty of work have been carried out by many scientist for Power-Law fluid model and Powell-Eyring fluid model such as Djukic et al. (1973,1974),Na et al.(1967), Patel et al.(2011,2012,2014,2009), Sirohi et al.(1984).

Very less information is available in literature about the fluid model proposed by Sisko (1958). Na and Hansen (1967) have examined the flow of Sisko fluid between two circular parallel disks. Bahraini et al. (1996) analysed the isothermal and axial laminar flow of Power-law fluid and Sisko fluid in an annuli. The thin film flow problem of Sisko fluid and Oldroyd fluid on a moving vertical belt was discussed nicely by Nemati et al. (2009). Homotopy analysis method applied to solve the flow problem in their work. The flow problem of Sisko fluid passing through an axisymmetric uniform tube was solved analytically using the perturbation method and HAM by Nadeem et al.(2010). Moallemi et al.(2011) have discussed the Homotopy perturbation method to solve the flow of a Sisko fluid in pipe.

Recently, The numerical solution for the unsteady free convective flow of Sisko fluid past flat plate moving through a binary mixture has been obtained by Olanrewaju et al. (2013). Also, Siddiqui et al. (2013) has examined the drainage of Sisko fluid film down a vertical belt. Asghar et al. (2014) have presented the equations for the peristaltic flow of MHD Sisko fluid in a channel. They have considered the effect of strong and weak magnetic fields both.

Mathematically, Stress tensor for Sisko Model can be written as,

$$\bar{\tau} = -\left\{ a + b\left|\sqrt{\frac{1}{2}\left(\bar{\Delta}:\bar{\Delta}\right)}\right|^{(n-1)} \right\}\bar{\Delta} \tag{5.1}$$

Where, $\bar{\tau}$ = stress tensor and $\bar{\Delta}$ = the rate of deformation tensor

a, b and n are constants and it is defined differently for different fluids.

Here we considered the problem of steady two dimensional MHD boundary layer flows of non-Newtonian fluids. Sisko fluid model is considered for stress-strain relationship.

Similarity and numerical solutions are obtained for the flow problem. All the cases are presented with graph.

5.3 PRESENT INVESTIGATION

This chapter develops to find the numerical solution of two-dimensional boundary layer equations for Sisko fluid model which is non-Newtonian fluid and characterized by the property that their components of deviatory stress is related to the rate of strain tensor component by an arbitrary continues function.

The similarity analysis is made of two-dimensional boundary layer equations for defined flow is purely viscous, steady and non-Newtonian fluids and under the influence of magnetic effect along with the boundary conditions for flat plat as well as in moving plat. Resultant similarity equation of defined flow problems are highly non linear Partial differential equations and so it is very difficult find its exact solution. Hence we try to find its numerical solution in this chapter. The governing non-linear partial differential equations of the laminar boundary layer flow of non-newtonian fluid are tranformed into non-linear ordinary differential equations using one parameter group tranformation method. Again given ordinary differential equation is also highly non linear so we converted into linear ordinary differential equation by quasilinearisation method and find it solution by Finite difference method and graphically presented by MATLAB solver.

5.4 GOVERNING EQUATION: (FOR THE FLOW PAST FLAT PLATE)

In present problem flow considered in two dimensional spaces parallel to *X*-direction and *Y*-axis is normal to it. The laminar MHD boundary layer flow of Sisko fluid past a semi-infinite flat plate. The governing equations (continuity equation and momentum equation) of defined fluid flow problem are:

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{5.2}$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial}{\partial y}\tau_{yx}+\sigma B_0^2(\mathrm{U-u}) \tag{5.3}$$

With boundary conditions

y=0: u=0, v=0 (5.4)

$y \to \infty$: u=U(x) (5.5)

For the Sisko Fluid Model in 2-D flow problems stress tensor τ is given by equation (5.1). So momentum equation (5.3) is converted into

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial}{\partial y}\left\{\left[a+b\left(\frac{\partial u}{\partial y}\right)^{n-1}\right]\frac{\partial u}{\partial y}\right\}+\sigma B_0^2(\mathrm{U-u}) \tag{5.6}$$

5.5 SIMILARITY SOLUTION: USING ONE PARAMETER GROUP THEORY METHOD

To obtain the similarity solutions of the continuity equation (5.2) and momentum equation (5.6) under the boundary conditions (5.4) - (5.5) we take one-parameter scaling group transformation as,

$$\mathrm{G}:\begin{cases} y = A^{\alpha_1}\,\overline{y} \\ x = A^{\alpha_2}\,\overline{x} \\ u = A^{\alpha_3}\,\overline{u} \\ v = A^{\alpha_4}\,\overline{v} \\ U = A^{\alpha_5}\overline{U} \end{cases} \tag{5.7}$$

Introducing equation (5.7) in equations (5.2), (5.6) and using simple chain rule formula we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=A^{\alpha_3-\alpha_2}\frac{\partial\,\overline{u}}{\partial\,\overline{x}}+A^{\alpha_4-\alpha_1}\frac{\partial\,\overline{v}}{\partial\,\overline{y}}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-U\frac{dU}{dx}-a\frac{\partial^2 u}{\partial y^2}-nb\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}=A^{2\alpha_3-\alpha_2}\,\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+A^{\alpha_4+\alpha_3-\alpha_1}\,\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}-$$

$$A^{2\alpha_5-\alpha_2}\,\bar{U}\frac{d\bar{U}}{d\bar{x}}-A^{\alpha_3-2\alpha_1}\,a\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}-nbA^{(n-1)\alpha_3+\alpha_3-(n-1)\alpha_1-2\alpha_1}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^{n-1}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+\sigma B_0^2(A^{\alpha_5}\bar{U}-A^{\alpha_3}\bar{u})$$

Above set of equations remains invariant under taken group transformation G. So we have relation between $\alpha's$ as,

$$\alpha_3-\alpha_2=\alpha_4-\alpha_1$$

$$2\alpha_3-\alpha_2=\alpha_4+\alpha_3-\alpha_1=2\alpha_5-\alpha_2=n\alpha_3-(n+1)\alpha_1=\alpha_5=\alpha_3$$

Solving above set of equations we get relation between α's as,

$$3\alpha_1=\alpha_2=3\alpha_3=-3\alpha_4=3\alpha_5$$

$$\text{put}\quad \alpha=\frac{\alpha_2}{\alpha_1}=\frac{1}{3}$$

$$\alpha=\alpha_3=-\alpha_4=\alpha_5$$

Now we define new independent variable η of two independent variables x and y, is called similarity independent variable and dependent variables $f(\eta)$, $g(\eta)$ and $h(\eta)$ of u, v and U respectively, are called similarity dependent variables Seshadri et al. (1985).

$$\eta=\frac{y}{x^{\alpha}}=\frac{y}{x^{\frac{1}{3}}}=x^{-\frac{1}{3}}y \tag{5.8}$$

$$f'(\eta)=\frac{u}{x^{\frac{1}{3}}} \tag{5.9}$$

$$g(\eta)=\frac{v}{x^{-\frac{1}{3}}} \tag{5.10}$$

$$h(\eta)=\frac{U}{x^{\frac{1}{3}}}=c_1 \tag{5.11}$$

Substituting the values in equations (5.8)-(5.11) in equations (5.2) and (5.6) will transform into the following equations:

$$f'(\eta)-\eta f''(\eta)+3g'(\eta)=0 \tag{5.12}$$

$$\begin{aligned}f'^2(\eta)-\eta f'(\eta)f''(\eta)+3g(\eta)f''(\eta)=c_1^2+3af'''(\eta)+3nb\left(f''(\eta)\right)^{n-1}f'''(\eta)\\+3\sigma B_0^2(c_1-f'(\eta))\end{aligned} \tag{5.13}$$

Now solving equation (5.12) for g,

$$g'(\eta) = \frac{1}{3}[\, f''(\eta)\eta - f'(\eta)\,]$$

$$\therefore\ g(\eta) = \frac{1}{3}\int [\, f''(\eta)\eta - f'(\eta)\,]\, d\eta$$

$$\therefore\ g(\eta) = \frac{1}{3}[\, f'(\eta)\eta - 2f(\eta)]$$

Now, putting the value g in equation (5.13) we obtained,

$$\begin{aligned} f'^{\,2}(\eta) - 2f(\eta) f''(\eta) - c_1^2 - 3af'''(\eta) - 3nb\left(f''(\eta)\right)^{n-1} f'''(\eta) \\ - 3\sigma B_0^2 (c_1 - f'(\eta)) = 0 \end{aligned} \tag{5.14}$$

And the boundary conditions in equation (5.3) and (5.4) are reduced as,

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1 \tag{5.15}$$

$$Put\ \sigma B_0^2 = M_0\ \ and\ C_1 = 1,$$

$$\begin{aligned} f'^{2}(\eta) - 2f(\eta) f''(\eta) - 3af'''(\eta) - 3nb\left(f''(\eta)\right)^{n-1} f'''(\eta) \\ - 3M_0(1 - f'(\eta)) - 1 = 0 \end{aligned} \tag{5.16}$$

We have equation (5.16) is non linear ordinary differential equation. First we convert equation (5.16) into linear ordinary differential equation by quasi linearization method as,

$$\begin{aligned} & f_n^{'2}(\eta) - 2f_n(\eta) f_n^{''}(\eta) - 3af_n^{'''}(\eta) - 3nb\left(f_n^{''}(\eta)\right)^{n-1} f_n^{'''}(\eta) - 3M_0(1 - f_n^{'}(\eta)) \\ & \quad -1 - 2f_n^{''}(\eta)(f_{n+1}(\eta) - f_n(\eta)) + 2f_n^{'}(\eta)(f_{n+1}^{'}(\eta) - f_n^{'}(\eta)) \\ & \quad + 3M_0(f_{n+1}^{'}(\eta) - f_n^{'}(\eta)) - 2f_n(\eta)(f_{n+1}^{''}(\eta) - f_n^{''}(\eta)) - 3af_{n+1}^{'''}(\eta) \\ & \quad + 3af_n^{'''}(\eta) - 3n(n-1)b\left(f_n^{''}(\eta)\right)^{n-2} (f_{n+1}^{''}(\eta) - f_{n+1}^{''}(\eta)) f_n^{'''}(\eta) \\ & \quad - 3nb\left(f_n^{''}(\eta)\right)^{n-1} (f_{n+1}^{'''}(\eta) - f_{n+1}^{'''}(\eta)) = 0 \end{aligned}$$

Simplifying above equation we get,

$$\begin{aligned} & -f_n^{'2} - 3M_0 - 1 - 2f_n''(\eta) f_{n+1}(\eta) + 2f_n'(\eta) f_{n+1}'(\eta) + 3M_0 f_{n+1}'(\eta) \\ & \quad - 2f_n(\eta) f_{n+1}''(\eta) + 2f_n(\eta) f_n''(\eta) - 3af_{n+1}'''(\eta) - 3n(n-1) \\ & \quad b\left(f_n''(\eta)\right)^{n-2} f_{n+1}''(\eta) f_n'''(\eta) + 3n(n-1)b\left(f_n''(\eta)\right)^{n-1} f_n'''(\eta) \\ & \qquad - 3nb\left(f_n''(\eta)\right)^{n-1} f_{n+1}'''(\eta) = 0 \end{aligned} \tag{5.17}$$

With boundary conditions:

$$f_n(0) = 0,\ f_n^{'}(0) = 0,\ f_n^{'}(\infty) = 1 \tag{5.18}$$

To fit the curve, consider the solution,

$$f_n = A_0\eta^2 + A_1\eta + A_2 \tag{5.19}$$

Under the boundary conditions in equation (5.18) we have the constants,

$$A_0 = 0.5,\quad A_1 = 0,\quad A_2 = 0$$

So, equation (5.19) can be written as, $f_n = 0.5\,\eta^2$

$$\therefore\quad f_n = 0.5\eta^2\ ,\quad f'_n = \eta,\quad f''_n = 1,\quad f'''_n = 0 \tag{5.20}$$

Substitute the values of equations (5.20) in equation (5.17) we get,

$$(3\mathrm{nb} + 3a)\ f'''_{n+1} + \eta^2\ f''_{n+1} + (2\eta + 3M_0)\,\mathrm{f}'_{n+1} - 2\,f_{n+1} = 1 + 3M_0 \tag{5.21}$$

5.6 NUMERICAL SOLUTION : FINITE DIFFERENCE METHOD

Here the equation (5.21) is linear ordinary differential equation now we will find its numerical solution using Finite difference method, Substitute the values of equations (3.51*a*) – (3.51*c*) in equation (5.21) at the ith node we get the system of linear equations.

$$\begin{aligned}&(3\mathrm{nb} + 3a)\ f_{i+2} + (-6nb - 6a + 2h\eta^2 - 2h^2\eta - 3h^2M_0)\,f_{i+1}\\&\quad + (-4h\eta^2 + 4h^3)\,f_i + (6nb + 6a + 2h\eta^2 + 2h^2\eta + 3h^2M_0)\,f_{i-1}\\&\quad + (-3\mathrm{nb} - 3a)\ f_{i-2} = (1 + 3M_0)2h^3\end{aligned} \tag{5.22}$$

Case I: Non- MHD Sisko fluid

If we take M_0 =0 in equation (5.22), then it will reduced in the case of non- magneto hydrodynamic Sisko fluid. Then the system of linear equations (5.22) will reduced in the following equation (5.23) with the same boundary conditions given in equation (5.18).

$$\begin{aligned}&(3\mathrm{nb} + 3a)\ f_{i+2} + (-6nb - 6a + 2h\eta^2 - 2h^2\eta)\,\mathrm{f}_{i+1} + (-4h\eta^2 + 4h^3)\,f_i\\&+(6nb + 6a + 2h\eta^2 + 2h^2\eta)\,f_{i-1} - (3\mathrm{nb} + 3a)\ f_{i-2} = 2h^3\end{aligned} \tag{5.23}$$

Case II: MHD Power-Law fluid

If we take *a*=0, *b*=1 in equation (5.22), then it will give us magneto hydrodynamic Power-Law fluid flow. And the system of linear equations in equation (5.22) will reduced in the following equation (5.24) with the same boundary conditions given in equation (5.18).

$$
\begin{aligned}
&3\mathrm{n}\, f_{i+2} + (-6n + 2h\eta^2 - 2h^2\eta - 3h^2 M_0)\, \mathrm{f}_{i+1} + (-4h\eta^2 + 4h^3)\, f_i \\
&+ (6n + 2h\eta^2 + 2h^2\eta + 3h^2 M_0)\, f_{i-1} - 3\mathrm{n}\, f_{i-2} = (1 + 3M_0)2h^3
\end{aligned}
\tag{5.24}
$$

Case III: Non-MHD Power-Law fluid

If we take *a*=0, *b*=1 and *Mo* = 0 in equation (5.22), then it will give us non magneto hydrodynamic Power-Law fluid. And the system of linear equations in equation (5.22) will reduced in the following equation (5.25) with the same boundary conditions given in equation (5.18).

$$
\begin{aligned}
&3\mathrm{n}\, f_{i+2} + (-6n + 2h\eta^2 - 2h^2\eta)\, f_{i+1} + (-4h\eta^2 + 4h^3)\, f_i \\
&+ (6n + 2h\eta^2 + 2h^2\eta)\, f_{i-1} - 3\mathrm{n}\, f_{i-2} = (1 + 3M_0)2h^3
\end{aligned}
\tag{5.25}
$$

Case IV: MHD Newtonian fluid

If we take *a*=1, *b*=0, n=1 in equation (5.22), then it will give us magneto hydrodynamic Newtonian fluid. And the system of linear equations in equation (5.22) will reduced in the following equation (5.26) with the same boundary conditions given in equation (5.18),

$$
\begin{aligned}
&3\, f_{i+2} + (-6 + 2h\eta^2 - 2h^2\eta - 3h^2 M_0)\, f_{i+1} + (-4h\eta^2 + 4h^3)\, f_i \\
&+ (6 + 2h\eta^2 + 2h^2\eta + 3h^2 M_0)\, f_{i-1} - 3\, f_{i-2} = (1 + 3M_0)2h^3
\end{aligned}
\tag{5.26}
$$

Case V: non- MHD Newtonian fluid

If we take *a*=1, *b*=0, n=1 and *Mo* = 0 in equation (5.22), then it will give us non magneto hydrodynamic Newtonian fluid. And the system of linear equations in equation (5.22) will reduced in the following equation (5.27) with the same boundary conditions given in equation (5.18).

$$
\begin{aligned}
&3\, f_{i+2} + (-6 + 2h\eta^2 - 2h^2\eta)\, f_{i+1} + (-4h\eta^2 + 4h^3)\, f_i \\
&\quad + (6 + 2h\eta^2 + 2h^2\eta)\, f_{i-1} - 3\, f_{i-2} = 2h^3
\end{aligned}
\tag{5.27}
$$

Which is system of linear equations we solve it in MATLAB by dividing [0,1] interval into 1000 subinterval having length h=0.001.solution is presented graphically.

5.7 GRAPHICAL PRESENTATION

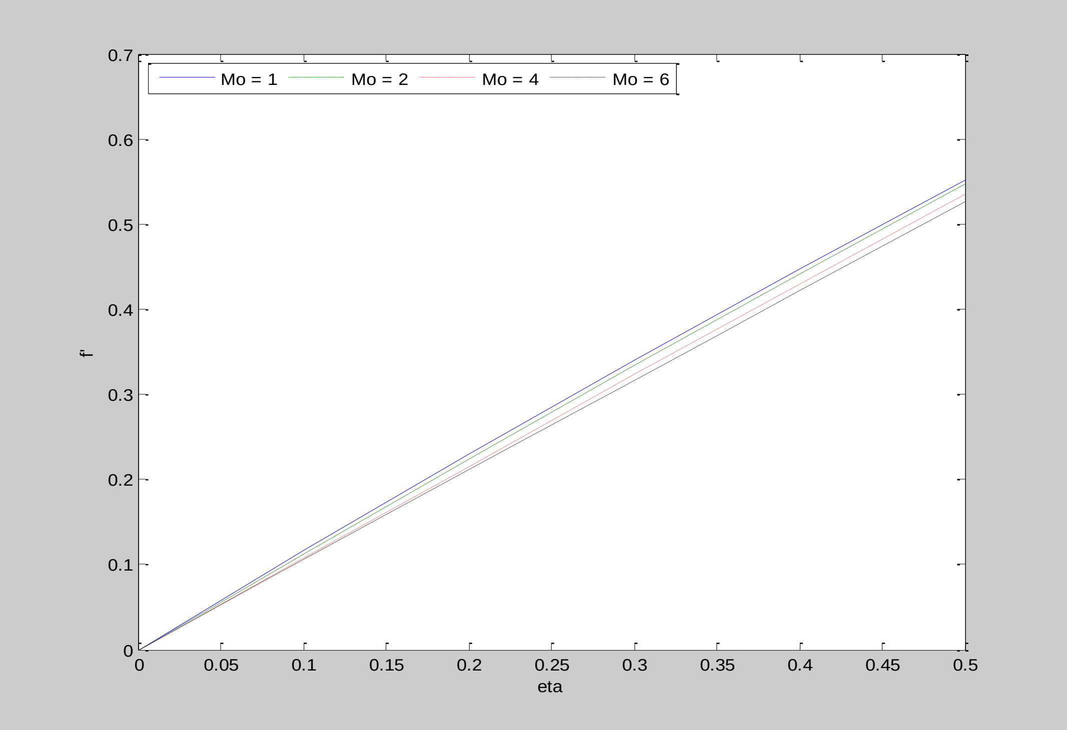


Figure 5.1 MHD Sisko fluid with *a*=*b*=0.5, *n*=0.5 and different values of *Mo*

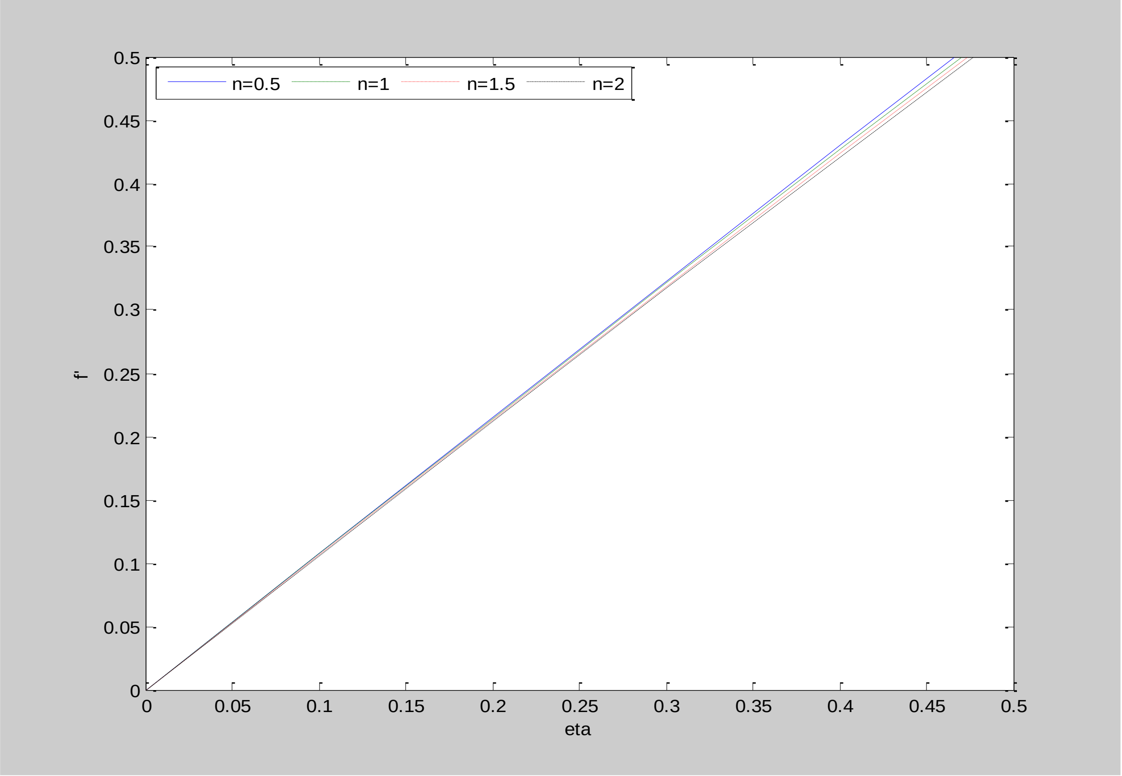


Figure 5.2 MHD Sisko fluid with *a*=*b*=0.5, *Mo* = 4 and different values of *n*

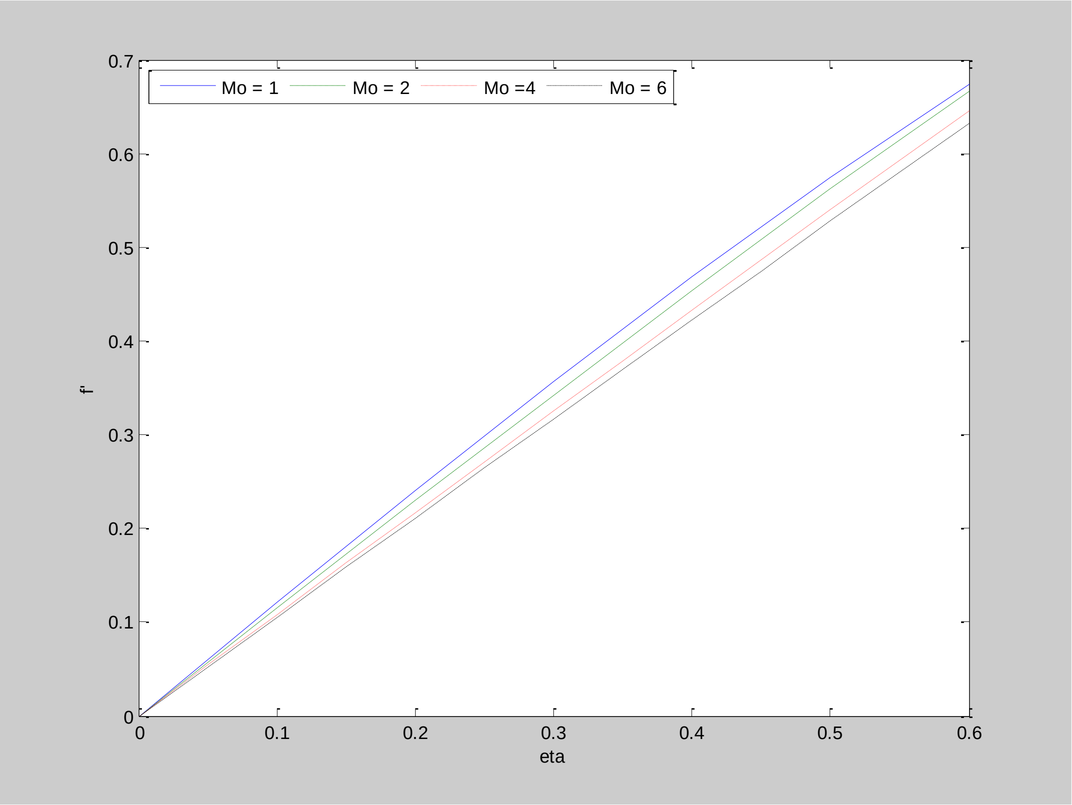


Figure 5.3 MHD Power-Law Fluid with *a*=0, *b*=1, *n*=0.5 and different values of *Mo*

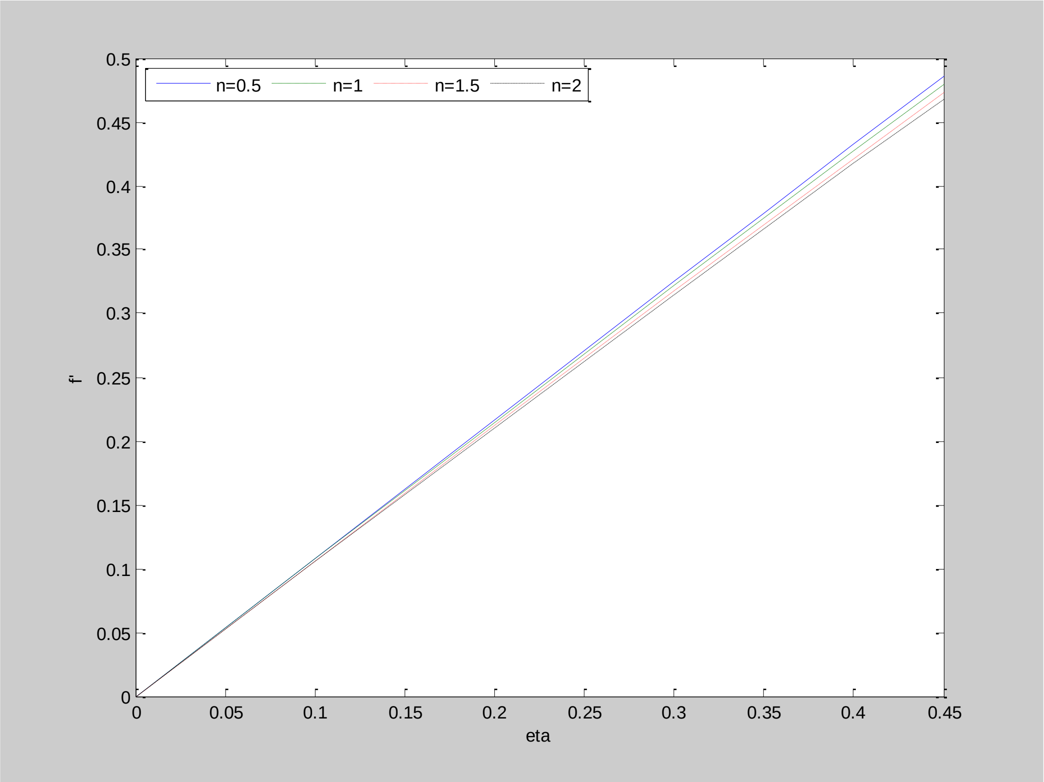


Figure 5.4 MHD Power-Law fluid for *a*=0, *b*=1, *Mo* = 4 and different *n*

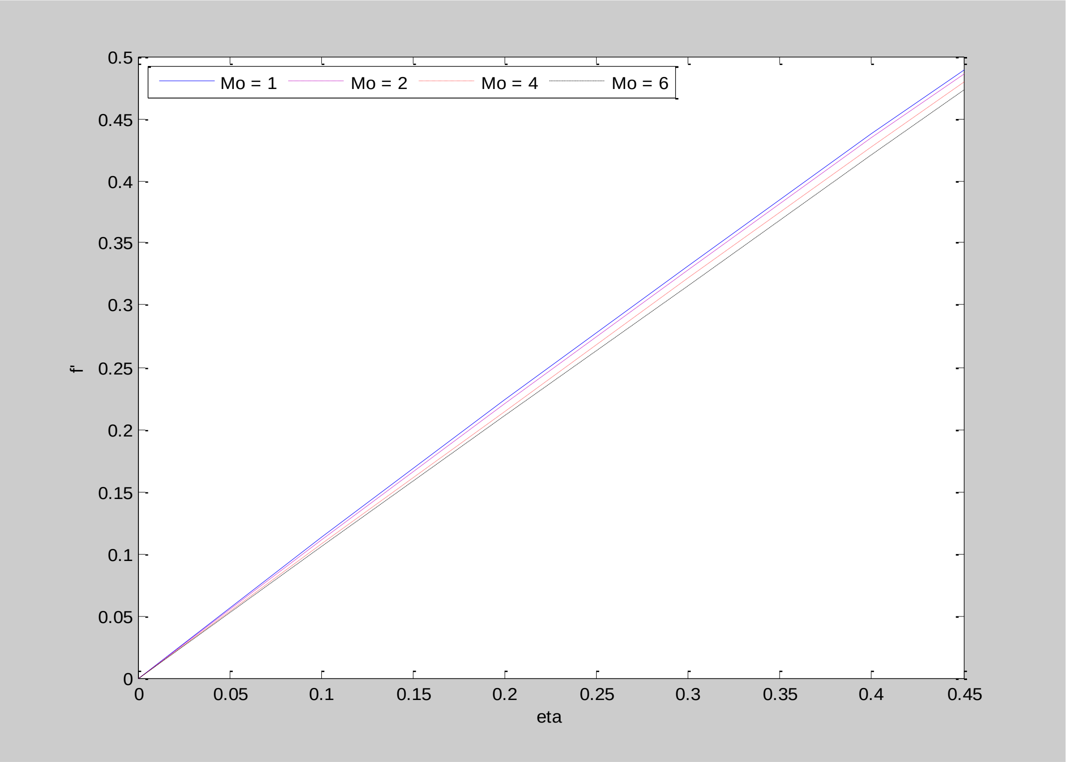


Figure 5.5 MHD Newtonian fluid with *a*=1, *b*=0, *n*=1 and different *Mo*

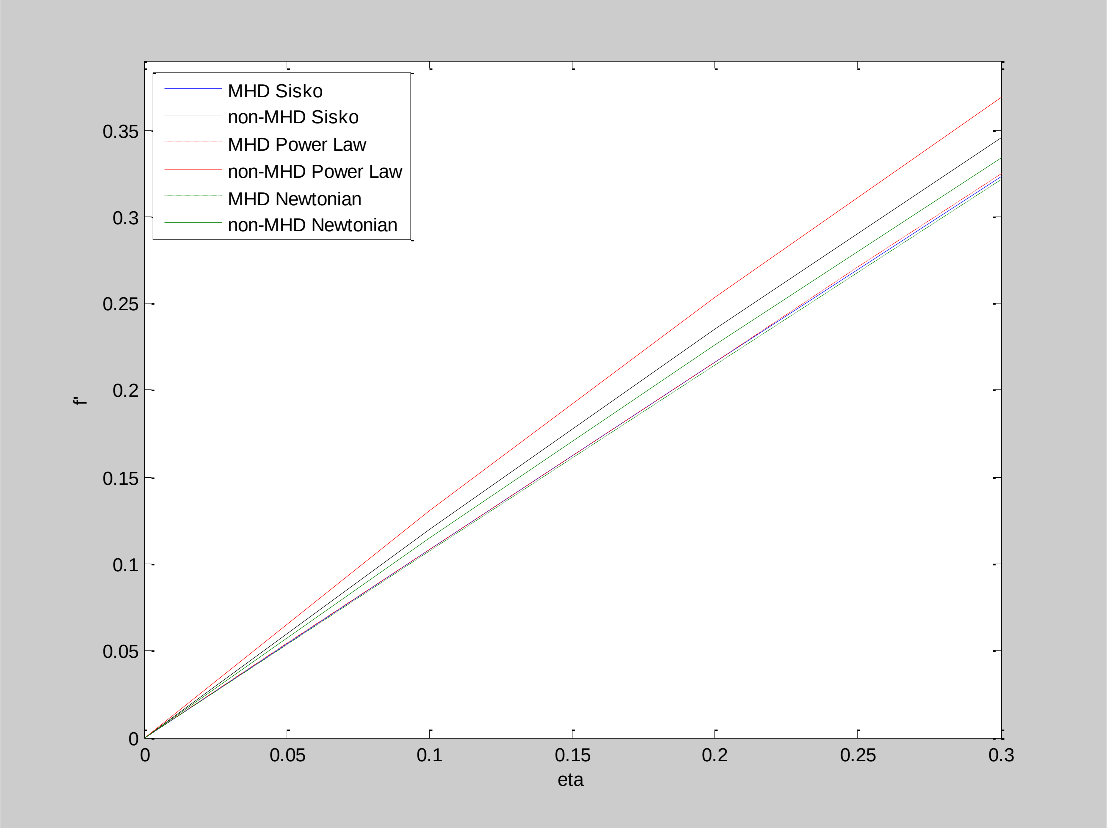


Figure 5.6 All fluid with and without Magneto field effect

5.8 RESULT AND DISCUSSION

The similarity solutions obtained for the defined flow problem are solved numerically and presented graphically. **Figure 5.1 and Figure 5.2** represent the velocity profile f ’ verses eta. of laminar magneto hydrodynamic boundary layer flow of Sisko fluid past a semi-infinite flat plate (equation (5.22)). In **Figure 5.1**, the magnetic constant Mo is varies and the flow behaviour index n taken as constant. While in **Figure 5.2**, n varies and Mo is considered as constant. It shows that the velocity profile increase with increase in the values of eta. and also the velocity profile decrease with increase in the value of the flow behaviour index n and the magneto number Mo. The solution of equation (5.24) of MHD boundary layer flow of Power-Law fluid is represented in **Figure 5.3 and Figure 5.4** graphically. In **Figure 5.3**, the magnetic constant Mo is varies and the flow behaviour index n is taken as constant. While in **Figure 5.4**, n varies and Mo is considered constant. **Figure 5.5** represents the velocity versus eta for magneto hydrodynamic Newtonian fluid (equation (5.26)) the velocity increase as eta increase. **Figure 5.6** is given to show the difference in velocity profiles for all above three types of fluids with and without the effect of magnetic field.

5.9 GOVERNING EQUATION :(FOR THE FLOW PAST MOVING SURFACE)

Consider the 2-D laminar MHD boundary layer flow of Sisko fluid past on moving surface. Let the flow parallel to *X*-direction and *Y*-axis is normal to it. The governing equations of motion for defined fluid flow problem are:

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5.28}$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial}{\partial y}\left\{\left[a+b\left(\frac{\partial u}{\partial y}\right)^{n-1}\right]\frac{\partial u}{\partial y}\right\}+\sigma B_0^2(U-u) \tag{5.29}$$

With boundary conditions

$y = 0$: $u=1$, $v=1$ (5.30)

$y \to \infty$: $u=0$ (5.31)

Solving both the equations (5.28) and (5.29) together using one parameter group theory method we get the similarity equation (discussed in 5.5).

$$\begin{aligned} f'^2(\eta) - 2f(\eta)f''(\eta) - 3af'''(\eta) - 3nb\left(f''(\eta)\right)^{n-1} f'''(\eta) \\ - 3M_0(1 - f'(\eta)) - 1 = 0 \end{aligned} \tag{5.32}$$

With boundary conditions

$$f'(0) = f(0) = 1, \;\; f'(\infty) \to 0 \tag{5.33}$$

Applying quasi linearization method to convert non linear differential equation (5.32) into linear ordinary differential equation under the boundary condition equations (5.33) as,

To fit the curve, consider the solution

$$f_n = A_0\eta^2 + A_1\eta + A_2 \tag{5.34}$$

Using the boundary conditions given in equation (5.33) and solve equation (5.34),

We have constants $A_0 = -0.4995, \;\; A_1 = 1, \;\; A_2 = 1$

So, equation (5.34) can be written as,

$$f_n = -0.4995\eta^2 + \eta + 1 \tag{5.35a}$$

$$\therefore \quad f_n' = -0.999\eta + 1 \tag{5.35b}$$

$$\therefore \quad f_n'' = -0.999 \tag{5.35c}$$

$$\therefore \quad f_n''' = 0 \tag{5.35d}$$

Substitute the values of equations (5.35a) - (5.35d) in the equation (5.17) we get the linear differential equation as,

$$\begin{aligned} [-3\mathrm{a} - 3\mathrm{nb}(-0.999)^{n-1}]\, f_{n+1}''' + [0.999\eta^2 - 2\eta - 2]\, f_{n+1}'' \\ + (2 + 3M_0 - 1.9998\eta)\, f_{n+1}' + 1.998\, f_{n+1} = 3.998 + 3M_0 \end{aligned} \tag{5.36}$$

Here the equation (5.36) is linear ordinary differential equation now we will find its numerical solution using Finite difference method. Substitute the values of equations

(3.51*a*)–(3.51*c*) in equation (5.36). So at the ith node we get the system of linear equations,

$$\begin{aligned}
&[-3a-3\mathrm{n}\,b(-0.999)^{n-1}]\ f_{i+2}+[6a+6nb(-0.999)^{n-1}+1.998hx^2-4hx\\
&\quad -4h+3h^2M_0+2h^2-1.998h^2x]f_{i+1}+[-3.996hx^2+8hx+8h+3.996h^3]f_i\\
&\quad +[-6a-6nb(-0.999)^{n-1}+1.998hx^2-4hx-4h-3h^2M_0-2h^2\\
&\quad +1.998h^2x]\ f_{i-1}+[3a+3\mathrm{n}\,b(-0.999)^{n-1}]\ f_{i-2}=(3.998+3M_0)2h^3
\end{aligned}\tag{5.37}$$

Case I: **Non- MHD Sisko fluid**

If we take M_0=0 and a=b=0.5 in equation (5.37), then it will give us Non-MHD fluid flow. And the system of linear equations (5.37) will reduce in the following equation (5.38) with the same boundary conditions given in equation (5.33).

$$\begin{aligned}
&[-1.5-1.5n(-0.999)^{n-1}]\ f_{i+2}\\
&+[3+3n(-0.999)^{n-1}+1.998hx^2-4hx-4h+2h^2-1.998h^2x]f_{i+1}\\
&+[-3.996hx^2+8hx+8h+3.996h^3]f_i\\
&+[-3-3n(-0.999)^{n-1}+1.998hx^2-4hx-4h-2h^2+1.998h^2x]\ f_{i-1}\\
&+[1.5+1.5n(-0.999)^{n-1}]\ f_{i-2}=7.996h^3
\end{aligned}\tag{5.38}$$

Case II: **MHD Power-Law fluid**

If we take a=0, b=1, M_0 non zero constant in equation (5.37), then it will give us MHD Power-Law fluid. And the system of linear equations (5.37) will reduce in the following equation (5.39) with the same boundary conditions given in equation (5.33).

$$\begin{aligned}
&[-3n(-0.999)^{n-1}]\ f_{i+2}\\
&+[6n(-0.999)^{n-1}+1.998hx^2-4hx-4h+3h^2M_0+2h^2-1.998h^2x]f_{i+1}\\
&+[-3.996hx^2+8hx+8h+3.996h^3]f_i\ +\\
&+[-6n(-0.999)^{n-1}+1.998hx^2-4hx-4h-3h^2M_0-2h^2+1.998h^2x]\ f_{i-1}\\
&+[3\mathrm{n}(-0.999)^{n-1}]\ f_{i-2}=[3.998+3M_0]2h^3
\end{aligned}\tag{5.39}$$

Case III: **Non-MHD Power-Law fluid**

If we take a=0, b=1 and Mo = 0 in equation (5.37), then it give us of non-MHD Power-Law fluid. And the system of linear equations (5.37) will reduce in the following equation (5.40) with the same boundary conditions given in equation (5.33).

$$\begin{aligned}
&[-3n(-0.999)^{n-1}]\ f_{i+2} \\
&+[6\,\mathrm{n}(-0.999)^{n-1}+1.998hx^2-4hx-4h+2h^2\ -1.998h^2x]f_{i+1} \\
&+[-3.996hx^2\ +8hx+8h+3.996h^3]f_i \\
&+[-6n(-0.999)^{n-1}+1.998hx^2-4hx-4h-2h^2\ +1.998h^2x]\ f_{i-1} \\
&+[3\,\mathrm{n}(-0.999)^{n-1}]\ f_{i-2}=7.996h^3
\end{aligned} \tag{5.40}$$

Case IV: MHD Newtonian fluid

If we take *n*=1, *a*=1, *b*=0 and M_0 non zero constant in equation (5.37), then it will give us MHD Newtonian fluid. Then the system of linear equations (5.37) will reduce in the following equation (5.41) with the same boundary conditions given in equation (5.33).

$$\begin{aligned}
&-3f_{i+2} \\
&+[6+1.998hx^2-4hx-4h+3h^2M_0+2h^2\ -1.998h^2x]f_{i+1} \\
&+[-3.996hx^2+8hx+8h+3.996h^3]f_i \\
&+[-6+1.998hx^2-4hx-4h-3h^2M_0-2h^2\ +1.998h^2x]\ f_{i-1} \\
&+3\ f_{i-2}=[3.998+3M_0]2h^3
\end{aligned} \tag{5.41}$$

Case V: **Non- MHD Newtonian fluid**

If we take *a*=1, *b*=0, *n*=1and M_0=0 in equation (5.37), then it give us non-MHD Newtonian fluid. Then the system of linear equations (5.37) will reduce in the following equation (5.42) with the same boundary conditions given in equation (5.33).

$$\begin{aligned}
&-3f_{i+2}+[6+1.998hx^2-4hx-4h+2h^2\ -1.998h^2x]f_{i+1} \\
&+[-3.996hx^2+8hx+8h+3.996h^3]f_i+[-6+1.998hx^2-4hx \\
&-4h-2h^2\ +1.998h^2x]\ f_{i-1}+3\ f_{i-2}=3.998h^3
\end{aligned} \tag{5.42}$$

This system of linear equations we solve it using MATLAB solver by dividing the interval [0,1] into 1000 subintervals having equal length of each subintervals h=0.001. Given numerical solution is presented graphically as below.

5.10 GRAPHICAL PRESENTATION

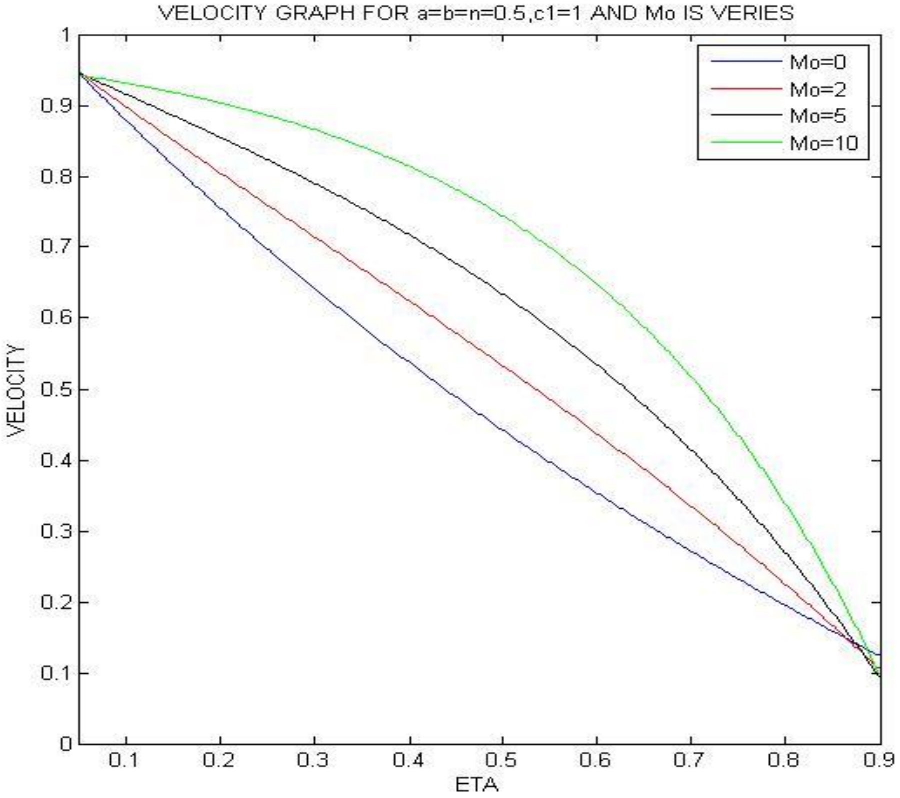


Figure 5.7 MHD Sisko fluid with *a*=*b*=*n*=0.5 and different values of *Mo*

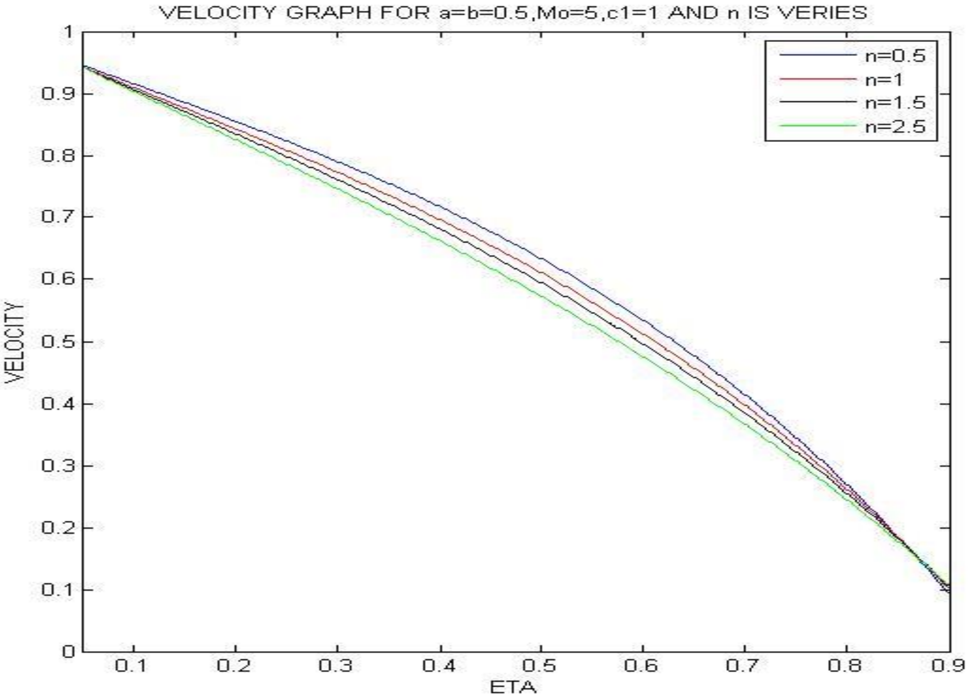


Figure 5.8 MHD Sisko fluid with *a*=*b*=0.5, M_0 =0.5 and different values of *n*

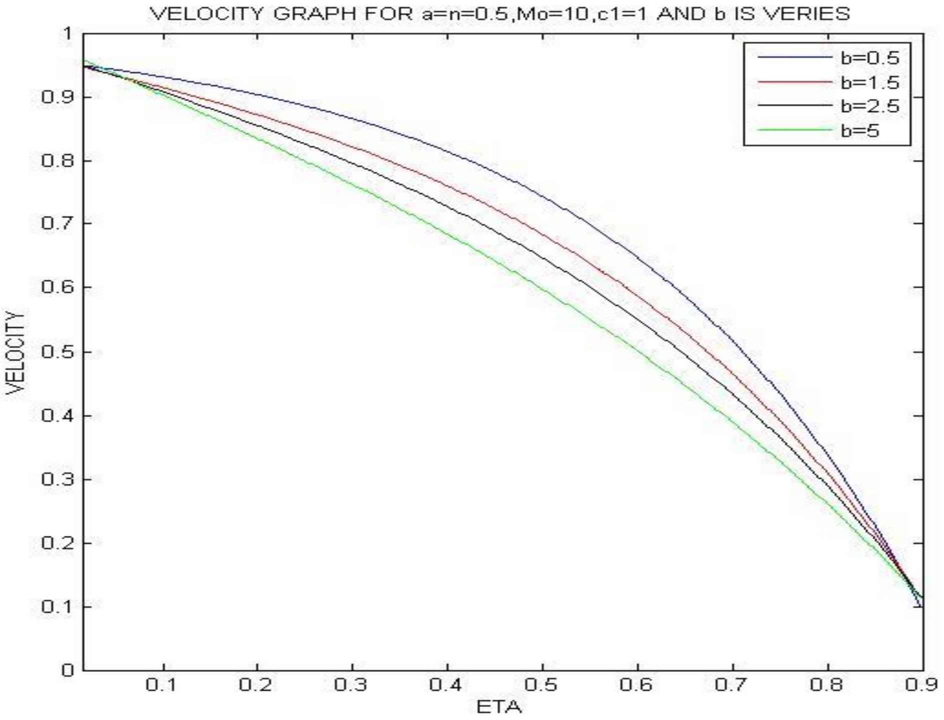


Figure 5.9 MHD Sisko fluid for *a*=n=0.5, *Mo* = 10 and different values of b

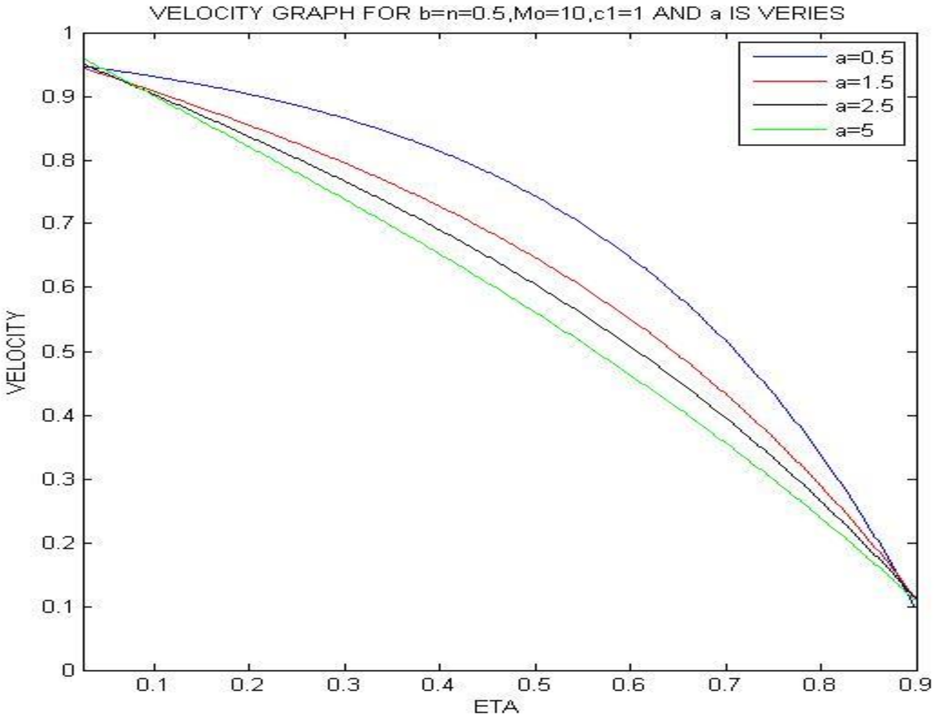


Figure 5.10 MHD Sisko fluid for *b*=n=0.5, *Mo* = 10 and different values of *a*

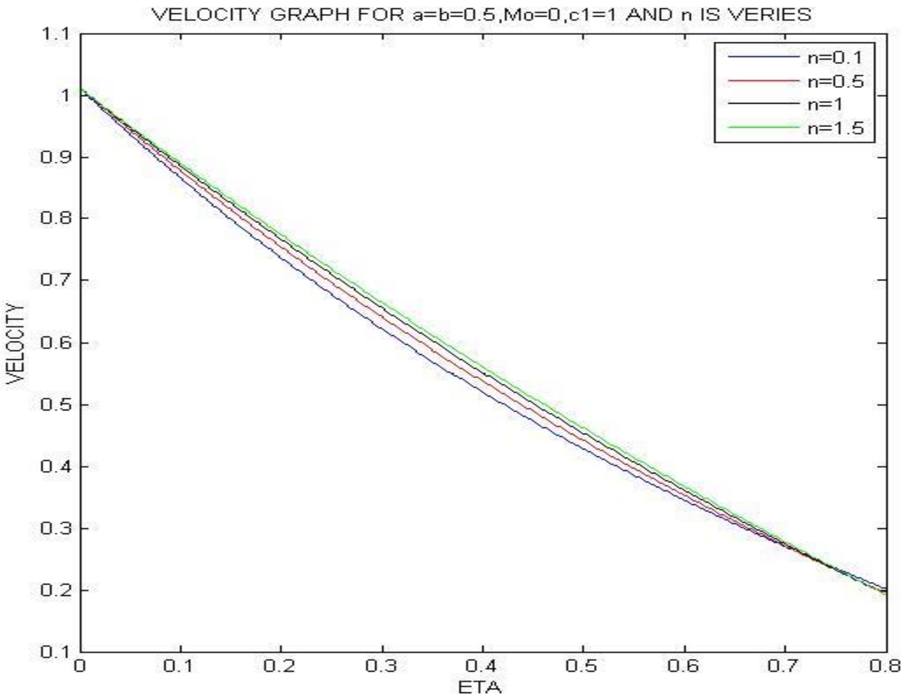


Figure 5.11 Non- MHD Sisko fluid for *a*=*b*=0.5 and different values of *n*

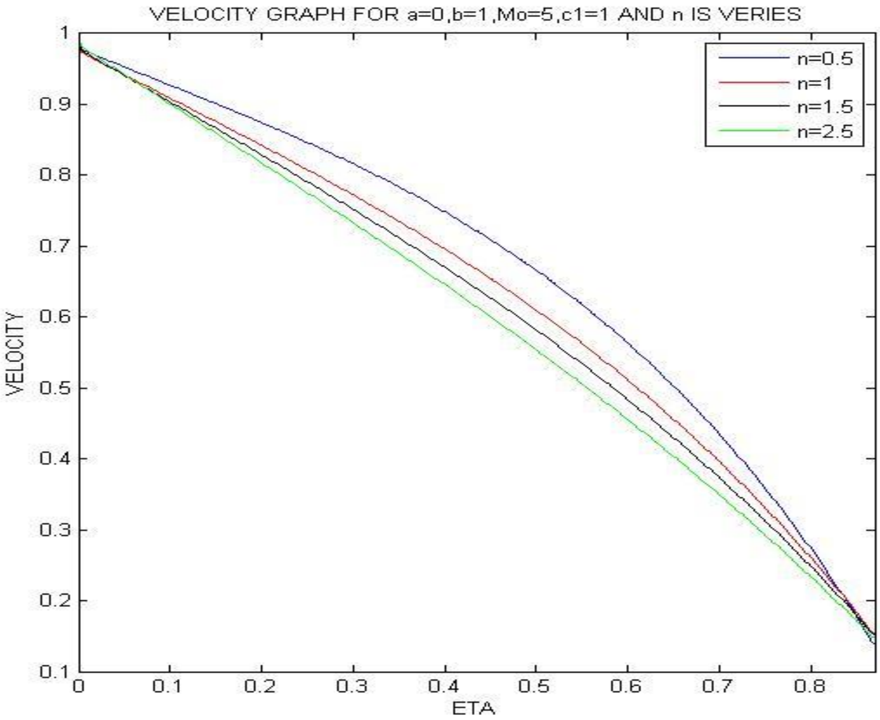


Figure 5.12 MHD Power-Law fluid for *a*=0, *b*=1, M_0 =5 and different values of *n*

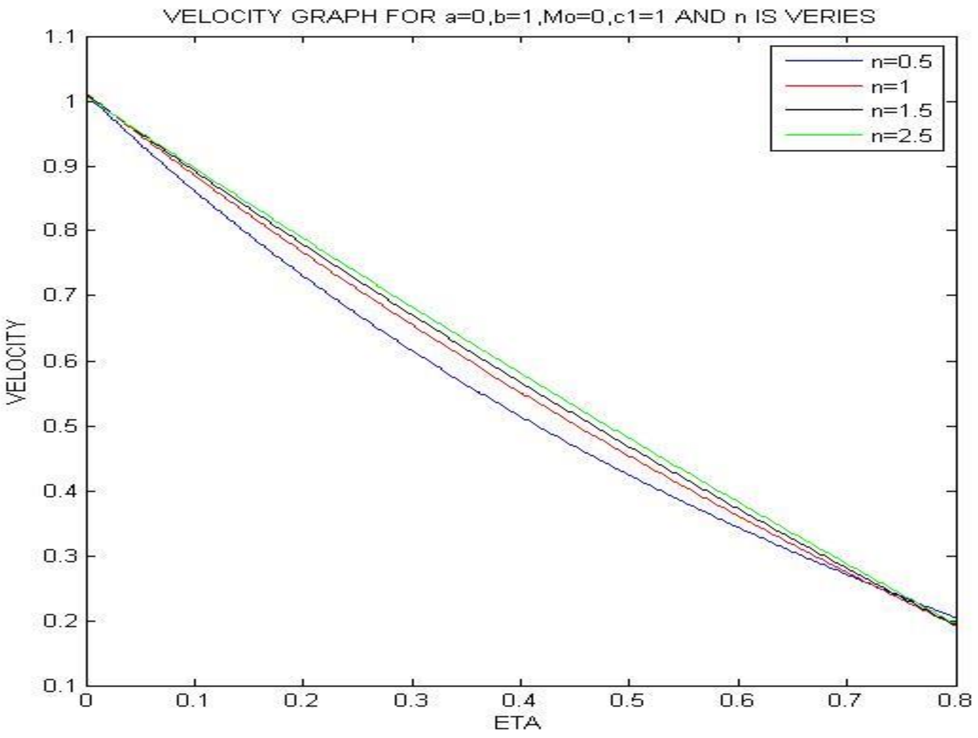


Figure 5.13 Non-MHD Power-Law fluid for *a*=0, *b*=1, M_0 =0 and different values of *n*

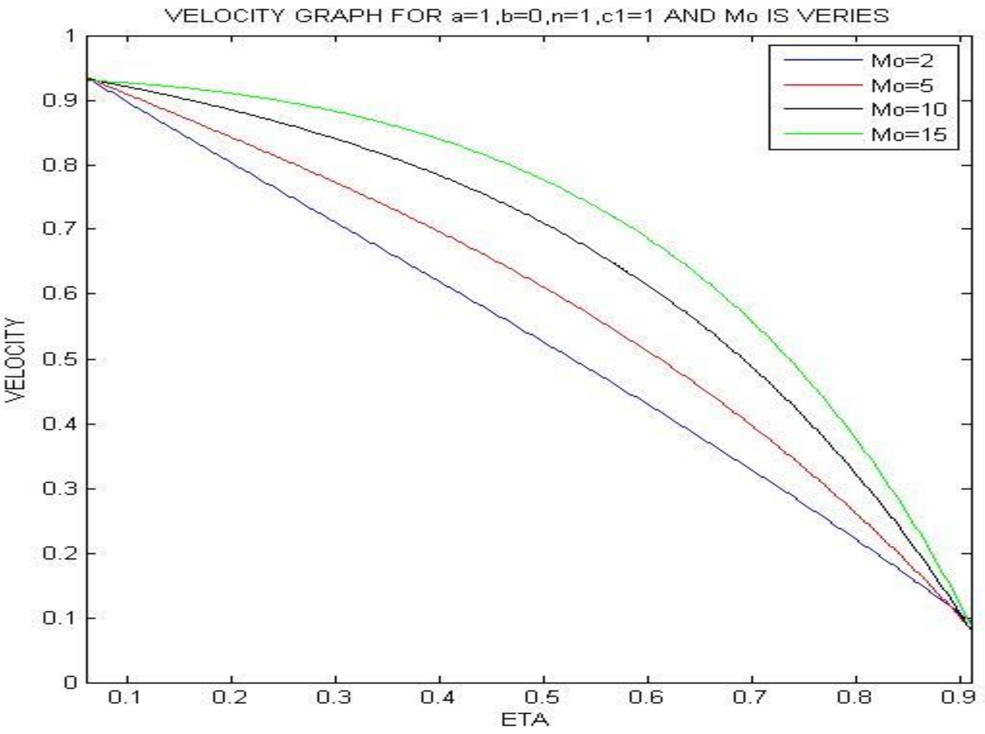


Figure 5.14 MHD Newtonian fluid for *a*=1, *b*=0, n=0 and different values of M_0

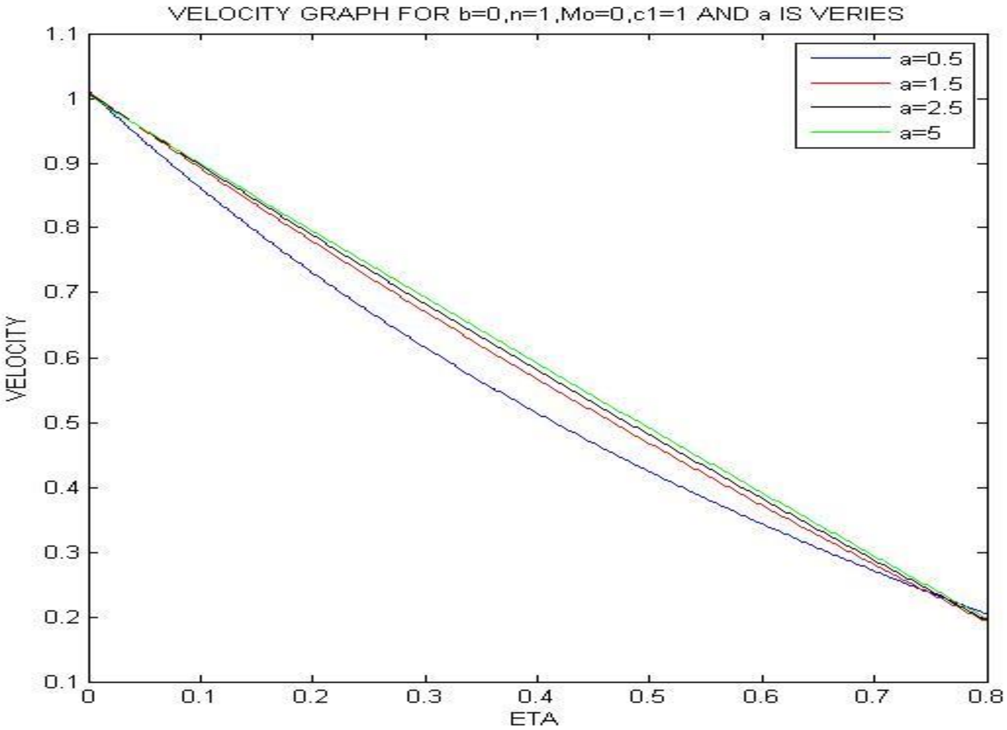


Figure 5.15 Non- MHD Newtonian fluid b=0, n=1, M_0=0 and different values of *a*

5.11 RESULT AND DISCUSSION

Using QLM we obtained the third order linear ODE which is then solved by FDM. If we change the values of flow parameter like *a*, *b*, M_o and the flow behavior index *n*, the fluid we considered is change to Power law and Newtonian. Which is indicated as subcases from Case-I to Case-V. We solve all sub cases and give graphical presentation of all. Our aim is to apply FDM on third-order ODE. Numerical values and graphs can now easily generated. In Figure 5.7, *a*=*b*=0.5 and the flow behavior index *n*=0.5 are constants and the magnetic constant *Mo* is varies, 0, 2, 5, 10. In Figure 5.8, *a*=*b*=0.5 and the magnetic parameter *Mo*=5 are constant and the flow behavior index n is varies 0.5, 1, 1.5, 2.5. In Figure 5.9, *a*=0.5, the magnetic number *Mo*=10 and the flow behavior index *n*=0.5 are constants and constant parameter *b* is varies, 0.5, 1.5, 2.5, 5. In Figure 5.10, *b*=0.5, the magnetic number. *Mo*=10 and the flow behavior index *n*=0.5 are constants and constant parameter *a* is varies, 0.5, 1.5, 2.5, 5. In Figure 5.11, *a*=*b*=0.5 and the magnetic number *Mo*=0 and the flow behavior index *n* is varies, 0.1, 0.5, 1, 1.5. In Figure 5.12, *a*=0, *b*=1 and the magnetic number *Mo*=5 is constant and the flow behavior index n is varies *n*=0.5, 1, 1.5, 2.5. In Figure 5.13, *a*=0, *b*=1 and the magnetic number *Mo*=0 is constant and the flow behavior index *n* is varies *n*=0.5, 1, 1.5, 2.5. In Figure 5.14, *a*=1, *b*=0 and the flow behavior index *n* is constant *n*=1 and the magnetic number *Mo* is varies *Mo*=2, 5, 10, 15. In Figure 5.14, *b*=0, the flow behavior index *n* is constant *n*=1 and the magnetic number *Mo* =0 the constant *a* is varies *a*=0.5, 1, 1.5, 2.5.

5.12 CONCLUSION

The governing equations of motion (partial differential equations) of the laminar MHD boundary layer flow of non-Newtonian fluid are solved numerically using FDM. When we consider flow over flat plate the velocity is increase uniformly with the increase in eta for variable fluid index, Magnetic induction and fluid parameters.

On the other side when we consider flow over moving surface the velocity is decrease uniformly with the increase in η for variable fluid index, Magnetic induction and fluid parameters. Increases in the magnetic number accelerate the fluid. Also for the case of Power-law fluid, shear thinning fluid velocity increase more rapidly than that of for shear thickning fluids.

CHAPTER-06

NUMERICAL SOLUTION OF BOUNDARY LAYER EQUATIONS OF WILLIAMSON FLUID FLOW

6.1 INTRODUCTION

Mostly non-Newtonian fluids are types of pseudo plastic fluids. Wide range of industrial application like suspension coated sheets (photographic films), high molecular weight polymers melts and its solutions etc… the boundary layer flow of such fluids are great interest in commercial point of view. All the rheological properties of such types of fluids are not explained by the Navier Stokes equations. Therefore, to overcome this type of deficiency different types of fluid models has been proposed by researchers. The power law model, Cross model, Carreaus model and Ellis model are it’s the examples but very rare information available about the Williamson fluid model.

In Williamson fluid viscosity decreases with increasing rate of shear stress due to this characteristics is known as a non-Newtonian fluid. Chyme in small intestine treated as Williamson fluid. The boundary layer in liquid film condensation process, In the industries extrusion of a polymer sheet from the die, emulsion coating on photographic films are examples of Williamsons fluid. Gastrointestinal tract in a human body, the peristaltic phenomenon plays a vital role throughout the digestion and absorption of food.

In this chapter we considered 2-D Williamson fluid flow over a moving plate. The governing equations of Williamson fluid model is simplified by using similarity transformations and boundary layer approach. The reduced equations are then numerically solved by finite difference method using MATLAB solver and velocity profile graphically presented.

6.2 HISTORICAL REVIEW

Williamson (1929) proposed a model for pseudoplastic materials when it moves and gives it equation of motion and also experimentally verified the results. Lyubimov and Perminov (2002) discussed in the presence of a gravitational field the flow of a thin layer over an inclined surface of a Williamson fluid. Dapra and Scarpi (2007) considered Williamson fluid injected into a rock fracture and gives the perturbation solution. Nadeem *et al*. (2012) discussed peristaltic flow of a Williamson fluid. Vasudev *et al*. (2010) discussed in details the peristaltic pumping of a Williamson fluid through a porous medium by considering heat transfer. Srivastava et al. (1995, 2007) discussed the flow in non-uniform geometry and gives theoretical model under the effects of an inserted endoscope on chyme movement in small intestine. Nadeem et al. (2013) find the solution flow of a Williamson Fluid over a stretching sheet by applying homotopy analysis method (HAM). L.M. Srivastava et al. (1983,1984) introduced casson model for peristaltic transport of blood and a physiological fluid. I. Dapra et al. (2007) discussed perturbation solution for pustule flow of a non-Newtonian Williamson fluid in a rock fracture.

6.3 PRESENT INVESTIGATION

In this chapter we tried to find the numerical solution of two-dimensional boundary layer equations for Williamson fluid model. The flow considered in two dimensional semi-infinite moving surfaces. Along the x-axis two opposite and equal forces are applied to produce stretching and keeping the fixed origin. The flow is generated due to the linear stretching.

Similarity equation of defined flow problems are highly non linear Partial differential equations hence we try to find its numerical solution by Finite difference method and graphically presented by MATLAB solver.

6.4 DERIVATION OF BOUNDARY LAYER EQUATION OF WILLIAMSON FLUID MODEL

The continuity and momentum equations for an incompressible fluid flow of the Williamson Fluid are: [Nadeem et al.,2013]

Continuity equation: $divV = 0$ (6.1)

Momentum equation:

$$\rho \frac{dV}{dt} = div\,S + \rho b \tag{6.2}$$

Where, ρ = Density,

V = Velocity vector,

S = Cauchy stress tensor,

b = Specific body force vector,

$\frac{d}{dt}$ = The material time derivative,

For Williamson fluid Cauchy stress tensor is given by equation,

$$S = -pI + \tau \tag{6.3}$$

$$\tau = \left[\mu_\infty + \frac{(\mu_0 - \mu_\infty)}{1 - \Gamma\gamma} \right] A_1 \tag{6.4}$$

Where, p = Pressure

I = Identity vector,

τ = Extra stress tensor,

μ_0 = Limiting viscosities at zero,

μ_∞ = Limiting viscosities at infinite,

Γ = Time Constant

A_1 = The First Rivlin- Ericksen tensor

And γ is given by the equation

$$\gamma = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \tag{6.5}$$

In equation (6.5), π denotes the second invariant strain tensor and it is given by the equation.

$$\pi = trace(A_1^2) \tag{6.6}$$

Now, The First Rivlin-Ericksen tensor A_1 is given by,

$$(\mathrm{A}_{ij})_1 = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \qquad 1 \le i, j \le n$$

$$\therefore\ A_1 = \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{pmatrix} \tag{6.7}$$

From equations (6.5)-(6.7) we get,

$$\therefore\ \gamma = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]^{\frac{1}{2}} \tag{6.8}$$

Here we considered only the case limiting viscosity at infinite is zero and $\Gamma\gamma < 1$.

So, the extra stress tensor takes the form

$$\tau = \left[\frac{\mu_0}{1-\Gamma\gamma}\right] A_1 \tag{6.9}$$

By taking binomial expansion and second and higher order terms neglected, we get,

$$\tau = \mu_0\left[1+\Gamma\gamma\right] A_1 \tag{6.10}$$

the extra stress tensor components are

$$\tau_{xx} = 2\mu_0\left[1+\Gamma\gamma\right]\frac{\partial u}{\partial x} \tag{6.11}$$

$$\tau_{xy} = \tau_{yx} = 2\mu_0\left[1+\Gamma\gamma\right]\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \tag{6.12}$$

$$\tau_{yy} = 2\mu_0\left[1+\Gamma\gamma\right]\frac{\partial v}{\partial y} \tag{6.13}$$

$$\tau_{xz} = \tau_{yz} = \tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \tag{6.14}$$

Let us consider two –dimensional, steady an incompressible flow of Williamson fluid over a semi infinite moving surface keeping the origin fixed in the absence of body forces. Due to the linear stretching flow is generated.

The governing equations of above defined flow problem in the direction of flow (x-axis) and normal to the flow (y-axis), respectively.

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} \tag{6.15}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y} \tag{6.16}$$

For Williamson fluid model it is given by substitute the values of equation (6.8) in equation (6.11)-(6.13).

$$\tau_{xx} = 2\mu_0 \left\{ 1+\Gamma \left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right]^{\frac{1}{2}} \right\} \frac{\partial u}{\partial x} \tag{6.17}$$

$$\tau_{xy} = \tau_{yx} = \mu_0 \left\{ 1+\Gamma \left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right]^{\frac{1}{2}} \right\} \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \tag{6.18}$$

$$\tau_{yy} = 2\mu_0 \left\{ 1+\Gamma \left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right]^{\frac{1}{2}} \right\} \frac{\partial v}{\partial y} \tag{6.19}$$

In the absence of a pressure gradient P and using equations (6.17)-(6.19) equations (6.15) and (6.16) are converted into,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \sqrt{2}\,\nu\,\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \tag{6.20}$$

Under the boundary conditions are:

$$u = Bx = U_\infty \quad and \;\; v = 0 \;\; at \;\; y = 0$$

$$u = 0 \quad at \;\; y \to \infty$$

Where, ν = kinematic viscosity,

U_∞ = velocity at the wall

$B > 0$ = stretching parameter along *x*-axis.

6.5 SIMILARITY SOLUTION: ONE PARAMETER GROUP THEORY METHOD

Appling similarity technique the relation between dependent variables and independent variable are as under [Nadeem et al.,2013].

$$u = Bxf'(\eta) \;\; , \;\; v = -\sqrt{B\nu}\, f(\eta) \;\; , \;\; \eta = \sqrt{\frac{B}{\nu}}\, y \tag{6.21}$$

Substitute transformations equations (6.21) in equation (6.20) we get non linear ordinary differential equation as,

$$f'''(\eta) - f'^2(\eta) + f(\eta) f'(\eta) + \lambda f''(\eta) f'''(\eta) = 0 \tag{6.22}$$

With the corresponding boundary conditions:

$$f(0) = 0,\ f'(0) = 1 \tag{6.23}$$

$$f'(\infty) = 0 \tag{6.24}$$

Where, $\lambda = \Gamma x \sqrt{\frac{2B^3}{\nu}}$ = the dimensionless Williamson parameter.

We convert equation (6.22) into linear ordinary differential equation by quasilinearization method as,

$$\begin{aligned} & f_n'''(\eta) - f_n'^2(\eta) - f_n f_n''(\eta) + \lambda f_n''(\eta) f_n'''(\eta) + f_n''(\eta)[f_{n+1}(\eta) - f_n(\eta)] \\ & -2 f_n'(\eta)[f_{n+1}'(\eta) - f_n'(\eta)] + f_n(\eta)[f_{n+1}''(\eta) - f_n''(\eta)] + \lambda f_n'''(\eta)[f_{n+1}''(\eta) \\ & - f_n''(\eta)] + f_{n+1}'''(\eta) - f_n'''(\eta) + \lambda f_n''(\eta)[f_{n+1}'''(\eta) - f_n'''(\eta)] = 0 \end{aligned} \tag{6.25}$$

Simplifying above equation we get,

$$\begin{aligned} & f_n'^2 + f_n''(\eta) f_{n+1}(\eta) - 2 f_n'(\eta) f_{n+1}'(\eta) + f_n(\eta) f_{n+1}''(\eta) \\ & \quad - f_n(\eta) f_n''(\eta) + \lambda f_n'''(\eta) f_n''(\eta) + f_{n+1}'''(\eta) \\ & \quad + \lambda f_n'' f_{n+1}'''(\eta) - \lambda f_n'' f_n''' = 0 \end{aligned} \tag{6.26}$$

To fit the curve, consider the solution.

$$f_n = A_0 \eta^2 + A_1 \eta + A_2 \tag{6.27}$$

With boundary conditions in (6.23) and (6.24)

We have constants, $A_0 = -0.5,\ \ A_1 = 1,\ \ A_2 = 0$

So, equation (6.27) can be written as,

$$f_n = -0.5\eta^2 + \eta$$

$$\therefore \quad f_n' = -\eta + 1$$

$$\therefore \quad f_n'' = -1$$

$$\therefore \quad f_n''' = 0$$

Substitute all above the values in equation (6.26),

$$[1-\lambda] f_{n+1}''' + [-0.5\eta^2 + \eta] f_{n+1}'' + [2\eta - 2] f_{n+1}' - f_{n+1} = -1 + \eta - 0.5\eta^2 \tag{6.28}$$

6.6 NUMERICAL SOLUTION : FINITE DIFFERENCE METHOD

Here the equation (6.28) is linear ordinary differential equation now we will find its numerical solution using Finite difference method. Substitute the values of equations (3.51*a*) – (3.51*c*) in equation (6.28) at the ith node we get the system of linear equations.

$$\begin{aligned}
&\left[1-\lambda\right] f_{i+2} + [-2+2\lambda - hx^2 + 2hx + 2xh^2 - 2h^2]\mathrm{f}_{i+1} \\
&\quad + [2hx^2 - 4hx - 2h^3] f_i + [2 - 2\lambda - hx^2 + 2hx - 2xh^2 + 2h^2] f_{i-1} \\
&\quad + \left[\lambda - 1\right] f_{i-2} = [-1 + x - 0.5x^2]2h^3
\end{aligned} \tag{6.29}$$

Which is system of linear equations we solve it in MATLAB by dividing [0,1] interval into 500 subinterval having length h=0.002.solution is presented graphically as below.

6.7 GRAPHICAL PRESENTATION

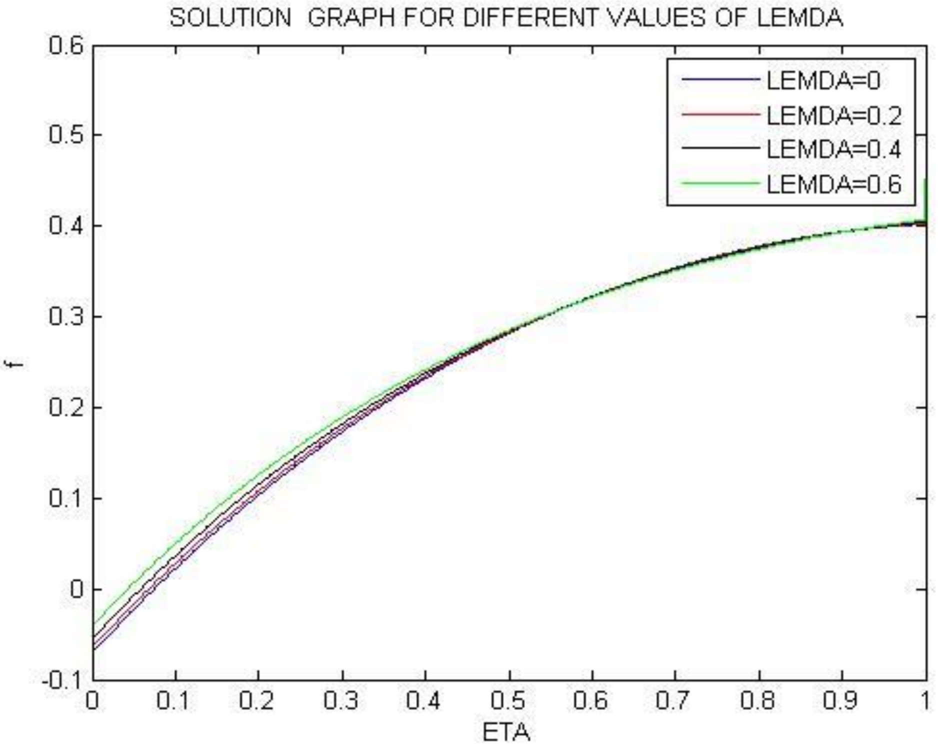


Figure 6.1 Solution graph f verses eta for different values of lambda

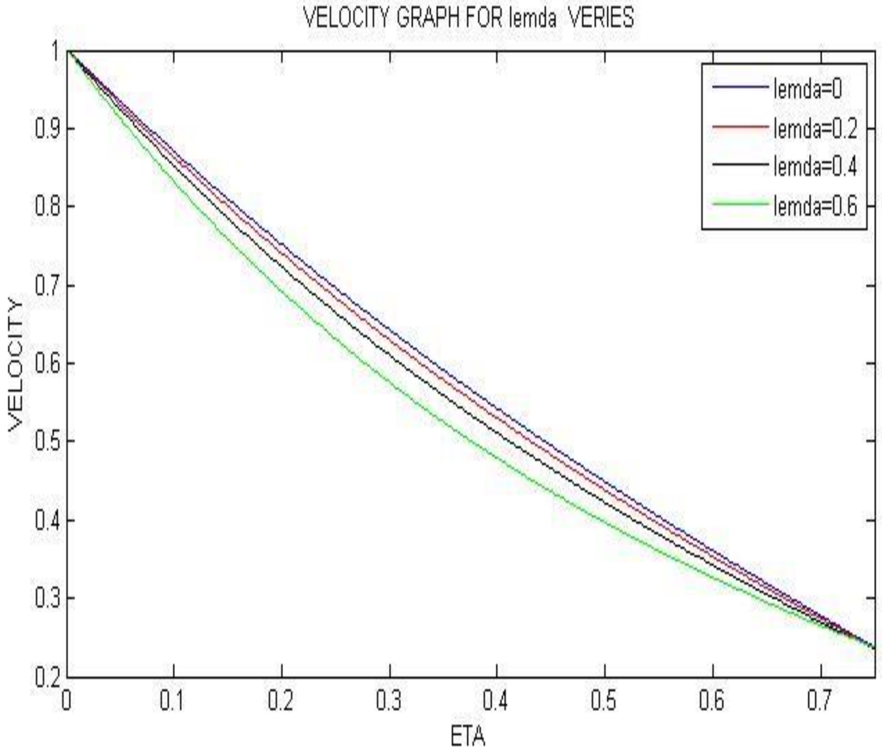


Figure 6.2 Velocity graph f’ verses eta for different values of lambda

6.8 CONCLUSION

Using Similarity group transformation method, the PDE of flow problems are transformed into ODE. The ODE are then solved numerically and presented graphically by MATLAB solver. **Figure 6.1** shows that f increase with an increase in lambda (λ). **Figure 6.2** represents the velocity profiles f' verses eta for different values of lambda (λ). It shows that velocity field decrease uniformly with increase the values of lambda (λ).

List of References

1. Akbar, N. S., Nadeem, S. and Lee, C. (2012), Int. J. Phys. Sci.,7,687.
2. A.W. Sisko,1958, The flow of lubricating greases. Industrial and Engineering Chemistry Research, 50(12), pp1789–1792.
3. A. M. Siddiqui, Hameed Ashraf, T. Haroon and A. Walait,2013, Analytic Solution for the Drainage of Sisko Fluid Film Down a Vertical Belt, Appl. Appl. Math., Vol. 8, Issue 2,pp. 465 – 480.
4. A. Alsaedia, Naheed Batoolb, H. Yasminb,∗ and T. Hayata,2013, Convective heat transfer analysis on Prandtl fluid model with peristalsis, Applied Bionics and Biomechanics 10 ,pp. 197–208. A. W. Sisko. ,1958, The flow of lubricating greases, Ind. Engng Chem. Vol 50 (12), pp.1789-1792.
5. A Bahraini, Siroos Azizmohhmmadi, Mohammad-Reza Golkar Narenji and Vahid Taghikhani. ,1996, Isothermal Axial Laminar flow of non-Newtonian fluids in concentric annuli (Power law and Sisko fluids), Italian Polymer Journal, vol. 5, pp. 271-278.
6. Bird R.B., Stewart W.E. and Lightfoot E.M.,1960, Transport phenomena, John Wiley, New York.
7. Blasius, H., 1908, Boundary layers in fluids with little viscosity, NACA TM – 1256.
8. Cramer, S. D. and Marchello , J. M. 1968, Numerical evaluation of models describing non-Newtonian behaviour. American Institute of Chemical Engineers Journal, 14, 980.
9. Djukic, Dj.S. ,1974, Hiemenz magnetic flow of power-law fluid, Journal of applied mechanics, Transaction of the ASME, pp.822-823.
10. Djukic, Dj.S. ,1973, On the use of Crocco’s equation for the flow of power-law fluids in a transverse magnetic field, AIChE Journal, vol 19, pp. 1159-1163.
11. Dapra and Scarpi, G., 2007, Perturbation solution for pulsatile flow of a non-Newtonian Williamson fluid in a rock fracture. International Journal of Rock Mechanics and Mining Sciences, 44, pp. 271.
12. E. S. Lee, 1968, Quasi linearization and Invariant Imbedding, Academic Press, New York, NY, USA.
13. Frank M. White,2011, Fluid Mechanics, 7th ed., McGraw- Hill, New York.
14. G. Sucharitha, S. Sreenadh and P. Lakshminarayana, 2012, Nonlinear peristaltic transport of a conducting prandtl fluid in a porous asymmetric channel, *International Journal of Engineering* Research and Technology 1, ISSN:2278–0181.
15. H. Parmar, M.G. Timol,2011, Deductive Group Technique for MHD coupled heat and mass transfer natural convection flow of non-Newtonian power law fluid over a vertical cone through porous medium, Int. J. Appl. Math. Mech. 7 (2) ,pp. 35–50.

16. Hansen A.G. and Na T.Y. 1968, Similarity solution of laminar, Incompressible boundary layer equations of non-Newtonian fluid, ASME Journal of basic engg.67, pp.71-74.

17. H. Nemati, M. Ghanbarpour, M. Hajibabayi, Hemmatnezhad, 2009, Thin film flow of non-Newtonian fluids on a vertical moving belt using Homotopy Analysis Method, Journal of Engineering Science and Technology Review, 2 (1) ,pp.118-122.

18. I. Dapra and G. Scarpi,2007, Perturbation solution for pulsatile flow of a non-Newtonian Williamson fluid in a rock fracture," International Journal of Rock Mechanics and Mining Sciences, vol. 44, no. 2, pp.271–278.

19. Jaffrin, M.Y. and Shapiro, A.H., 1971, Ann. Rev. Fluid Mech, 3, 13.

20. K.A. Adnan, A.H. Hasmani, M.G. Timol, 2011, A new family of similarity solutions of three dimensional MHD boundary layer flows of non-Newtonian fluids using new systematic grouptheoretic approach, Appl. Math. Sci. 5 (27), pp. 1325–1336.

21. Kapur J. N., Bhatt B.S. and Sacheti N.C.,1982, Non– Newtonian fluid flows, Pragati Prakashan, Meerut (India).

22. Lyubimov, D. V. and Perminov , 2002, A. V., Motion of a thin oblique layer of a pseudoplastic fluid. Journal of Engineering Physics and Thermophysics, 75, No. 4, 920.

23. Lee S.Y. and Ames W.F. ,1966, Similar solutions for non-Newtonian fluids, AIChE J. 12, pp.700-708.

24. L. M. Srivastava, V. P. Srivastava, and S. N. Sinha,1983, Peristaltic transport of a physiological fluid: part I. Flow in non-uniform geometry, Biorheology, vol. 20, no. 2, pp. 153–166.

25. L. M. Srivastava and V. P. Srivastava,1984, Peristaltic transport of blood: casson model. II," Journal of Biomechanics, vol. 17, no. 11, pp. 821–829.

26. M. Patel, H. Surati, M. S. Chanda and M. G. Timol,2013. Models of Various Non-Newtonian Fluids, International E-journal of education and mathematics (IEJEM), 2(4), pp. 21-36.

27. M. Patel, Jayshri Patel and M. G. Timol,2015, Laminar Boundary Layer Flow of Sisko Fluid, Applications and Applied Mathematics(AAM), Vol. 10, Issue 2 , pp. 909-918, ISSN: 1932-9466.

28. M. Patel and M. G. Timol, 2009, Numerical solution of the equation for unsteady boundary layer flow of non-Newtonian fluids past semi-infinite plate. International Journal of Applied Mechanics and Mathematics (IJAMM), Vol. 5, no 3, pp.22-29.

29. M. Patel and M. G. Timol, 2010, Similarity Analysis in MHD Heat and Mass Transfer of Non-Newtonian Power Law Fluids Past a Semi-infinite Flate Plate, International Journal of Applied Mechanics and Engineering (IJAME), vol.15,pp 1205-1214.

30. M. Patel and M. G. Timol ,2010, The general stress–strain relationship for some different visco-inelastic non-Newtonian fluids, International Journal of Applied Mechanics and Mathematics (IJAMM), 6 (12), pp.79-93.

31. M. Patel, H. Surati, M. S. Chanda and M. G. Timol, 2013, Models of Various Non-Newtonian Fluids, International E-journal of education and mathematics (IEJEM),2(4), pp.21-36.

32. Manisha Patel and M Timol. ,2011, Magneto hydrodynamic orthogonal stagnation point flow of a power-law fluid toward a stretching surface American Journal of Computational Mathematics, volume 1(issue 2), pp.129-133.

33. M. Patel and M. G. Timol, 2011, Orthogonal Stagnation Point Flow of A Power-Law Fluid Toward A Stretching Surface, International Journal of Applied Mechanics and Mathematics (IJAMM), 7(3), pp.31-37.

34. M. Patel, R. Patel and M. G. Timol, 2012, Group Invariance for non-Linear PDE's : 3-D MHD stagnation point flow of non-Newtonian power-law fluids, International Journal of Mathematics and Scientific Computing (IJMSC), vol 2, no. 2, pp.72-80.

35. M. Patel and M. G. Timol, 2014, Numerical treatment of MHD Power-law fluid flow using the method of satisfaction of asymptotic boundary conditions (MSABC), International Journal of Applied Mechanics and Mathematics (IJAMM), 10 (8), pp.61-77.

36. M. J. Moran , R. A. Gaggioli, , 1968, A new symmetric formalism for similarity analysis,with application to boundary layer flows, Technical summary report No.918, mathematical Research center, U. S. Army, Madison, Wisconsin.

37. M. J. Moran, R. A. Gaggioli,1968, Similarity analysis via group theory, AIAA J.6 ,pp. 2014-2016.

38. M. Patel, 2007, (Ph.D. Thesis) Similarity solutions of three dimensional boundary layer flows of non-Newtonial fluids, Veer Narmad University, Surat-India.

39. Metzner, A.B., (1956): Advances in chemical engg., vol. I, Academic Press, New York.

40. Nagendra, N., Madhava Reddy, C., Subba Reddy, M. V. and Jayaraj B.,2008, Journal of Pure and Applied Physics, , 20, 189.

41. Noreen Sher Akbar,2014, Blood flow analysis of Prandtl fluid model in tapered stenosed arteries, Ain Shams Engineering Journal.

42. Nita Jain and M.G. Timol,2015, Similarity solutions of quasi three dimensional power law fluids using the method of satisfaction of asymptotic boundary conditions Alexandria Engineering Journal -54, 725–732.

43. N.S. Akbar, S. Nadeem and C. Lee,2012, Peristaltic flowof a prandtl fluid in an asymmetric channel, International Journal of Physical Sciences 7 , pp.687–695.

44. Na T.Y. and Hansen A.G. ,1967, Similarity solutions of a class of laminar three-dimensional boundary layer equations of power-law fluids, Int. J. Non-linear Mech., v.2, pp. 373 – 385.

45. N. Moallemi, Shafieenejad and a. B. Novinzadeh. ,2011, Exact solutions for flow of a Sisko fluid in pipe ,Special Issue of the Bulletin of the Iranian Mathematical Society, Vol. 37, No. 2 Part 1 ,pp. 49-60.

46. P. O. Olanrewaju, A.A. Adigun, O.D. Fenwa, A. Oke, and Ajala, Funmi,2013, Unsteady Free Convective Flow of Sisko Fluid with Radiative Heat Transfer past a Flat Plate Moving Through a Binary Mixture, Thermal Energy and Power Engineering, Vol. 2, Issue 4, pp.147-155.

47. Prandtl, L.,1904, Uber Flussigkeit – shewegung bei sehr kleiner Reibung (on fluid motions with very small friction); verhandlunger, des. III, Internationlen Mathematiker congress, Heidelburg.

48. P. Gangavathi, Dr. M V Subba Reddy,2014, Dr. Ramakrishna Prasad, Peristaltic flow of a Prandtl fluid in an asymmetric channel under the effect of magnetic field, International Journal of Conceptions on Computing and Information Technology Vol.2, Issue 1, ISSN: 2345 – 9808.

49. Patel Manisha and Timol M.G.,2005, Similarity solution of laminar incompressible boundary layer equations of non-Newtonian fluids by generalized dimensional analysis, Varahmihir Journal of Mathematical Sciences, vol.5, No.2, pp.387-394.

50. Patel Manisha and Timol M.G., 2008, Similarity solutions of three-dimensional MHD boundary layer flow of non-Newtonian fluids, ISH Journal of Hydraulic Engineering, Special Issue, Vol.14, no.3, pp.82-93.

51. R.P. Chhabra and J.F. Richardson, 1999, Non-Newtonian Flow in the Process Industries Fundamentals and Engineering Applications, 1st ed., Butterworth-Heinemann, Woburn.

52. R. E. Bellman and R. E. Kabala,1965, Quasi linearization and Nonlinear Boundary-Value Problems, Elsevier Publishing Company, New York.

53. R. Kalaba,1959, On nonlinear differential equations, the maximum operation and monotone convergence, J. Math. Mech. 8, 519.

54. R. M. Darji, M.G. Timol, 2013, Group symmetry and similarity solutions for MHD mixed-convection flow of power-law fluid over a non-linear stretching surface, Int. J. Math. Sci. Comput. 3 (2) , pp. 32–36.

55. Sastry S. S.,2008, Engineering Mathematics, 4th eds., Prentice-Hall of India, New Delhi.

56. Seshandri R. and Na T. Y.,1985, Group Invariance in Engineering Boundary Value Problems. Springer-Verlag.

57. Subba Reddy, M. V., Ramachandra Rao, A. and Sreenadh, S.(2007), International Journal of Non-Linear Mechanics, 42, 153.

58. S. K. Som and G. Biswas, 2008, Introductory Fluid Mechanics and Fluid Machines, 2nded. McGraw-Hill, New York.

59. S. Navaneeswara Reddya, G. Viswanatha Reddyb and M.V.Subba Reddy,2012, Peristaltic flow of a prandtl fluid through a porous medium in a channel, *International Journal of Mathematical Archive* 11, pp. 4072–4080.

60. S. Jothia, A.R. Prasad and M.V. Subba Reddy,2012, Peristaltic flow of a prandtl fluid in a symmetric channel under the effect of a magnetic field, Advances in Applied Science Research 3 , pp. 2108–2119.

61. S. Nadeem, Hina Sadafa, Noreen Sher Akbar, 2014, Analysis of peristaltic flow for a Prandtl fluid model in an endoscope Journal of Power Technologies 94 (2), pp. 1–11.

62. Sohail Nadeem, Noreen Sher Akbar. ,2010, Peristaltic flow of Sisko fluid in a uniform inclined tube, Acta Mech Sin,26,pp.675–683.

63. Saleem Asghar, Tayyaba Minhas, and Aamir Alia,2014, Existence of a Hartmann layer in the peristalsis of Sisko fluid, Chin. Phys. B Vol. 23, No. 5,pp.1-5.

64. S. Nadeem, S.T. Hussain and Changhoon Lee,2013, Flow of a Williamson Fluid over a stretching sheet ,Brazilian Journal of Chemical Engineering, Vol. 30, No. 03, pp. 619 - 625.

65. Sohail Nadeem, Sadaf Ashiq and Mohamed Ali,2012, Williamson Fluid Model for the Peristalic Flow of Chyme in small intestine, Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2012, Article ID 479087.

66. T.Y. Na and A. Hansen.,1967, Radial flow of viscous non-newtonian fluids between disks, Int. J. Non-Linear Mech. Vol. 2, pp. 261-273.

67. Timol M.G. and Kalthia N.L. ,1986, Similarity solutions of three-dimensional boundary layer equations of non-Newtonian fluids, Int. J. non-Linear Mechanics, vol.21, No.6, pp. 475- 481.

68. Timol M.G. and Patel Manisha. ,2004, On the class of similarity solutions for three dimensional boundary layer flows of non-Newtonian fluids, J. of Veer Narmad South Gujarat University, II-B, pp.103-109.

69. V. Sirohi, M.G. Timol and N.L. Kalathia. 1984, Numerical treatment of Powell-Eyring fluid flow past a 90 wedge , Reg J. Energy Heat & Mass Trans., vol. 6, no. 3, pp.219-223.

70. V. P. Srivastava and M. Saxena,1995, A two-fluid model of non-Newtonian blood flow induced by peristaltic waves, Rheologica Acta, vol. 34, no. 4, pp. 406–414.

71. V. P. Srivastava, 2007, Effects of an inserted endoscope on chyme movement in small intestine—a theoretical model,Applications and Applied Mathematics, vol. 2, no. 2, pp. 79–91.

72. Vasudev, C., Rao, U. R., Reddy, M. V. S. and Rao, G. P., 2010, Peristaltic pumping of Williamson fluid through a porous medium in a horizontal channel with heat transfer. American Journal of Scientific and Industrial Research, 1, No. 3, pp.656.

73. Williamson, R. V., 1929,The flow of pseudoplastic materials. Industrial & Engineering Chemistry Research, 21, No. 11, pp.1108.

74. Wells C.S. ,1964, Unsteady boundary layer flow of a non-Newtonian fluid on a flat plate, AIAA Journal, vol.2 (5),pp. 951-952. M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan,2002, Solution of the Rayleigh problem for a power law non-Newtonian conducting fluid via group method, Int. J. Eng. Sci. 40 ,pp.1599–1609.

75. Yunus A Cengel and John M. Cimbala, 2006, Fluid Mechanics Fundamental Application, 1st ed. McGraw- Hill, New York.

76. Young, Donald F., Bruce R. Munson, Theodore H. Okiishi, Wade W. Huebsch, 2010, A Brief Introduction to Fluid Mechanics, 5th ed., John Wiley & Sons. p. 95. ISBN: 978-0-470-59679-1.

77. Y. Nakayama and R.F. Boucher, 2000, Introduction to Fluid Mechanics, 2nd ed., Butterworth - Heinemann, Woburn.

List of Publications

1. Laminar Boundary Layer Flow of Sisko Fluid, International Journal of Applications and Applied Mathematics (AAM), Vol. 10, Issue 2(December 2015), pp. 909-918. ISSN: 1932-9466.

2. Magneto hydro dynamic flow of Sisko fluid, Journal of Applied Research and Social Sciences (ARS), Vol.3, Issue.14 (July 2016), ISSN 2350-1472.

3. Finite Difference Method on third order non-linear Differential Equation: Magneto hydro dynamic flow of Sisko fluid, Indian Journal of Industrial and Applied Mathematics (IJIAM), Vol. 8, No. 2 (July–December 2017), pp. 191–200, ISSN: 0973-4317.

4. On the Solution of Boundary Layer Flow of Prandtl Fluid Past a Flat Surface, Journal of Advanced Mathematics and Applications (JAMA),Vol. 6, (2017), pp.1–6, 2156-7565.

5. Similarity Solution of Forced Convection Flow of Powell-Eyring & Prandtl-Eyring Fluids by Group-Theoretic Method, Mathematical Journal of Interdisciplinary Sciences,Vol-5, No-2, (March 2017),pp. 151–165,ISSN:2278-9561.

6. Numerical Solution of 3rd order ODE Using FDM: On a Moving surface in MHD flow of Sisko fluid, International Journal of Applications and Applied Mathematics (AAM), Accepted (Feb.-2019).