

**SIMILRITY AND NONSIMILARITY
ANALYSIS FOR ENGINEERING
BOUNDARY VALUE PROBLEMS**

A Thesis submitted to Gujarat Technological University

for the Award of

Doctor of Philosophy

in

SCIENCE-MATHS

By

HEMANGINI S. SHUKLA

Enrollment No.149997673002

under

supervision of

Dr. HEMA C. SURATI



**GUJARAT TECHNOLOGICAL
UNIVERSITY
AHMEDABAD**

AUGUST– 2020

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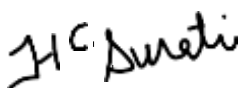
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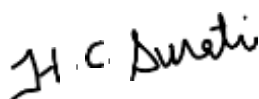


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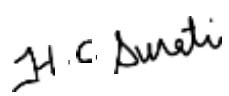


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ABSTRACT

The new era of engineering and technology had given many splendid inventions to the world. Because of the innovative advances in technology, the demand for the qualitative products and enhancement of the performance of products increases day to day. Many industrial products like beauty products, food products, medical products, plastic products, fibre products, etc. have an unlimited application of fluid. It is essential to study the principles, characteristics, and parameters of fluid flow which is addressed by the subject Fluid Mechanics. Mathematical models characterize the fluid flow problems which arise in the real world and in the industrial process. These types of problems deal with either linear or non-linear partial differential equations or ordinary differential equations.

One of the reasons for nonlinearity in the governing equations is stress-strain expression defined for non-Newtonian type fluid model. To handle the nonlinear models for finding an exact solution is an arduous job. Many researchers attempted to find the solution of the non-linear partial differential equations analytically. It is proved that similarity methods are one of the most powerful analytic tools currently available in the study of nonlinear partial differential equations.

The present research work is based on the application of group-theoretic methods for finding similarity solution for engineering boundary value problems. It also deals with another application of the local non-similar method to find the non-similarity solution. Two-dimensional model for Sisko fluid, Powell-Eyring and Prandtl Eyring fluids are studied. Also, three-dimensional Nanofluids for Newtonian and non-Newtonian models are studied. After reducing partial differential equations into ordinary differential equations by applying group-theoretic similarity methods and non-similarity methods, further procedure is carried out by numerical techniques using Maple software and MATLAB software.

There are some partial differential equations that do not possess invariant conditions and hence in such cases similarity solutions are not available. In such a case, only non-similar solutions or Local non-similar solutions may be available. In literature, it is found that for governing boundary layer flow of non-Newtonian fluids past a wedge, a similarity solution exists only for a 90-degree wedge. Powell-Eyring model is explored by considering free-stream velocity in power-law form as a part of this thesis. The non-similarity solution is found using the local non-similarity method at other than the 90-degree wedge.

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List of Abbreviation

- PDE -Partial differential equation
- ODE-Ordinary differential equation
- BVP- Boundary Value Problem
- IVP- Initial Value Problem
- MHD-Magneto Hydro Dynamics
- SWCNT-Single Wall carbon Nanotube
- DWCNT- Double Wall carbon Nanotube
- MWCNT- Multi Wall carbon Nanotube
- Cu-Copper
- Ag-Silver
- Au-Gold
- Al₂O₃ -Aluminum oxide
- CuO -Copper oxide
- AlN- Aluminum nitride
- SiN-Silicon nitride
- SiC- Silicon carbide
- TiO₂-Titanium dioxide

List of Symbols

κ	Thermal conductivity
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{K}, \lambda, A, B$	Fluid characteristic constant
p	Pressure
τ	Shear stress
μ	Coefficient of viscosity
Q	Amount of heat transfer
A	Surface area
ΔT	Difference in temperature
c_p	Specific heat at constant pressure
h	Convective heat transfer
L	Characteristic length of surface
α	Thermal diffusivity
c_p	Specific heat at constant pressure
Bi, Bi_1, Bi_2	Biot number
Re	Reynolds number
D_B	Brownian diffusion coefficient
C	Fluid concentration
D_T	Thermophoresis diffusion coefficient
D_B	Brownian diffusion coefficient
ρ	Fluid density
n	Flow index
σ	Electrical conductivity of the fluid
pr	Prandtl number

M_1, M_2	Matrix
Le	Lewis number,
N_b	Brownian motion parameter,
N_t	Thermophoresis parameter.
τ_{ij}	stress in the j -direction exerted on a plane perpendicular to the i -axis
f_x, f_y, f_z	Force components in X,Y,Z directions
Π, π	similarity variables
$a, a_1, a_2, A_1, A_2, A_3, A_4$	Parameter of group transformation
Re	Reynolds Number
a, b	Positive constants
G	Group
u, v, w	Velocity component in X, Y and Z directions respectively
x, y, z	Cartesian co-ordinates
u_w, v_w	Flat surface velocity in X and Y directions respectively
T	Temperature of fluid
η	Independent similarity variable
T_w	Surface temperature
T_∞	Ambient temperature
C_∞	Ambient concentration
B	Magnetic field
μ_f	Effective viscosity of the fluid
μ_{nf}	Effective viscosity of the nanofluid
$\rho_f, \rho_s, \rho_{nf}$	Effective density of the fluid, solid particles and nanofluid respectively
$\alpha_f, \alpha_s, \alpha_{nf}$	Thermal diffusivity of the fluid, solid particles and nanofluid respectively
$\mathcal{P}^{x_i}, Q^{x_i}, \mathcal{P}^{y_j}, Q^{y_j}$	Real valued differentiable functions in real argument a .
$H, I, J, L, M, N, O, E, R, Q$	Functions of parameters a_1, a_2

$\vartheta_f, \vartheta_{nf}$	Viscosity of fluid and nanofluid
ϕ	Solid volume fraction of the nanofluid,
k_f, k_s, k_{nf}	Thermal conductivity of the fluid, solid particles and nanofluid respectively
$(\rho c_p)_f, (\rho c_p)_s, (\rho c_p)_{nf}$	Heat capacitance of the fluid, solid particles and nanofluid respectively
h_f	Heat transfer coefficient
h_s	Convective mass transfer coefficient
T_f	Fluid temperature
τ_1	Heat capacitance ratio
$f, g, h, F_1, F_2, F_3, \pi_4, \pi_5, g_1, g_2, g_3, g_4, g_5, \gamma_i$	Functions
ξ	Variable
$m_1, m_2, n_1, n_2, \Lambda, \varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, c_1, c_2, c_3, c_4, c_5, c_6, \alpha_1, \alpha_2, \dots, \alpha_{14}, \beta_1, \beta_2, \dots, \beta_{14}, \lambda_{ij}$	Constants

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Dedicated to my late mother Hanshaben Shukla.....

CHAPTER-1

Introduction

1.1 Brief description on the state of the art of the research topic

The living organisms are surrounded by the fluid and knowingly or unknowingly different fluid characteristics apply to them. Natural fluid flow like air flow, water flow in rivers and oceans, blood flow in vein, etc. are following the principle of Fluid Mechanics.

The rapid advances in Fluid Mechanics started in the 19th Century. An idea of the boundary layer was given by Prandtl (1904) [1] as a very thin layer in the vicinity of the body. He proved, theoretically and practically, significant role of friction in this layer and in the remaining region outside this layer friction can be neglected.

The real fluid is subcategorized as Newtonian and non-Newtonian fluid according to its properties. Bird et. al. [2], Wilkinston [3], Skelland [4], have defined a different model for the different behaviour of a non-Newtonian fluid. The relationship between stress and strain for viscous-inelastic fluids is discussed by Patel and Timol [5].

Newtonian and non-Newtonian fluid flow boundary layer problems are modelled mathematically by a system of non-linear partial differential equations. Similarity methods are powerful systematic techniques to solve nonlinear partial differential equations analytically. The similarity techniques have the most powerful characteristics to reduce independent variables. Applications of the different group-theoretic techniques to find similarity solutions are shown in this research work.

Mathematician Lie [6] is a pioneer of the technique which gives an invariant solution. This technique, known as group-theoretic methods, is the application of modern algebra to solve the non-linear problem which gives invariant solutions. In the last decades, the rapid evolution of the development of a deductive similarity theory based on finite and infinitesimal continuous transformation groups has taken place. This is because of the advantage to reduce one or more than one independent variables from differential equations.

Bluman and Cole [7], Hansen and Na [8], Brikhoff [9] and Morgan [10], Moran and Gajjoli ([11], [12]), Moran and Masherek [13] had explored the theories on similarity solution. The literature survey says that, Brikhoff [9], Morgan [10], Hansen and Na [8], Ames [14], Sheshadri [15] have contributed for a similarity solution of PDEs in the applied field of science using group theory. Most recently Timol and his co-worker [16], [17], [18] had presented a significant contribution to analyse different fluid models in different geometry using group-theoretic similarity methods.

Klamrin [19] contributed to the method for converting BVP into IVP. In the book of Sheshadri and Na [15], a detailed discussion about the application of the group-theoretic method to transform BVP into IVP is explored.

One of the classical similarity methods, is the dimensional analysis method. Buckingham [20] had given a very simple and popular theorem on dimensional analysis known as the Buckingham-pi theorem. But this result has a limitation that, it cannot give suggestion for the form of absolute invariants, which is required for analysis. Moran and Marshek [13] had suggested interesting results for the required form of absolute invariants. The absolute invariants depend on the rank of the associated matrix of independent and physical variables. Limitation of similarity solution is that, it exists only at 90-degree wedge for a non-Newtonian fluid model ([8], [18]). So, for the non-similar solution, one of the simplest methods is local non-similarity introduced by Sparrow, Quack, and Boerner [21]. They have also implemented the same on velocity boundary value problems which are non-similar. Sparrow and Yu [22] had extended and applied the method for thermal boundary layer problem as well.

The concept of flow over stretching surfaces has gained the interest of researchers due to its wide industrial applications. Credit goes to Crane [23] and Skiadas [24] for the study of flow pass through stretching sheets.

The word nanofluid is introduced by Choi [25] for suspension of base fluid and nanoparticles of nanometer size. There are many applications of nanofluid in different areas such as in coolants, brake fluid and as gear lubrication in automobiles application, as industrial cooling, solar devices, cancer drug, etc. [26]. Because of the wide applications of nanofluid many researchers have recently focused on it. Choi et al. [25], Xuan et al. [27] and [28], Wang et al. [29], have attempted on the enhancement of thermal conductivity and augmented heat transfer analysis for the low volume of nanoparticles in nanofluid and have got unimagined success. So, it is also called new generation heat transfer fluid.

1.2 Definition of the Problem

There are various similarity and non-similarity techniques available in the literature to solve non-linear PDEs. Many inventors focused on these techniques to explore theoretical and practical applications in the area of science and engineering. From that versatile, elegant and easily approachable concept, is that of group-theoretical procedure in the solution of non-linear PDEs.

Newtonian and non-Newtonian fluid flow BVPs are generally described in the form of system of PDEs. The non-linear relationship of stress and strain for several fluids is defined as different non-Newtonian fluid models like Powell-Eyring, Prandtl-Eyring, Sisko, Power-law, Reiner-Rivlin fluids, Bingham plastics, Ellis Fluids, Reiner-Philippoff fluid, Williamson fluids, Oldroyd fluid, Rivlin–Ericksen fluids, Walters fluid, Maxwell fluids, etc. Here, Powell-Eyring, Prandtl-Eyring, Sisko, Power-law fluid models are taken into consideration for the analysis in two and/or three dimensions. The humble endeavour to examine Newtonian and non-Newtonian 3-D nanofluid model is successful in this thesis. Several group-theoretic techniques, like Generalised Dimensional Analysis, one and two-parameter deductive group-theoretic methods, are implemented to transform governing non-linear boundary value type PDEs into ODEs. Similarity group-theoretic procedure is utilized to transfer BVP into IVP.

The concept of local non-similarity is implemented for achieving a non-similar solution. The obtained non-similar equations are treated as the type of similarity solutions. The numerical solution of the transformed ODEs is achieved by different software like Maple and MATLAB. Impacts of different physical parameters on fluid flow are depicted graphically for each BVP.

1.3 Objectives

The prime objective of this research is

- to show effectiveness and significance of similarity methods to analyse BVPs arising in engineering field which are formed in the system of nonlinear PDE and to study effects of different physical parameters on fluid flow for the Newtonian and non-Newtonian fluid model.

- to demonstrate utilization of potent and simple technique called as a group-theoretic method to convert BVP into IVP.
- to suggest a non-similarity solution when similar solutions are not possible or to find a solution other than a 90-degree wedge for a non-Newtonian fluid model.
- to obtain a numerical solution of converted ODEs by applying simple Taylor's series method and by using computer software.

1.4 Future scope

- In this study, steady flow boundary value problems using different similarity methods have been considered. In the future, these methods can be applied for time-dependent flow problems.
- In this research work, two and three-dimensional flow in a Cartesian coordinate system has been considered for flat plates and over a linearly stretching surface. The work can be expanded by considering flow over a nonlinearly stretching sheet, over moving boundary for the cylindrical and spherical coordinate system. A two-equation model for a local non-similarity method has been used. It can be extended for three-equation model as well.
- The study is carried out for some of the non-Newtonian fluid models like Powell-Eyring, Prandtl-Eyring, Sisko, Power-law model. A further study can be carried out for other fluid models by constructing appropriate group and invariant solutions.
- Here, one and two-parameter group-theoretic method is applied for bringing a solution to boundary value problems. It can be extended for multiparameter group method.

1.5 Original contribution by the thesis

The contribution of this thesis to the field of similarity and non-similarity methods on the application side is as follows:

The thesis consists of seven chapters. The first chapter concerns with the general introduction of the thesis. The second chapter is the foundation of the whole thesis.

The second chapter includes basic definitions and fundamental principles of Fluid Mechanics, concepts of similarity methods and related results.

In the third chapter of the thesis, the Generalized Dimensional Analysis method is applied to find the similarity solution. The steady, two-dimensional boundary layer flow is analysed for non-Newtonian Sisko fluid.

The non-linear boundary value problem (BVP) is transformed into an initial value problem (IVP) by applying one-parameter linear group method. Simple numerical technique namely Taylor's series is applied for solution of IVP.

In the fourth chapter, two non-Newtonian fluid models namely Powell-Eyring and Prandtl-Eyring fluid are considered for similarity analysis. Generalized group-theoretical method with one parameter is used to obtain a similarity solution for forced convection flow past a semi-infinite flat plate. MATLAB bvp4c solver is used for graphical representation of velocity and temperature profiles for different physical parameters. Also, the comparison for both fluid models is done graphically.

The fifth chapter analyses the three-dimensional Newtonian Nanofluid flow over a flat surface, which is continuously stretching linearly in two lateral directions. Similarity solution is found by reducing two independent variables, using two-parameter deductive group-theoretic method. The impact on velocity and temperature of nanoparticle volume fraction is examined.

In the sixth chapter, the three-dimensional MHD nanofluid flow over linearly stretching sheet, by considering Brownian motion and thermophoresis effect for non-Newtonian Power-law model has been investigated. Similarity solution for three-dimensional MHD flow over linearly stretching sheet under the effect of Brownian motion and thermophoresis are found by applying two-parameter deductive group-theoretic method. The impact of various physical parameters on flow is examined.

The seventh chapter includes the non-similar solution of non-Newtonian fluid for Powell-Eyring model using the local non-similarity method. Non-similar solutions are obtained for different values of power-law index m for free-stream velocity in power-law form $U = x^m$ and streamwise location ξ . Influence of various parameters on velocity and temperature field are represented graphically using MATLAB bvp4c solver.

1.6 Methodology of research and results

Different similarity and non-similarity methods and fluid flow boundary value problems arising in the engineering field were studied as a part of literature survey. Based on the literature survey study, it was concluded that most of the research work was done by

assuming similarity variables. Some advantages and limitations of different similarity methods were also observed. This thesis instead has derived the similarity variables systematically and using the same the PDEs are converted to ODEs.

In this thesis, Generalised Dimensional Analysis method for analysis of a Sisko fluid model is applied to reduce variables. The basic advantage of this method is that, it is very simple to apply and it requires only knowledge of the rank of a dimensional matrix associated with independent and physical variables. It is also useful for a linear change of scale. But as a limitation, sometimes physical variables are reduced instead of independent variables depending on rank. The equation obtained with reduced variable was still BVP. This BVP is converted to IVP using the method of linear group transformations. The transformed IVP is further solved by simple Taylor's series method.

Similarity analysis for Powell-Eyring and Prandtl-Eyring fluid models using the one-parameter deductive group-theoretic method was started with the aim to reduce one independent variable from governing equations. The procedure is systematic and no special form of the group is initially assumed. The appropriate group is deduced and auxiliary conditions are also taken into account for the deduction of the group. Limitation of the method is that if we want to reduce two or more than two variables then we have to repeat the same procedure. So, in order to reduce two or more than two variables, multiparameter group-theoretic method is to be applied.

As a part of this thesis the effect of nanoparticles volume fraction on temperature and velocity profile is investigated for three-dimensional Newtonian nanofluid flow. The flow is over flat surface, stretched continuously in two lateral directions. This flow is investigated by applying two-parameter deductive group-theoretic method. As a result, two independent variables are reduced from governing equations.

The mathematical Newtonian model is further extended to the non-Newtonian mathematical model for analysis. The two-parameter deductive group-theoretic method is implemented to investigate three-dimensional MHD nanofluid flow over linearly stretching sheet by considering Brownian motion and thermophoresis effect for non-Newtonian power-law model.

In this work, the Powell-Eyring model for non-similarity is modified. The local non-similarity method with the two-equation model of Sparrow and Yu is applied to achieve local solutions, which are independent of the upstream information for non-Newtonian fluid.

The general group of transformations is considered and then absolute invariants are gained under which system of nonlinear PDEs are transformed into ODEs. All BVPs under

consideration, governed by a system of nonlinear PDEs, are converted into a system of ODEs using different similarity and non-similarity procedures.

Systems of transformed ODEs are solved numerically by Taylor's series method or using software like Maple and MATLAB. All obtained results are expressed numerical and/or graphical form.

1.7 Achievements with respect to objectives

Newtonian and non-Newtonian fluid flow boundary value problems are studied with governing equations in the form of a system of non-linear PDEs. Similarity solutions using various group-theoretic methods successfully achieved. One interesting application of the similarity group-theoretic method to convert BVP into IVP is successfully demonstrated. An interesting outcome was found by employing the local non-similarity method systematically, for flow other than a 90-degree wedge. The impact of different physical parameters on fluid flow is analysed. The analysis involved numerical and graphical results, which are achieved by software Maple and MATLAB.

1.8 Conclusion

Similarity methods are categorised in two ways

1. Direct methods, which are very simple
2. Group-theoretic methods, which are more elegant and satisfies invariant postulate.

Group-theoretic methods are subcategorized as

(a) the assumed group of transformation and

(b) the deductive group of transformations. Deductive group-theoretic methods are those for which auxiliary conditions are also taken into account for invariance analysis.

- As a part of the conclusion of this thesis, the similarity solution exists for free-stream velocity which is in the form $U = x^m$, where $m = 1/3$ for non-Newtonian models. For other values of m , we have to find a non-similar solution. The local non-similarity method of Sparrow and Yu is very simple and systematic. The obtained non-similar equations can be treated as similar equations consequently.
- The main outcome of the analysis in this thesis, is that nonlinear PDEs are successfully transformed invariantly into ODEs by applying similarity methods.

- A second observation is that software likes Maple and MATLAB are powerful enough to give numerical solution and graphical representation of BVPs.
- The Sisko fluid model analysis in this thesis, earned an outcome that, the velocity increases as the similarity variable increases.
- It is also concluded that, the velocity distribution in Prandtl-Eyring fluid is quite higher than the Eyring Powell fluid.
- It is also observed from the analysis of 3-D Newtonian nanofluid model, that thermal conductivity enhances as the amount of (copper) nanoparticles increases. The velocity, which opposes the trend of thermal conductivity, decreases as amount of (copper) nanoparticles increases, for 3-D Newtonian nanofluid model.
- Concentration and temperature both are affected by Lewis number, Biot number, Thermophoresis parameter, Brownian motion parameter, for 3-D non-Newtonian power-law model.
- Here, non-similar solution for shape other than 90-degree wedge is derived. It is observed that streamwise co-ordinate ξ significantly influences the flow.
- Conclusion of each BVP is given in the concerned chapter of the thesis.

CHAPTER-2

Foundation of the Thesis

2.1 Introduction

Study of Fluid Mechanics provides the knowledge of the behaviour of fluids either they are in motion or at rest under the influence of different forces acting on it. The general definition of the fluid is “A substance that undergoes continually deformation when shear stress or an external force is applied to it.” Out of four states of matter, fluids contain three states of matter liquid, gaseous and plasma.

The three aspects of the fluid are static, kinematics, and dynamics aspects.

- 1) Fluid statics: Fluid statics is the study of the fluid at the rest.
- 2) Fluid kinematics: Fluid kinematics is the study of the moving fluid without considering the effect of external pressures.
- 3) Fluid dynamics: Fluid dynamics is the study of the moving fluid including the effect of all internal as well as external pressures exerting on it.

2.1.1 Importance of the study of Fluid Mechanics

Fluids are very necessary for our day-to-day life. The potential of fluids has been observed in the number of new applications and various functions in Engineering and Technology. Some of these include:

- 1) There are some fluids such as petrol, diesel, kerosene which when burnt, produce lots of heat, and are used for various applications.
- 2) Some fluids have characteristics to exert very high pressure or force. Such fluids like oils are used for lifting various heavy loads. For example, the fluids used in hydraulic machines and hydraulic lifters.

- 3) Various fluids are used for the lubrication of various machines due to their very good flow properties.
- 4) The kinetic and potential energy contained in the water is used for the generation of electricity in hydroelectric power plants.
- 5) The fluids play an important role as refrigerants in refrigerators and air-conditioners.
- 6) Compressed air is used for the operation of various types of equipment and automatic valves.
- 7) Earlier in the design of heat engine, air was used as a fluid to generate power in automobiles, and it was believed that the efficiency of an engine depends on the type of fluid used, but later it was shown by Sadi Carnot, that the efficiency of an engine is not dependent on the type of the fluid, but rather, the temperature of the fluid. Thus, the progress of the human race fully depends on Fluids.

The study area of Fluid Mechanics deals with problems such as surface tension, fluid statics, flow in enclosing bodies or flow around bodies (solid or otherwise), flow stability, etc. For example, various aircraft and rocket engines involve fluid flow fields, reaction forces. The knowledge of Fluid Mechanics is required in process design and for the design of the transport system in the food and chemical process industries. That means almost any reaction of humankind has involved some kind of a Fluid Mechanics problem.

2.2 History of fluid [30]

The real-world situations, like water supply, irrigation, navigation, and water power, have given rise in the development of the study of Fluid Mechanics. The fundamental principles of hydrostatics and dynamics were given by Archimedes in his work on floating bodies around 250 BC which is considered to be as true today on some 2250 years ago. After the fall of the Roman Empire (476 A.D.), there is no record of progress made in Fluid Mechanics until the time of Leonardo da Vinci (1500A.D.), who designed the first chambered canal lock. Leonardo performed many experiments and based on it he developed the theory which contains accurate descriptions of waves, jets, hydraulic jumps and both low and high drag designs.

A Frenchman, Edme Mariotte (1620-184) built the first wind tunnel and tested models in it. Afterward, Galileo, Torricelli, Euler, Newton, Bernoulli family, and D'Alembert has also given a very great contribution to the development of Fluid Mechanics. And it was concluded by D'Alembert that —The theory of fluids must necessarily be

based upon experiment. For example, the concept of ideal liquid that leads to motion with no resistance conflicts with reality. Euler, La Grange, Helmholtz, Kirchhoff, Rayleigh, Rankine, and Kelvin have worked on the theoretical aspects whereas Brahms, Bossut, Chezy, Dubuat, Fabre, Coulomb, Dupuit, d'Aubisson, Hagen, and Poisseuille worked on experimental aspects mainly for pipes and open channels. In the middle of the nineteenth century, Navier and Stokes gave the equation for the fluid in a motion known as Navier-Stokes Equation.

William Froude (1810-1879) and his son Robert (1864-1924) developed laws of model testing; Lord Rayleigh (1842-1919) proposed the technique of dimensional analysis and Osborne Reynolds (1842-1912) published the classic pipe experiments in 1883 which showed the importance of the dimensionless Reynolds number. After him Ludwig (1875-1953) Prandtl gave the idea of the Boundary layer. Thus, Prandtl and his students Blasius, VonKarman, Taylor, Buckingham, and many others, transformed the Fluid Mechanics to today modern science. In the year 1687, Isaac Newton (1642-1727) postulated his laws of motion and the law of viscosity of the linear fluids is called Newtonian.

Fluid Mechanics is a subject to study the fluid behaviour of fluid at rest or in motion. One must have a very good concept regarding the rules and laws of Fluid Mechanics for innovative work. Thus, due to the demanding real-world applications in Fluid Mechanics, it has become the thrust area of research. Due to the introduction of computers, today many problems based on fluid can be analysed by using numerical tools.

2.3 Fluid

A substance in liquid/gas phase is referred to as 'fluid'. The distinction between solid and fluid is made on the basis of the substance's ability to resist an applied shear (tangential) stress that tends to change its shape. A solid can resist an applied shear by deforming its shape whereas a fluid deforms continuously under the influence of shear stress, no matter how small is its applied stress. In solids, stress is proportional to strain, but in fluids, stress is proportional to 'strain rate.'

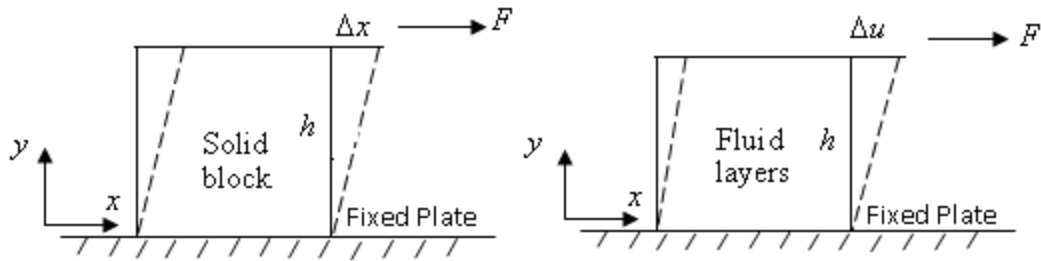


FIGURE 2. 1 Illustration of solid and fluid deformation

[Ref: NPTEL – Mechanical – Principle of Fluid Dynamics]

Generally, fluid is the material that deforms continuously under action of shear stress, regardless of how small this applied stress is. The shear stress is the ratio of the force applied to material and the material's cross-sectional area. Fluids have a property to attain the shape of the container where they are poured. Due to the difference in the physical structure of fluids and solids, the study of fluids is different from that of solids which are shown in Fig 2.1.

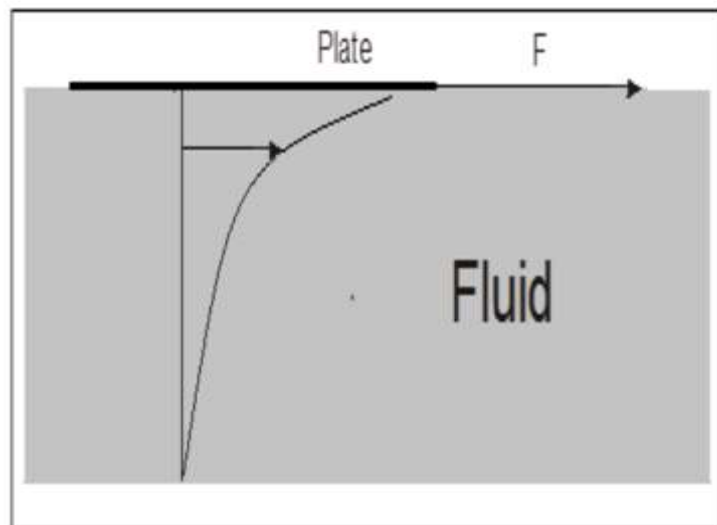


FIGURE 2. 2 Fluid moves with the plate

Figure 2.2 shows the motion of the fluid when a force F is applied to a plate on top of the fluid causing the plate to move. With the movement of the plate, the fluid continues to flow or deform since the fluid next to the plate is under the action of a shear stress equal to the force F divided by the surface area of the plate. When a fluid is at rest means no relative motion between the fluid elements, there can be no shear stress.

2.4 Types of fluid

2.4.1 Ideal fluid

Inviscid and incompressible fluids are assumed as Ideal fluid. Fluids like air and water are considered as an Ideal fluid but actually, there does not exist any Ideal fluid in nature.

2.4.2 Real fluid

A fluid with some viscosity and compressibility is defined as a real fluid.

2.4.3 Newtonian fluid

A fluid which follows the rule known as Newton's law of viscosity is called Newtonian fluid. This rule says that the shear stress is linearly proportional to the shear rate.

$$\text{Means, } \tau = \mu \frac{du}{dy}, \quad (2.1)$$

where τ is shear stress and μ is the coefficient of viscosity.

Air, water, kerosene, thin lubricating oils are examples of Newtonian fluids.

2.4.4 Non-Newtonian fluid

A fluid which does not follow Newton's law of viscosity is called non-Newtonian fluid. That means the shear stress is not linearly proportional to the shear rate.

Face cream, face wash, nail paint, ketchup, polymers are non-Newtonian fluids.

2.5 Types of Non-Newtonian fluid

2.5.1 Shear thinning

A fluid for which the apparent viscosity decreases as increasing stress is called shear thinning or pseudo plastic. Milk, paint, blood, syrup, paper pulp in water are examples of shear thinning fluid.

2.5.2 Shear thickening

A fluid in which the apparent viscosity increases as increasing stress is called shear thickening or dilatant. Concentration of corn starch, concentrated solution of sugar, suspension of rice starch are examples of shear thickening fluid.

2.6 Non-Newtonian fluid models

The fluids under consideration are of the form in which shearing stress tensor τ_{yx} is related to the rate of strain by the arbitrary function of the type

$$F\left(\tau_{yx}, \frac{\partial u}{\partial y}\right) = 0 \quad (2.2)$$

2.6.1 Power law model

Mathematically, Power law model is written as

$$\tau_{yx} = \mathcal{K} \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y}, \quad (2.3)$$

where \mathcal{K} is the fluid characteristic constant and n is power-law index.

2.6.2 Sisko model

Mathematically, Sisko model is written as

$$\tau_{yx} = \mathcal{A} \frac{\partial u}{\partial y} + \mathcal{B} \left| \frac{\partial u}{\partial y} \right|^n, \quad (2.4)$$

where \mathcal{A} and \mathcal{B} are constants characteristic of the model and n is power-law index.

2.6.3 Powell-Eyring model

Mathematically, Powell-Eyring model is written as

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} + \frac{1}{\mathcal{B}} \sinh^{-1} \left(\frac{1}{\mathcal{C}} \frac{\partial u}{\partial y} \right), \quad (2.5)$$

where \mathcal{B} and \mathcal{C} are constants characteristic of the model.

2.6.4 Prandtl-Eyring model

Mathematically, Prandtl-Eyring model is written as

$$\tau_{yx} = B \operatorname{Sinh}^{-1}\left(\frac{1}{C} \frac{\partial u}{\partial y}\right), \quad (2.6)$$

where B and C are constants characteristic of the model.

2.7 Types of fluid flow

2.7.1 Steady flow

The fluid flow is said to be steady flow if its characteristics like velocity (v), pressure (p), density (ρ), etc. remain constant with respect to time at any point in the flow.

That means, $\frac{\partial v}{\partial t} = 0$, $\frac{\partial p}{\partial t} = 0$, $\frac{\partial \rho}{\partial t} = 0$ at fixed point (x_0, y_0, z_0) in fluid flow.

2.7.2 Unsteady flow

The fluid flow is said to be unsteady flow if its characteristics like velocity (v), pressure (p), density (ρ), etc. changes with respect to time at any point in the flow.

That means, $\frac{\partial v}{\partial t} \neq 0$, $\frac{\partial p}{\partial t} \neq 0$, $\frac{\partial \rho}{\partial t} \neq 0$ at fixed point (x_0, y_0, z_0) in fluid flow.

2.7.3 Laminar flow

Laminar flow is that in which fluid particles travel smoothly in a well-defined manner such as one layer over another adjacent layer.

2.7.4 Turbulent flow

Turbulent flow is that in which fluid particles travel in unpredictable paths or fluid travels with fluctuations.

2.7.5 Compressible flow

Compressible flow is that in which the density changes because of variations in pressure and temperature from one point to another point in fluid flow.

2.7.6 Incompressible flow

Incompressible flow is that in which density does not change means constant density for the fluid flow.

2.8 Fluid properties

2.8.1 Viscosity

Viscosity is the property of the fluid that measures the resistance of movement of one layer of fluid over another adjacent layer of fluid.

Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}}, \quad (2.7)$$

where μ is called absolute viscosity or the coefficient of viscosity or dynamic viscosity.

For water, value of μ is 10^{-3} N.s/m^2 and for air it is $0.0181 \times 10^{-3} \text{ N.s/m}^2$.

2.8.2 Density

Density is the ratio of mass with volume.

Mathematically,

$$\rho = \frac{\text{mass of fluid}}{\text{volume of fluid}} \quad (2.8)$$

2.8.3 Kinematic viscosity

Kinematic viscosity is the ratio of dynamic viscosity with density.

$$\vartheta = \frac{\mu}{\rho} \quad (2.9)$$

Fluid properties

For water value of ν is $1 \times 10^{-6} m^2/s$ and for air value of ν is $15 \times 10^{-6} m^2/s$.

2.8.4 Thermal conductivity

The thermal conductivity is a property of material which measures the amount of conducting heat or transferring heat. Means, The material with lower thermal conductivity transfer heat at a lower rate and the material like metals with higher conductivity transfer heat at a high rate.

$$\text{Mathematically, } \kappa = \frac{QL}{A\Delta T}, \quad (2.10)$$

where Q the amount of heat transfer, A the surface area, ΔT the difference in temperature, L length of the surface.

2.8.5 Thermal diffusivity

The ratio of the thermal conductivity to the heat storage capacity of the material is defined as thermal diffusivity. If the thermal diffusivity is higher, then the time taken for the diffusion of heat is less, because of either high thermal conductivity or low heat capacity.

$$\alpha = \frac{\kappa}{\rho c_p}, \quad (2.11)$$

where κ is the thermal conductivity, ρ density, c_p specific heat at constant pressure.

2.8.6 Velocity Boundary layer

Prandtl [1] had introduced the concept of a boundary layer. The Boundary layer is a very thin layer near to surface where the velocity varies from zero at the surface to the free-stream velocity. Viscous effects are not negligible inside the boundary layer. Outside the boundary layer, velocity is constant and the same as free-stream velocity. The fluid is inviscid outside the boundary layer.

2.8.7 Thermal Boundary layer

Heated surface affects the fluid temperature in a thin layer of surface and does not so much affect on fluid temperature to the large distance from the surface. This thin layer is called a thermal boundary layer.

2.8.8 MHD

Magneto hydrodynamics is a study of electrically conducting fluid motion in the presence of a magnetic field.

2.9 Heat Transfer

The process in which the thermal energy transmitted from one region to another region because of the temperature difference is called heat transfer.

The second law of thermodynamics states that heat transfers from the hot region to cold region. There are three modes of heat transfer referred to convection, conduction, and radiation.

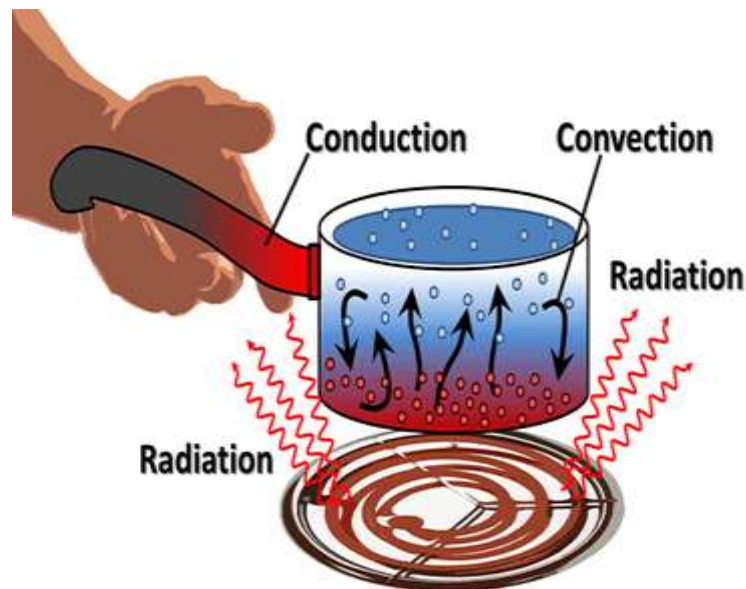


FIGURE 2. 3 Difference between conduction, convection and radiation

[Source:<https://www.machinedesign.com/whats-difference-between/what-s-difference-between-conduction-convection-and-radiation>]

2.9.1 Convection

Convection is the process of heat transfer by the flow of fluid over a solid surface with different temperature of the fluid and solid surface.

Convection is classified in main two types one as free convection and second as forced convection.

2.9.2 Free convection

Free convection is the mechanism of heat transfer in which fluid motion occurs because of density difference due to the temperature difference. In free convection, motion of the fluid is due to buoyancy effects naturally means hot fluid rise and cold fluid fall down.

2.9.3 Forced convection

In this type of heat transfer, fluid motion is occurred by external forces like a fan or pump.

2.9.4 Mixed convection

Mixed convection is a combination of both forced and free convection. For example, if fluid is moving vertically upward along the moment of the vertical stretching sheet is forced convection while in the same phenomena fluid is freely falling due to the gravity which is free convection. When these two phenomena appear in the same model then such kind of flow is mixed convection.

2.9.5 Conduction

This type of energy transport takes place because of the collisions of particles and the movement of free electrons. Conduction occurs in all states of matters like solid, liquid and gas.

2.9.6 Radiation

This type of heat transfer takes place because of transmission or emission of particles or waves in space or a medium.

2.10 Nanofluid

A nanofluid is a suspension of base fluid (conventional fluid) and nanoparticles which are of nanometer-sized. Nanofluid is also called next generation heat transfer fluid because of its characteristics of enhancing heat transfer. Generally, the size of the nanoparticles is less

than 100nm. Generally, water, ethylene glycol, oil is taken as base fluid and metals Cu, Ag, Au; Metallic oxides like Al_2O_3 , CuO; Nitrides like AlN, SiN; Carbides like SiC, TiC; semi-conductors like TiO_2 , SiC; and different types of carbon nanotubes like SWCNT, DWCNT, MWCNT used for nanoparticles.

Application of nanofluids are found in different areas such as coolants, brake fluid and as gear lubrication in automobiles, also in industrial cooling, solar devices, in cancer drug.

2.11 Physical parameters

Here we define physical parameters which we will use in subsequent chapters.

2.11.1 Prandtl number

Prandtl number is the ratio of kinematic viscosity to the thermal diffusivity from which we can compare thickness of velocity boundary layer and thermal boundary layer which is denoted mathematically as,

$$Pr = \frac{\nu}{\alpha}, \quad (2.12)$$

where ν is the kinematic viscosity and the α is the thermal diffusivity.

2.11.2 Biot number

The ratio of resistance of heat transfer inside the body to the surface of body is defined as Biot number.

$$Bi = \frac{hL}{\kappa}, \quad (2.13)$$

Here, h is convective heat transfer, L represents the characteristic length and κ the thermal conductivity of the fluid.

2.11.3 Brownian diffusion coefficient

Brownian diffusion occurs due to continuous collision between the molecules and nanoparticle of the fluid. The Brownian diffusion coefficient D_B is given by

$$D_B = \frac{k_B T C_C}{3\pi\mu d_p}, \quad (2.14)$$

Stream function

where K_B , T , Cc and μ represents Boltzmann constant, temperature, correction factor and viscosity respectively.

2.11.4 Thermophoretic diffusion coefficient

Thermophoresis diffusion occurs when particles diffuse due to the effect of temperature gradient. The thermophoretic diffusion coefficient is given by

$$D_T = \frac{-v_{th}T}{\vartheta\Delta T}, \quad (2.15)$$

where v_{th} , T , ϑ and ΔT denote thermophoretic velocity, temperature, kinematic viscosity and temperature gradient respectively.

2.11.5 Reynolds number

The dimensionless parameter Re is defined as the ratio of inertial forces to viscous forces, i.e.

$$Re = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{\frac{\rho U^2}{L}}{\frac{\mu U}{L^2}} = \frac{UL}{\vartheta}, \quad (2.16)$$

where U , L , ρ and μ are some characteristic values of the velocity, length, density and viscosity of the fluid respectively. It is a parameter for viscosity. If Reynolds number Re is small, the viscous force will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand, if Reynolds number Re is large, the inertial forces will be predominant and in such a case, the effect of viscosity can be considered to be confined in a thin layer, known as velocity boundary layer, adjacent to a solid boundary. However, if Reynolds number Re is very large, the flow ceases to be laminar and becomes turbulent.

2.12 Stream function

The function $\Psi(x, y)$ is a function which satisfies the continuity equation and reduces one dependent variable from governing equation by replacing velocities u and v by $\frac{\partial \Psi}{\partial y}$ and $-\frac{\partial \Psi}{\partial x}$ respectively.

2.13 Conservation laws

Here, the three important conservation laws are defined to obtain the governing equations for analysing fluid flow. The differential equations of flow are derived by considering a differential volume element of fluid and describing mathematically.

- a) The conservation of mass of fluid entering and leaving the control volume; the resulting mass balance is called the equation of continuity.
- b) The conservation of momentum entering and leaving the control volume; this energy balance is called the equation of motion.

2.13.1 Conservation of mass

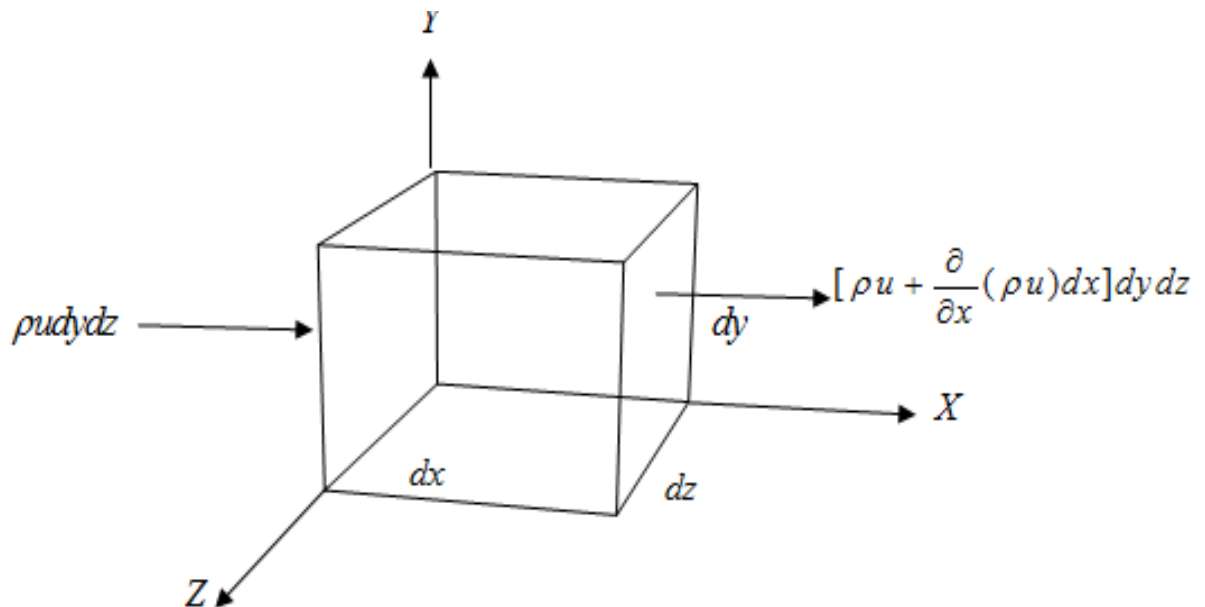


FIGURE 2. 4 Cubical fluid element with volume $dx dy dz$

The principle of mass conservation states that the rate of mass entering the system will always be equal to the mass flow rate leaving the system for a flow that is incompressible. Consider a cubical element having a volume of $dx \cdot dy \cdot dz$ in a flow as shown in the Figure 2.4.

Applying mass conservation principle,

Conservation laws

Mass accumulation rate = Mass flow rate in to the element – Mass flow rate out of the element

Mass accumulation rate in x-direction

= Mass flow rate in to the element in x-direction – Massflow rate out of the element in x-direction

$$= \rho u \, dy \, dz - \left(\rho + \frac{\partial \rho}{\partial x} dx \right) \left(u + \frac{\partial u}{\partial x} dx \right) dy dz \quad (2.16)$$

$$= \rho u \, dy \, dz - \left(\rho u + \frac{\partial(\rho u)}{\partial x} dx \right) dy dz \quad (2.17)$$

$$= - \frac{\partial(\rho u)}{\partial x} dx dy dz \quad (2.18)$$

Similarly, we can find mass accumulation rate in y and z direction as

$$- \frac{\partial(\rho v)}{\partial y} dx dy dz, - \frac{\partial(\rho w)}{\partial z} dx dy dz \text{ respectevily.}$$

$$\text{Total mass accumulation rate} = \frac{\partial(\rho dx dy dz)}{\partial t} \quad (2.19)$$

Mass accumulation rate = Mass flow rate in to the element – Mass flow rate out of the element

$$\frac{\partial(\rho dx dy dz)}{\partial t} = \left(- \frac{\partial(\rho u)}{\partial x} dx dy dz \right) + \left(- \frac{\partial(\rho v)}{\partial y} dx dy dz \right) + \left(- \frac{\partial(\rho w)}{\partial z} dx dy dz \right) \quad (2.20)$$

$$\frac{\partial \rho}{\partial t} dx dy dz = \left(- \frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} \right) dx dy dz \quad (2.21)$$

Thus we obtained continuity equation in differential form as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (2.22)$$

Density ρ is constant for incompressible, steady fluid so,

$$\rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} = 0 \quad (2.23)$$

which gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.24)$$

2.13.2 Conservation of momentum

Here we apply Newton's second law

$$Force = (mass) (acceleration) \quad (2.25)$$

There are two types of forces acts on fluid element in flow. One type is body forces like magnetic, electric and gravitational forces. Other type is surface forces due to the pressure distribution acting on the surface and the shear and normal stress distributions acting on the surface.

Body force per unit mass is denoted by f ; and f_x, f_y, f_z are three components acting on fluid element in x, y and z -direction respectively. The volume of the fluid element is $dx dy dz$.

$$\text{Therefore, body force in the } x\text{-direction on the fluid element} = \rho f_x dx dy dz \quad (2.26)$$

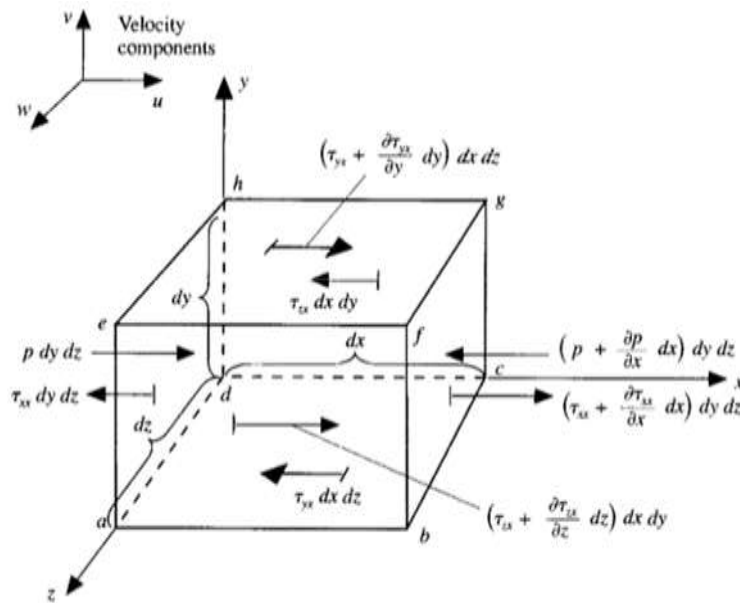


FIGURE 2. 5 Forces acting on fluid element

In above Figure 2.5, forces acting in x -direction are shown. τ_{ij} denotes a stress in the j -direction exerted on a plane perpendicular to the i -axis. On bottom face $abcd$ the shear stress $\tau_{yx} dx dz$ is acting. Face $efgh$ is apart from $abcd$ face at distance dy parallelly. Force acting on face $efgh$ is $\left[\tau_{yx} + \left(\frac{\partial \tau_{yx}}{\partial y} \right) dy \right] dx dz$.

Positive and negative sign of the forces are taken as the direction of axis. Positive increase in velocities occurs in positive directions of axes. On face $efgh$ velocity u is higher than at

face $abcd$. So, the fluid element pulled in positive X -direction for face $efgh$ and on face $abcd$ in negative X -direction. In same manner, directions of all viscous stresses are shown in Figure 2.4.

On face $adhe$, which is perpendicular to the X -axis, the only forces in the X -direction are the pressure forces $pdydz$, which always acts in the direction into the fluid element.

So, from above explanation we can say that,

Net surface force in the X -direction

$$= \left[p - \left(p + \frac{\partial p}{\partial x} dx \right) \right] dydz + \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx \right) - \tau_{xx} \right] dydz \\ + \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) - \tau_{yx} \right] dx dz + \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) - \tau_{zx} \right] dx dy \quad (2.27)$$

The total force in the X -direction = Body forces + Surface forces

$$f_x = \rho f_x dx dy dz - \frac{\partial p}{\partial x} dx dy dz + \frac{\partial \tau_{xx}}{\partial x} dx dy dz + \frac{\partial \tau_{yx}}{\partial y} dy dx dz + \frac{\partial \tau_{zx}}{\partial z} dx dy dz \quad (2.28)$$

The mass of the fluid element is equal to

$$m = \rho dx dy dz \quad (2.29)$$

The acceleration of the fluid element is defined as the time-derivative of its velocity. The symbol a_x is component of acceleration in the X -direction. Since we are following a moving fluid element, this time derivative is given by the substantial derivative. So,

$$a_x = \frac{Du}{Dt} \quad (2.30)$$

$$\text{We know that } f_x = ma_x \quad (2.31)$$

$$(\rho dx dy dz) \frac{Du}{Dt} = \rho f_x dx dy dz - \frac{\partial p}{\partial x} dx dy dz + \frac{\partial \tau_{xx}}{\partial x} dx dy dz + \frac{\partial \tau_{yx}}{\partial y} dx dy dz \\ + \frac{\partial \tau_{zx}}{\partial z} dx dy dz \quad (2.32)$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x, \quad (2.33)$$

which is the X -component of the momentum equation for a viscous flow.

In a similar fashion, the Y and Z components can be obtained as

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \quad (2.34)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \quad (2.35)$$

These three equations (2.32) - (2.34) are component of momentum equation in X , Y and Z -directions respectively.

In a physical sense, the substantial time derivative of a quantity designates its time derivative (i.e. rate of change) evaluated along a path that follows the motion of the fluid. In general terms, the substantial time derivative of a variable F is defined by the following expression:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} \quad (2.36)$$

$$\frac{D(\rho u)}{Dt} = \frac{\partial(\rho u)}{\partial t} + u \frac{\partial(\rho u)}{\partial x} + v \frac{\partial(\rho u)}{\partial y} + w \frac{\partial(\rho u)}{\partial z} \quad (2.37)$$

$$\frac{\partial(\rho u)}{\partial t} + u \frac{\partial(\rho u)}{\partial x} + v \frac{\partial(\rho u)}{\partial y} + w \frac{\partial(\rho u)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \quad (2.38)$$

$$\frac{\partial(\rho u)}{\partial t} = -\left(u \frac{\partial(\rho u)}{\partial x} + v \frac{\partial(\rho u)}{\partial y} + w \frac{\partial(\rho u)}{\partial z}\right) - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \quad (2.39)$$

Thus, we obtain the above equation for the equation of motion in the x -direction of momentum.

For steady flow

$$-\left(u \frac{\partial(\rho u)}{\partial x} + v \frac{\partial(\rho u)}{\partial y} + w \frac{\partial(\rho u)}{\partial z}\right) - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x = 0 \quad (2.40)$$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \quad (2.41)$$

For incompressible, Newtonian fluids at constant viscosity, the general equations of motion can be simplified further by replacing the shear stress functions by the Newton law of viscosity.

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho f_x \quad (2.42)$$

Similarly, we can find for y -and z -components respectively of the momentum equation.

Vector notation of momentum equation:

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{B}, \quad (2.43)$$

where \vec{B} is the body force.

2.14 The concept of similarity

“A symmetry can be defined as the invariance in the pattern that is observed when some transformation is applied to it.”

The Nobel Prize winner Yang (1957) stated the invariance concept of symmetry.

A symmetry, or symmetry transformation of an object is a transformation which leaves the object apparently unchanged (invariant). Meaning of symmetry for differential equation is that under the transformation the differential equations are invariant. These types of transformations form a group then they are called group of transformations.

2.14.1 Similarity transformation

Similarity transformations are those which reduce at least one independent variable from the original partial differential equations.

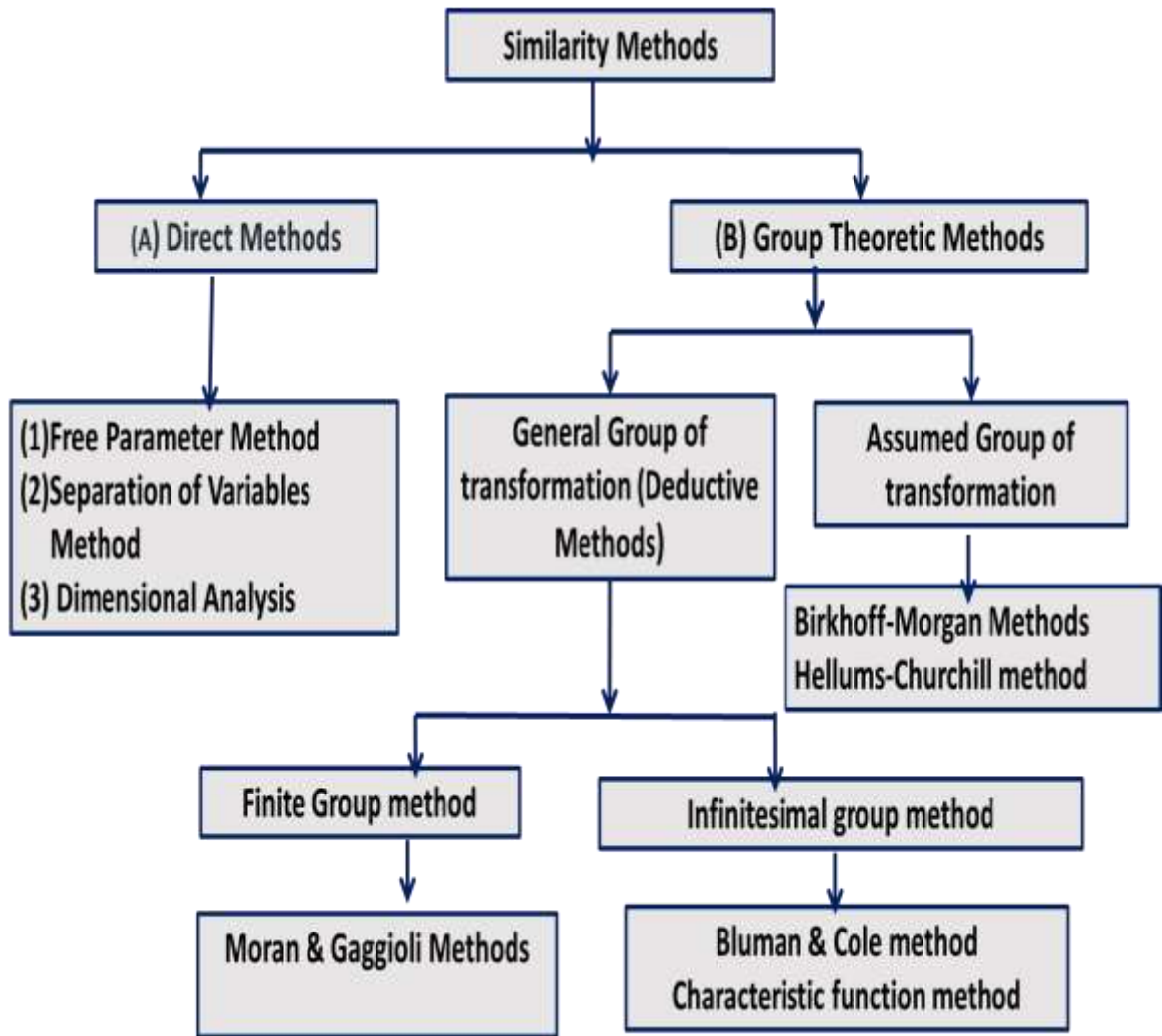
Similarity method: The method for finding such transformation is known as similarity method.

2.14.2 Similarity solution

The solution obtained by applying similarity transformations is called similarity solution.

2.14.3 Different similarity methods

There are different effective similarity methods available in literatures which are classified as follows:



2.15 Group

The set of elements G which is nonempty and satisfies following four axioms under the operation $*$ is called Group.

1 Closure Property:

The set is closed, that is, if b_1 and b_2 are two elements of the set G , then $b_1 * b_2 \in G$

2 Existence of Identity:

There exists an identity element e such that $b_1 * e = e * b_1 = b_1$

3 Existence of Inverse:

Every element b_1 in G has an inverse, say b_1^{-1} in G such that $b_1 * b_1^{-1} = b_1^{-1} * b_1 = e$

4 Associative Law:

The operation $*$ is associative. That is $b_1 * (b_2 * b_3) = (b_1 * b_2) * b_3$ for all $b_1, b_2, b_3 \in G$

2.15.1 Transformation Groups

Consider the continuous set of functions in x_i and a_j both variables denoted by $f_i(x_1, x_2, \dots, x_n; a_1, \dots, a_r), (i = 1, \dots, n)$ with continuous derivatives of functions. Here, parameters of functions f_i are a_1, \dots, a_r . This a_1, \dots, a_r are called essential parameter if there does not exist $(r - 1)$ functions $\alpha_j: \alpha_1(a), \dots, \alpha_r(a)$ such that

$$f_i(x_1, x_2, \dots, x_n; a_1, \dots, a_r) = F_i(x_1, x_2, \dots, x_n; \alpha_1, \dots, \alpha_{r-1}). \quad (2.44)$$

If the a_j are not essential parameters, it means that fewer parameters can be constructed from the a_j to serve the same purpose in a function.

Considering f_i as a set of transformations which depends on the parameters a_1, \dots, a_r , and transform a point (x_1, x_2, \dots, x_m) into $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m)$

$$\text{i.e. } \bar{x}_i = f_i(x_1, x_2, \dots, x_n; a_1, \dots, a_r) \quad (2.45)$$

Successive transformations employing various set of functions are considered as the operation of set and this operation if satisfy all four axioms of group (described above) is known as transformation groups.

For an inverse of $f_i(x_1, x_2, \dots, x_n; a_1, \dots, a_r)$, the Jacobian $\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} \neq 0$ in neighbourhood of (x_1, x_2, \dots, x_n) then an inverse transformation $f_i^{-1}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m; a_1, \dots, a_r)$ exists such that

$$x_i = f_i^{-1}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m; a_1, \dots, a_r) \quad (2.46)$$

For a particular set of a_j , say, a_{11}, \dots, a_{1r} transformations

$$\bar{x}_i = f(x_1, x_2, \dots, x_n; a_{11}, \dots, a_{1r}) \text{ is written as } T_{a_1} x = \bar{x}.$$

Thus, if $\{a_{21}, \dots, a_{2r}\}$ is a set of values distinct from $\{a_{11}, \dots, a_{1r}\}$, we consider T_{a_1} and T_{a_2} to be different transformations.

Now the conditions are,

1. The set is closed.

2. There exists an identity transformation such that $IT_{a_1} x = T_{a_1} I x = T_{a_1} x$

3. Given any transformation T_{a_1} , an inverse transformation $T_{a_1}^{-1}$, exist such that

$$T_{a_1} T_{a_1}^{-1} x = T_{a_1}^{-1} T_{a_1} x = I x = x$$

4. The associative law is true, i.e., $(T_{a_1}(T_{a_2}T_{a_3}))x = ((T_{a_1}T_{a_2})T_{a_3})x$

Example of linear transformations forms group

$$\text{Define } T_a X = \begin{cases} a^{\alpha_1} x \\ a^{\alpha_2} y \end{cases}, \text{ where } X = \begin{cases} x \\ y \end{cases} \quad (2.47)$$

Take three transformations $T_{a_1}X, T_{a_2}X, T_{a_3}X$ defined as

$$T_{a_1}X = \begin{cases} a_1^{\alpha_1} x \\ a_1^{\alpha_2} y \end{cases} \text{ and } T_{a_2}X = \begin{cases} a_2^{\alpha_1} x \\ a_2^{\alpha_2} y \end{cases} \text{ and } T_{a_3}X = \begin{cases} a_3^{\alpha_1} x \\ a_3^{\alpha_2} y \end{cases} \quad (2.48)$$

1 Closure Property:

$$\begin{aligned} & T_{a_2}T_{a_1}X \\ &= T_{a_2}(T_{a_1}X) \\ &= T_{a_2} \begin{cases} a_1^{\alpha_1} x \\ a_1^{\alpha_2} y \end{cases} \\ &= \begin{cases} a_2^{\alpha_1} a_1^{\alpha_1} x \\ a_2^{\alpha_2} a_1^{\alpha_2} y \end{cases} \\ &= \begin{cases} (a_2 a_1)^{\alpha_1} x = a_4^{\alpha_1} x \\ (a_2 a_1)^{\alpha_2} y = a_4^{\alpha_2} y \end{cases}, \text{ where } a_4 = a_2 a_1 \end{aligned}$$

2 Existence of Identity:

$$\begin{aligned} & \text{Take } I_a X = \begin{cases} a^0 x \\ a^0 y \end{cases} \\ & I_a T_a X = \begin{cases} a^0 a^{\alpha_1} x = a^{\alpha_1} x \\ a^0 a^{\alpha_2} y = a^{\alpha_2} y \end{cases} = T_a X \end{aligned}$$

3 Existence of Inverse:

$$\begin{aligned} & \text{Define } T_a^{-1} X = \begin{cases} a^{-\alpha_1} x \\ a^{-\alpha_2} y \end{cases} \\ & T_a^{-1} T_a X \\ &= T_a^{-1} \begin{cases} a^{\alpha_1} x \\ a^{\alpha_2} y \end{cases} \\ &= \begin{cases} a^{-\alpha_1} a^{\alpha_1} x \\ a^{-\alpha_2} a^{\alpha_2} y \end{cases} \\ &= \begin{cases} a^0 x \\ a^0 y \end{cases} \end{aligned}$$

Group

$$= I_a X$$

4 Associative Law:

$$\begin{aligned} & T_{a_3}(T_{a_2}T_{a_1}X) \\ &= T_{a_3} \left\{ \begin{array}{l} (a_2 a_1)^{\alpha_1} x \\ (a_2 a_1)^{\alpha_2} y \end{array} \right. \\ &= \left\{ \begin{array}{l} (a_3 a_2 a_1)^{\alpha_1} x \\ (a_3 a_2 a_1)^{\alpha_2} y \end{array} \right. \\ &= \left\{ \begin{array}{l} (a_3 a_2)^{\alpha_1} a_1^{\alpha_1} x \\ (a_3 a_2)^{\alpha_2} a_1^{\alpha_2} y \end{array} \right. \\ &= (T_{a_3} T_{a_2}) T_{a_1} X \end{aligned}$$

2.15.2 The Concept of an Infinitesimal Transformation

A group is said to be continuous if its elements are identified by a set of continuous parameters. Here, a set of transformation $\bar{x} = \varphi(x, y, a)$; $\bar{y} = \Omega(x, y, a)$ is a one-parameter continuous group of transformation.

Since φ and Ω are continuous functions, the transformation can be written as

$$\bar{x} = \varphi(x, y, a_0 + \epsilon); \bar{y} = \Omega(x, y, a_0 + \epsilon) \quad (2.49)$$

where, ϵ is an infinitesimal quantity which changes x and y by an infinitesimal amount and is defined as an infinitesimal transformation and a_0 is the value of the parameter corresponding to the identical transformation, i.e.

$$\bar{x} = \varphi(x, y, a_0) = x; \quad \bar{y} = \Omega(x, y, a_0) = y$$

Taylor's series expansion of equation (2.49) is

$$\bar{x} = \varphi(x, y, a_0) + \frac{\epsilon}{1!} \left(\frac{\partial \varphi}{\partial a} \right)_{a_0} + \frac{\epsilon^2}{2!} \left(\frac{\partial^2 \varphi}{\partial a^2} \right)_{a_0} + \dots \quad (2.50 \text{ a})$$

$$\bar{y} = \Omega(x, y, a_0) + \frac{\epsilon}{1!} \left(\frac{\partial \Omega}{\partial a} \right)_{a_0} + \frac{\epsilon^2}{2!} \left(\frac{\partial^2 \Omega}{\partial a^2} \right)_{a_0} + \dots \quad (2.50 \text{ b})$$

Since ϵ is an infinitesimal quantity, equation (2.50) then becomes

$$\bar{x} = x + \epsilon \xi(x, y) + o(\epsilon^2), \quad \bar{y} = y + \epsilon \eta(x, y) + o(\epsilon^2), \quad (2.51)$$

$$\text{where } \xi(x, y) = \left(\frac{\partial \varphi}{\partial a} \right)_{a_0}, \quad \eta(x, y) = \left(\frac{\partial \Omega}{\partial a} \right)_{a_0} \quad (2.52)$$

A given function $f(x, y)$ would be changed to $f(\bar{x}, \bar{y})$ if it is subjected to the infinitesimal transformation defined by equation (2.51). Expanding in Taylor series, $f(\bar{x}, \bar{y})$ becomes

$$f(\bar{x}, \bar{y}) = f(x + \epsilon\xi, y + \epsilon\eta) = f(x, y) + \frac{\epsilon}{1!} \mathfrak{U}f + \frac{\epsilon^2}{2!} \mathfrak{U}^2f + \dots \quad (2.53)$$

where, $\mathfrak{U}f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}$ (2.54)

and $\mathfrak{U}^n f$ represents repeating the operator n times.

2.15.3 The Concept of Invariance

A function $f(x, y)$ is said to be invariant under the infinitesimal transformation defined by equation (2.51), if it is unaltered by the transformation,

i.e., $f(\bar{x}, \bar{y}) = f(x, y)$ (2.55)

Equation (2.53) shows that (2.55) will be satisfied if $\mathfrak{U}f, \mathfrak{U}^2f, \mathfrak{U}^3f, \dots$ etc. is simultaneously equal to zero. However, since $\mathfrak{U}^2f = \mathfrak{U}(\mathfrak{U}f)$ and $\mathfrak{U}^3f = \mathfrak{U}(\mathfrak{U}^2f)$ and it follows that the condition

$$\mathfrak{U}f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0 \quad (2.56)$$

is both necessary and sufficient requirement for invariance of $f(x, y)$.

Now, the “invariant function” under a given group of transformation represented by $\mathfrak{U}f$ can be solved from equation (2.56). From elementary theories of partial differential equations, we have

$$\frac{dx}{\xi} = \frac{dy}{\eta} \quad (2.57)$$

The solution to this equation is given by $\mathcal{G}(x, y) = \text{constant}$, which is the required invariant function corresponding to an operator \mathfrak{U} . Since equation (2.57) has only one independent solution depending on a single arbitrary constant, a one-parameter group of transformations in two variables has one and only one independent invariant.

2.15.4 Different group transformations

- Linear: $\bar{x} = a^{\alpha_1}x, \bar{y} = a^{\alpha_2}y$, where a is a parameter.
- Spiral group of transformation: $\bar{x} = x + \alpha_1 a, \bar{y} = ye^{\alpha_2 a}$
- Translation group of transformation: $\bar{x} = x + \alpha_1 a, \bar{y} = y + \alpha_2 a$

- Scaling group of transformation: $\bar{x} = e^{a\alpha_1}x$, $\bar{y} = e^{a\alpha_2}y$

2.16 The similarity method of Morgan

The general theory of Morgan [10] and Michal [31] for developing similarity solutions of partial differential equations was succinctly discussed by Ames [14]. This significant contribution to applied mathematics employed one-parameter continuous groups of transformations. Later extensions by Manohar [32] to special forms of n -parameter groups are also found by Ames [14]. In this section, the basic definitions and theorems of Morgan [10] are presented. Consider the one-parameter groups S , G and E_k of the form

$$E_k: \begin{cases} G = \begin{cases} S: \{\bar{x}_i = f_i(x_1, x_2, \dots, x_m; a); i = 1, \dots, m, m \geq 2 \\ \bar{y}_j = f_j(y_j; a); j = 1, \dots, n, n \geq 1 \end{cases} \\ \left(\frac{\partial^l y_j}{\partial(x_1)^{l_1} \dots \partial(x_m)^{l_m}} \right) = f^{(l)}_{(j; l_1, \dots, l_m)} \left(\frac{\partial^l y_j}{\partial(x_1)^{l_1} \dots \partial(x_1)^{l_1-1} \partial(x_2)^{l_2} \dots \partial(x_m)^{l_m} \partial(x_1)^{l_1-1} \dots y_j, x_1, \dots, x_m; a} \right), \end{cases} \quad (2.58)$$

$\sum_i l_i = l \leq k$, where, the functions are continuous in the parameter a .

The identity element is denoted by a_0 , thus, e.g., $x_i = f_i(x_1, x_2, \dots, x_m; a_0)$.

The value of the parameter for the transformation inverse to that given by a is denoted by \bar{a} ; thus, e.g. if $\bar{y}_j = f_j(y_j; a)$, then $y_j = f_j(\bar{y}_j; \bar{a})$.

The transformations $x_i \rightarrow \bar{x}_i$ from a subgroups S of G .

Given any set of function $\{y_j\}$ in class C^K such that $y_j = y_j(x_1, \dots, x_m)$ be differentiable in x_i up to order k and append to the transformations of G , the transformations of the partial derivatives of the y_j with respect to the x_i .

Consider the set of function $\{\bar{y}_j\}$ defined by

$$\bar{y}_j(\bar{x}_1, \dots, \bar{x}_j) = f_j[y_j(f_1(\bar{x}_1, \dots, \bar{x}_m; \bar{a}), \dots, f_m(\bar{x}_1, \dots, \bar{x}_m; \bar{a})); a] \quad (2.59)$$

E_K is continuous group, called the k th enlargement of G for $\{y_j\}$, where the functions are defined so that

$$\overline{\left(\frac{\partial^l y_j}{\partial(x_1)^{l_1} \dots \partial(x_m)^{l_m}} \right)} = \frac{\partial^l \bar{y}_j}{\partial(\bar{x}_1)^{l_1} \dots \partial(\bar{x}_m)^{l_m}} \quad (2.60)$$

By elementary group theory (Eisenhart [33]), G has $m + n - 1$ functionally independent absolute invariant:

$$\eta_r(x_1, \dots, x_m) = \eta_r(\overline{x_1}, \dots, \overline{x_m}) \text{ for } r = 1, \dots, m - 1 \quad (2.61)$$

$$g_j(y_1, \dots, y_n, x_1, \dots, x_m) = g_j(\overline{y_1}, \dots, \overline{y_n}, \overline{x_1}, \dots, \overline{x_m})$$

The g_j can be selected so that the Jacobian $\frac{\partial(g_1, \dots, g_n)}{\partial(y_1, \dots, y_n)} \neq 0$ and only such g_j will be considered here and the rank of the Jacobian matrix is $\frac{\partial(\eta_1, \dots, \eta_{m-1})}{\partial((x_1, \dots, x_m))}$ equal to $m - 1$.

2.16.1 The concept of constant conformally invariant and absolutely invariant

Assuming that y_j are the dependent variables and x_i are independent variables in the one-parameter group of the transformation G the system of partial differential equations

$$\phi_i \left(x_1, \dots, x_m, y_1, \dots, y_n, \frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial^k y_n}{\partial (x_m)^k} \right) = 0 \quad (2.62)$$

is invariant under this group of transformation, G , if each of the ϕ_i is “conformally invariant” under the transformation E_k , (the enlarged group). This means that

$$\phi_i(\overline{z_1}, \overline{z_2}, \dots, \overline{z_p}) = F_i(z_1, z_2, \dots, z_p; a) \phi_i(z_1, z_2, \dots, z_p)$$

where the arguments z_1, z_2, \dots, z_p are the variables x_1, \dots, x_m , functions y_1, \dots, y_n and the derivatives thereof up to the k th order.

If $F_i = f(a)$, ϕ_i is said to be constant conformally invariant under G ,

If $F_i = f(a) = 1$, ϕ_i is said to be absolutely invariant under this group of transformation.

2.16.2 Morgan’s theorem

Suppose that the forms ϕ_i are conformally invariant under the group E_k , then the invariant solutions of $\phi_i = 0$ can be expressed in terms of the solutions of a new system of PDEs.

$$\phi_i(\eta_1, \dots, \eta_{m-1}; F_1, \dots, F_n; \dots, \frac{\partial^k F_n}{\partial (\eta_{m-1})^k}) = 0 \quad (2.63)$$

The η_i are the absolute invariants of the subgroup of transformation on the z_i alone and the variables F_i are such that

$$F_j(\eta_1, \dots, \eta_{m-1}) = \text{functions of } (z_1, \dots, z_m) \quad (2.64)$$

CHAPTER-3

Similarity Analysis of Non-Newtonian Sisko Fluid Boundary Value Problem to Initial Value Problem

3.1 Introduction

The viscosity is the most important characteristic of fluid using which we can classify fluids as Newtonian and non-Newtonian types. Because of its global applications, scientists are keenly interested in non-Newtonian fluids. There are various non-Newtonian fluid models defined in literature based on its different properties. Different fluid models are addressed by Manisha et al. ([5], [34])

Boundary value flow problems of Newtonian fluid are easy to analyse as compared to non-Newtonian fluids. So, limited work is done on non-Newtonian fluids, most of which comprises of Power-law fluid model. Pakedemirli [35], Djukic [36], Kapur [37], Na and Hansen [38], Nagler [39], Patel et al ([5], [40], [41]) have contributed to the analysis of Power-law fluid model.

The Sisko fluid model suggested by Sisko [42], is a combination of the Newtonian and Power-law fluid model. Lubricating greases, cement slurries, polymeric suspensions are examples of Sisko fluids which are utilized in many industries. Sisko fluid flow between two circular parallel disks is studied by Na and Hansen [38]. Gozde and Pakdemirli [43] had analysed the flow of Sisko fluid by Lie group method. Olanrewaju and his co-worker [44] have studied the unsteady free convective flow of Sisko fluid with radiative heat transfer. Manisha Patel [45] had studied Sisko fluid flow for laminar boundary layer by applying a scaling group of transformations with one-parameter.

Dimensional analysis concept for ‘Mathematical Similarity’ is introduced by Helmholtz [46] for solving partial differential equations. The Russian mathematician S. Lie [6] at the same time had presented a famous Lie algorithm based on the continuous group transformations for the same type of simplification of partial differential equations.

Brikhoff’s [9] first discussed the application of group-theoretical technique on Dimensional Analysis technique for engineering issues. Ovsjannikov [47], Moran [48], Morgan [10] led to this method later on. The method of Generalized Dimensional Analysis established by Moran [48] is systematically applied on a dimensional analysis by applying the group-theoretical technique.

Three significant findings have been obtained by M. J. Moran and K. M. Marshek [13] to suggest the type of necessary absolute invariants using classical Dimensional Analysis combined with the group-theoretical method. The classical Dimensional Analysis was expanded to Generalized Dimensional Analysis. Moran criteria are limited to only Newtonian fluid [49]. M.J. Moran ([13], [48]) concluded, using diffusion equation and Bausius problem, that the outcome gained via Generalized Dimensional Analysis are more generalized than those gained through conventional Dimensional Analysis. This method was later expanded for non-Newtonian fluids also by Timol and his co-worker [50].

In this chapter, we have studied boundary value problem associated with most valuable non-Newtonian fluid model namely Sisko fluid model. The governing partial differential equations are solved for similarity using Generalised Dimensional Analysis method to convert into differential equations with only one independent variable. The associated auxiliary conditions are boundary conditions and we cannot solve these types of problems using well established numerical methods. So, we have applied linear group of transformations to convert this BVP into IVP. In this chapter, we numerically analyse Newtonian fluid model, Power-law fluid model and Sisko fluid model by Taylor’s series method.

3.2 Similarity Method: Dimensional Analysis

There are two methods of dimensions known as Rayleigh’s Method and Buckingham’s pi theorem. Using Rayleigh’s method, we can express a variable which depends on maximum three or four variable. This maximum dependability of variables is the restriction on the method. The limitation of Rayleigh’s method is eradicated in Buckingham’s pi theorem where there is no restriction on the dependency of number of variables.

Buckingham's pi theorem:

If there are n variables (independent and dependent variable) in a physical phenomenon and if these variables contain m fundamental dimensions, then the variables are arranged into $(n - m)$ dimensionless terms. Each term is called pi term.

Consider n variables $X_1, X_2, X_3, \dots, X_n$ which are concerned with a physical problem under consideration from which X_1 is dependent and X_2, X_3, \dots, X_n are independent.

$$X_1 = f(X_2, X_3, \dots, X_n) \tag{3.1a}$$

Or

$$g(X_1, X_2, X_3, \dots, X_n) = 0 \tag{3.1b}$$

If these variables have m fundamental dimensions then as per Buckingham's pi theorem, equation (3.1b) can be written in the following form with $(n - m)$ dimensionless pi terms,

$$g(\pi_1, \pi_2, \pi_3, \dots, \pi_{n-m}) = 0 \tag{3.2}$$

Generally, we are taking M, L, T as fundamental dimensions.

The number of fundamental dimensions or we can say repeating variables are three.

Take X_2, X_3, X_4 as repeating variables then we get

$$\pi_1 = X_2^{h_1} X_3^{f_1} X_4^{g_1} X_1 \tag{3.3}$$

$$\pi_2 = X_2^{h_2} X_3^{f_2} X_4^{g_2} X_5 \tag{3.4}$$

$$\pi_{n-m} = X_2^{h_{n-m}} X_3^{f_{n-m}} X_4^{g_{n-m}} X_n \tag{3.5}$$

3.3 Dimensional similarity analysis [13]

Consider r -parameter group of transformations

$$\begin{cases} \bar{z}_j = A_1^{a_{j1}} \dots A_r^{a_{jr}} z_j & (j = 1, \dots, n) & \text{Dependent variable} \\ \bar{x}_k = A_1^{b_{k1}} \dots A_r^{b_{kr}} x_k & (k = 1, \dots, m) & \text{Independent variable} \\ \bar{y}_e = A_1^{c_{e1}} \dots A_r^{c_{er}} y_e & (e = 1, \dots, p) & \text{physical variable} \end{cases} \tag{3.6}$$

Theorem 3.1: If the function I_j is invariant in form, under r -parameter group, so $z_j = I_j(x_1, \dots, x_m; y_1, \dots, y_p)$ transforms to $\bar{z}_j = \bar{I}_j(\bar{x}_1, \dots, \bar{x}_m; \bar{y}_1, \dots, \bar{y}_p)$ then z_j is equivalent to a relationship in fewer variables .

$$\Pi_j(z_j, x_1, \dots, x_m; y_1, \dots, y_p) =$$

$$F_j\left(\pi_1(x_1, \dots, x_m; y_1, \dots, y_p), \dots, \pi_3(x_1, \dots, x_m; y_1, \dots, y_p)\right)$$

here in $\mathfrak{z} = [m + p - r] > 0$ and $\{\Pi_j, \pi_1, \dots, \pi_3\}$ are independent absolute invariants of the above group.

The statement of Theorem 3.1 does not suggest a preferred form for the absolute invariants. The rank of dimensional matrix M_1 associated with independent and physical variables is r and s for matrix M_2 , which is associated with physical variables, is required to be determined.

Theorem 3.2: If and only if $r = s$, the set of $[n + m + p - r]$ independent absolute invariants required by Theorem 3.1 may be obtained in the following form

$$\Pi_j = z_j [y_1]^{\lambda_{j1}} \dots [y_r]^{\lambda_{jr}} \quad (j = 1, \dots, n)$$

$$\widehat{\pi}_k = x_k [y_1]^{\nu_{k1}} \dots [y_r]^{\nu_{kr}} \quad (k = 1, \dots, m)$$

$$\pi_\eta = y_\eta [y_1]^{\delta_{\eta 1}} \dots [y_r]^{\delta_{\eta r}} \quad (\eta = [r + 1], \dots, p)$$

Theorem 3.3: If and only if $r > s$, the set of $[n + m + p - r]$ independent absolute invariants required by Theorem 3.1 may be obtained in the following form

$$\Pi_j = z_j [x_\varepsilon]^{A_{j\varepsilon}} \dots [x_m]^{A_{jm}} [y_1]^{\lambda_{j1}} \dots [y_s]^{\lambda_{js}} \quad (j = 1, \dots, n)$$

$$\widehat{\pi}_\delta = x_\delta [x_\varepsilon]^{\Gamma_{\delta\varepsilon}} \dots [x_m]^{\Gamma_{\delta m}} [y_1]^{\nu_{\delta 1}} \dots [y_s]^{\nu_{\delta s}} \quad (\delta = 1, \dots, [m + s - r])$$

$$\pi_\eta = y_\eta [y_1]^{\delta_{\eta 1}} \dots [y_r]^{\delta_{\eta r}} \quad (\eta = ([s + 1], \dots, p))$$

where in $\varepsilon = [m + s - r + 1] \leq m$.

3.4 Boundary Value Problem of Sisko Fluid

Here, Sisko fluid flow in two-dimension X and Y past a semi-infinite flat plate for laminar boundary layer is considered. The governing equations for this boundary layer flow which consist of continuity and momentum equations are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.7)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{\partial}{\partial y} \left\{ \left[\mathcal{A} + \mathcal{B} \left(\frac{\partial u}{\partial y} \right)^{n-1} \right] \frac{\partial u}{\partial y} \right\} \quad (3.8)$$

Boundary conditions are as follows:

$$y = 0 : u = 0, v = 0 ; y = \infty : u = U(x) \quad (3.9)$$

where parameter of the Sisko fluid are \mathcal{A} and \mathcal{B} .

We can reduce one dependent variable from governing equations by defining stream function $\psi(x, y)$, which satisfies equation (3.7).

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (3.10)$$

Equation (2) is transformed in following equation with associated boundary conditions.

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U \frac{dU}{dx} + \frac{\partial}{\partial y} \left\{ \left[\mathcal{A} + \mathcal{B} \left(\frac{\partial^2 \psi}{\partial y^2} \right)^{n-1} \right] \frac{\partial^2 \psi}{\partial y^2} \right\} \quad (3.11)$$

Subject to the boundary conditions:

$$y = 0 \Rightarrow \frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0$$

$$y = \infty \Rightarrow \frac{\partial \psi}{\partial y} = U(x) \quad (3.12)$$

3.5 Application of Generalized Dimensional Analysis Method

Consider the following group of transformation:

$$G: \begin{cases} \bar{y} = A_2 y; \bar{x} = A_3 x & (\text{Independent variables}) \\ \bar{\psi} = A_1 \psi & (\text{Dependent variable}) \\ \bar{U} = A_4 U & (\text{Physical variable}) \end{cases} \quad (3.13)$$

where group parameters are A_1, A_2, A_3, A_4 .

Equations (3.11) and (3.12) remain invariant under group of transformations defined by G in equation (3.13).

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y} \partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = \bar{U} \frac{d\bar{U}}{d\bar{x}} + \frac{\partial}{\partial \bar{y}} \left\{ \left[\mathcal{A} + \mathcal{B} \left(\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} \right)^{n-1} \right] \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} \right\} \quad (3.14)$$

$$\begin{aligned} & A_1 A_2^{-1} A_1 A_2^{-1} A_3^{-1} \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - A_1 A_3^{-1} A_1 A_2^{-2} \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} \\ &= A_4 A_4 A_3^{-1} U \frac{dU}{dx} + \frac{\partial}{\partial y} \left\{ \left[\mathcal{A} + \mathcal{B} \left(A_1 A_2^{-2} \frac{\partial^2 \psi}{\partial y^2} \right)^{n-1} \right] A_1 A_2^{-2} \frac{\partial^2 \psi}{\partial y^2} \right\} A_2^{-1} \end{aligned} \quad (3.15)$$

Equation (3.11) and (3.12) remains invariant under group of transformations defined by G in equation (3.13) provided A_i 's have following relations.

$$A_1^2 A_2^{-2} A_3^{-1} = A_4^2 A_3^{-1} \text{ and } A_1^2 A_2^{-2} A_3^{-1} = A_1 A_2^{-3} \quad (3.16)$$

$$A_1^1 A_2^{-1} = A_4 \text{ and } A_1^1 A_2^1 = A_3 \quad (3.17)$$

Two-parameter group of transformations

$$G: \begin{cases} \bar{y} = A_1^0 A_2^1 y; \bar{x} = A_1^1 A_2^1 x & (\text{Independent variables}) \\ \bar{\psi} = A_1^1 A_2^0 \psi & (\text{Dependent variable}) \\ \bar{U} = A_1^1 A_2^{-1} U & (\text{Physical variable}) \end{cases} \quad (3.18)$$

Now it is necessary to identify the rank of dimensional matrix M_1 connected with independent and physical variables and of matrix M_2 connected with physical variables, according to the pi theorem, reported by Moran and Murshek [13].

The associated dimensional matrix will be

$$M_1: \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ where last row corresponds matrix } M_2.$$

Rank of matrix M_1 is two and of matrix M_2 is one. ($r = 2 > s = 1$)

So, by theorem 3.3 stated by Moran and Murshek [13], $n + m + p - r = 2$ absolute invariants exist in the following form

$$\Pi = \psi[x]^{m_1}[U]^{m_2} \text{ and } \pi = y[x]^{n_1}[U]^{n_2} \quad (3.19)$$

Solving corresponding equations of (3.19) as follows.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + m_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + m_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.20)$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + n_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + n_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.21)$$

Exponents $m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2}, n_1 = -\frac{1}{2}, n_2 = \frac{1}{2}$ are obtained.

So, we have similarity transformations

$$\eta = yx^{-\frac{1}{2}}U^{\frac{1}{2}} \quad (3.22)$$

$$f(\eta) = \psi x^{-\frac{1}{2}}U^{-\frac{1}{2}} \quad (3.23)$$

It should be noted that similarity solutions exist only for the flow past 90-degree wedge for this kind of non-Newtonian fluids.

$$\text{Take } U = x^{\frac{1}{3}} \quad (3.24)$$

Using chain rule of differentiation for transformations given in equations (3.22), (3.23), (3.24), equations (3.11), (3.12) are transformed into ODE in the following form

$$f'^2(\eta) - 2f(\eta)f''(\eta) - 3\mathcal{A}f'''(\eta) - 3\mathcal{B}n(f''(\eta))^{n-1}f'''(\eta) - 1 = 0 \quad (3.25)$$

$$\text{with boundary conditions } f(0) = 0, f'(0) = 0 \text{ and } f'(\infty) = 1 \quad (3.26)$$

3.6 Reduction of BVP to IVP

3.6.1 Case(i) Newtonian fluid model

Put $\mathcal{A} = 1, \mathcal{B} = 0$

Equation (3.25) is transformed as

$$f'^2(\eta) - 2f(\eta)f''(\eta) = 1 + 3f'''(\eta) \quad (3.27)$$

To convert equation (3.27) with boundary condition into an initial value problem, a linear transformation group of one-parameter is described as below.

$$\eta = a^{\alpha_1}\bar{\eta}, \quad f = a^{\alpha_2}\bar{f} \quad (3.28)$$

where a is a parameter of transformation and α_1, α_2 are constants which have to be determined.

Substituting the transformation in equation (3.27)

$$a^{2\alpha_2-2\alpha_1}\bar{f}'^2(\eta) - 2a^{2\alpha_2-2\alpha_1}\bar{f}(\eta)\bar{f}''(\eta) - 1 - 3a^{\alpha_2-3\alpha_1}\bar{f}'''(\eta) = f'^2(\eta) - 2f(\eta)f''(\eta) - 1 - 3f'''(\eta)$$

Now under this transformation equation (3.27) must remain invariant. This is possible only if

$$2\alpha_2 - 2\alpha_1 = \alpha_2 - 3\alpha_1 = 0, \quad \alpha_1 + \alpha_2 = 0 \quad (3.29)$$

To get missing initial condition,

$$\eta = 0, \quad f''(\eta) = a$$

$$\text{So, } \bar{\eta} = 0, \quad a^{\alpha_2-2\alpha_1}\bar{f}''(\eta) = a$$

The boundary condition is independent of α_1, α_2 and a if

$$\alpha_2 - 2\alpha_1 = 1 \quad (3.30)$$

which then gives $\bar{\eta} = 0$, $\bar{f}''(\eta) = 1$

From (3.29) and (3.30) $\alpha_1 = -\frac{1}{3}$, $\alpha_2 = \frac{1}{3}$

Finally, the value of a can be computed from the boundary condition at the second point, i.e., $f'(\infty) = 1$

$$a^{\alpha_2 - \alpha_1} \bar{f}'(\infty) = 1$$

$$a^{\frac{2}{3}} \bar{f}'(\infty) = 1$$

$$a = \left(\frac{1}{\bar{f}'(\infty)} \right)^{\frac{3}{2}} \quad (3.31)$$

Thus, boundary value problem is transformed into initial value problem

$$\bar{f}'^2(\bar{\eta}) - 2\bar{f}(\bar{\eta})\bar{f}''(\bar{\eta}) = 1 + 3\bar{f}''''(\bar{\eta}) \quad (3.32)$$

with initial conditions

$$\bar{f}(0) = 0, \bar{f}'(0) = 0 \text{ and } \bar{f}''(0) = 1$$

$$3\bar{f}''''(\bar{\eta}) = \bar{f}'^2(\bar{\eta}) - 2\bar{f}(\bar{\eta})\bar{f}''(\bar{\eta}) - 1$$

$$\bar{f}''''(0) = -\frac{1}{3}$$

Differentiating (3.32) with respect to η

$$3\bar{f}^{(iv)}(\eta) = -2\bar{f}'(\eta)\bar{f}''''(\eta) \quad (3.33)$$

$$\bar{f}^{(iv)}(0) = 0$$

Differentiating (3.33) with respect to η

$$3\bar{f}^{(v)}(\eta) = -2\bar{f}''(\eta)\bar{f}''''(\eta) - 2\bar{f}'(\eta)\bar{f}^{(iv)}(\eta) \quad (3.34)$$

$$\bar{f}^{(v)}(0) = 0$$

Differentiating (3.34) with respect to η

$$3\bar{f}^{(vi)}(\eta) = -2\bar{f}'''(\eta)\bar{f}''''(\eta) - 4\bar{f}'(\eta)\bar{f}^{(iv)}(\eta) - 2\bar{f}(\eta)\bar{f}^{(v)}(\eta)$$

$$\bar{f}^{(vi)}(0) = \frac{2}{9}$$

Using Taylor's series expansion and ignoring higher order terms we get,

$$\bar{f}(\bar{\eta}) = \frac{\bar{\eta}^2}{2} - \frac{\bar{\eta}^3}{18} \quad (3.35)$$

$$\bar{f}'(\eta) = \bar{\eta} - \frac{\bar{\eta}^2}{6} \quad (3.36)$$

TABLE 3. 1 Values of $\bar{\eta}$, $\bar{f}(\bar{\eta})$ and $\bar{f}'(\bar{\eta})$ for Newtonian fluid

$\bar{\eta}$	$\bar{f}(\bar{\eta})$	$\bar{f}'(\bar{\eta})$
0	0	0
0.5	0.1180	0.4583
1	0.444	0.8333
1.5	0.9375	1.125
2	1.555	1.3333
2.5	2.257	1.4583
3	3	1.5

Thus $a = \left(\frac{1}{\bar{f}'(3)}\right)^{\frac{3}{2}} = 0.544$

$$\eta = a^{\alpha_1} \bar{\eta} \Rightarrow \eta = 1.2249 \bar{\eta} \text{ and } f = a^{\alpha_2} \bar{f} \Rightarrow f = 0.816 \bar{f} \text{ and } f' = 0.6664 \bar{f}'$$

The calculated values of η , $f(\eta)$ and $f'(\eta)$ are given in Table3.2.

TABLE 3. 2 values of η , $f(\eta)$ and $f'(\eta)$ for Newtonian fluid

η	$f(\eta)$	$f'(\eta)$
0	0	0
0.6124	0.09629	0.3054
1.2249	0.3623	0.55531
1.8373	0.765	0.7947
2.4498	1.2688	0.8883
3.06225	1.8417	0.97183
3.6747	2.448	0.9996

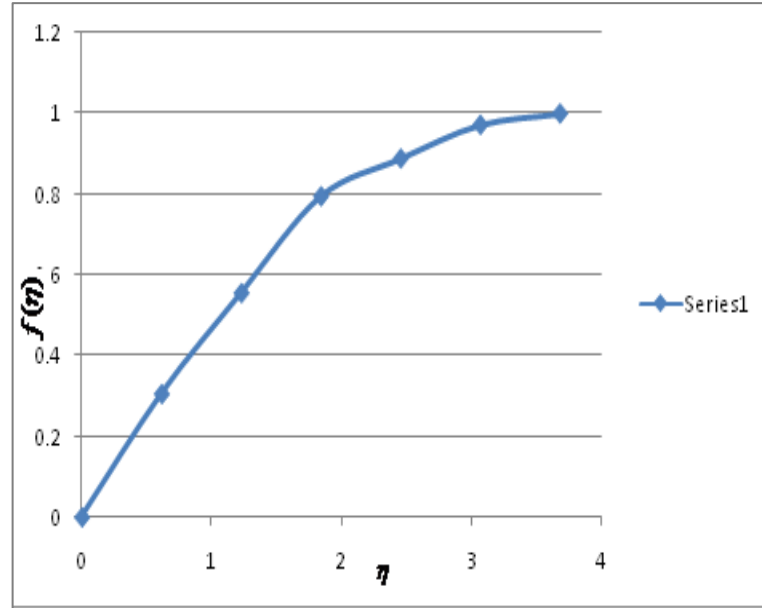


FIGURE 3. 1 Velocity profile for Newtonian fluid model

3.6.2 Case(ii) Power law fluid model

Put $\mathcal{A} = 0$, $\mathcal{B} = 1$, $n = 1/3$

Equation (3.25) is transformed as

$$f'^2(\eta) - 2f(\eta)f''(\eta) = 1 + 3(f''(\eta))^{-\frac{2}{3}}f'''(\eta) \quad (3.37)$$

with boundary conditions

$$f(0) = 0, f'(0) = 0 \text{ and } f'(\infty) = 1$$

Using the same method as discussed above in case (i) to convert boundary value problem into initial value problem,

$$\bar{f}'^2(\bar{\eta}) - 2\bar{f}(\bar{\eta})\bar{f}''(\bar{\eta}) = 1 + 3(\bar{f}''(\bar{\eta}))^{-\frac{2}{3}}\bar{f}'''(\bar{\eta}) \quad (3.38)$$

with initial conditions

$$\bar{f}(0) = 0, \bar{f}'(0) = 0 \text{ and } \bar{f}''(0) = 1 \quad (3.39)$$

and $a = \left(\frac{1}{\bar{f}''(\infty)}\right)^{\frac{11}{6}}$ is obtained.

Differentiating (3.38) successively and neglecting higher order derivatives the following equation using Taylor's series expansion is obtained.

$$\bar{f}(\bar{\eta}) = \frac{\bar{\eta}^2}{2} - \frac{\bar{\eta}^3}{6} \tag{3.40}$$

$$\bar{f}'(\bar{\eta}) = \bar{\eta} - \frac{\bar{\eta}^2}{2} \tag{3.41}$$

TABLE 3. 3 Values of $\bar{\eta}$, $\bar{f}(\bar{\eta})$ and $\bar{f}'(\bar{\eta})$ for Power-law fluid

$\bar{\eta}$	$\bar{f}(\bar{\eta})$	$\bar{f}'(\bar{\eta})$
0	0	0
0.1	0.0048	0.095
0.2	0.0186	0.18
0.3	0.0405	0.255
0.4	0.0693	0.32
0.5	0.1042	0.375
0.6	0.144	0.42
0.7	0.1878	0.455
0.8	0.2346	0.48
0.9	0.2835	0.495
1	0.333	0.5

Thus $a = \left(\frac{1}{\bar{f}'(1)}\right)^{\frac{3}{2}} = 3.563$

$\eta = a^{\alpha_1} \bar{\eta} \Rightarrow \eta = 0.5612 \bar{\eta}$ and $f = a^{\alpha_2} \bar{f} \Rightarrow f = 1.1224 \bar{f}$

and $f' = 1.9998 \bar{f}'$

The calculated values of $\eta, f(\eta)$ and $f'(\eta)$ are given in Table 3.4.

TABLE 3. 4 Values of η , $f(\eta)$ and $f'(\eta)$ for Power-law fluid

η	$f(\eta)$	$f'(\eta)$
0	0	0
0.0561	0.005382	0.1899
0.1122	0.02087	0.3599
0.1683	0.0454	0.5099
0.2245	0.07778	0.6399
0.2806	0.1169	0.7499
0.3367	0.1616	0.8399
0.3928	0.21078	0.9099
0.44896	0.2633	0.9599
0.50508	0.3182	0.9899
0.5612	0.3737	0.999

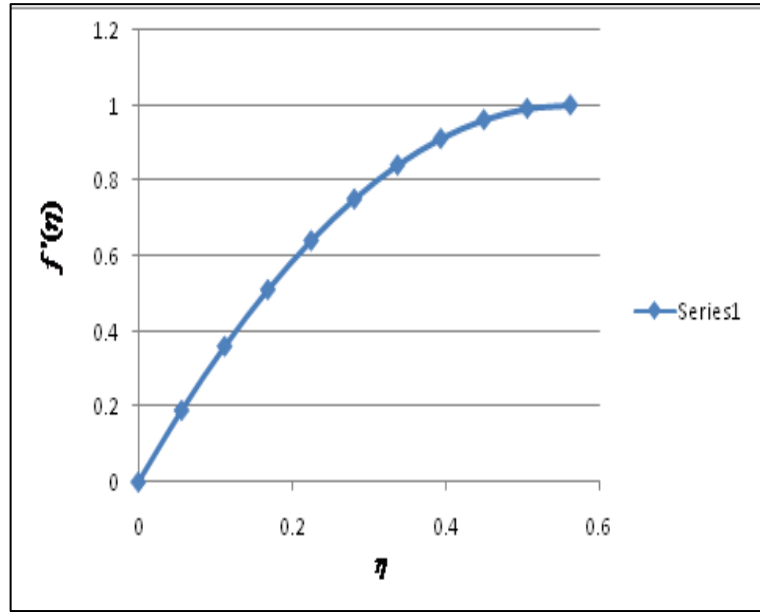


FIGURE 3. 2 Velocity profile for Power-law model

3.6.3 Case(iii) Sisko fluid model

Put $\mathcal{A} = \frac{1}{2}$, $\mathcal{B} = \frac{1}{2}$, $n = \frac{2}{3}$

Equation (3.25) is transformed as

$$f'^2(\eta) - 2f(\eta)f''(\eta) = 1 + \frac{3}{2}f'''(\eta) + 3(f''(\eta))^{-\frac{1}{3}}f'''(\eta) \quad (3.42)$$

with boundary conditions $f(0) = 0, f'(0) = 0$ and $f'(\infty) = 1$ (3.43)

Using the same method to convert boundary value problem into initial value problem

$$\bar{f}'^2(\bar{\eta}) - 2\bar{f}(\bar{\eta})\bar{f}''(\bar{\eta}) = 1 + \frac{3}{2}\bar{f}'''(\bar{\eta}) + 3(\bar{f}''(\bar{\eta}))^{-\frac{1}{3}}\bar{f}'''(\bar{\eta}) \quad (3.44)$$

with initial conditions

$$\bar{f}(0) = 0, \bar{f}'(0) = 0 \text{ and } \bar{f}''(0) = 1 \quad (3.45)$$

and $a = \left(\frac{1}{\bar{f}'(\infty)}\right)^{\frac{3}{2}}$ is obtained.

Differentiating (3.44) successively and neglecting higher order derivatives the following Taylor's series expansion is derived.

$$\bar{f}(\bar{\eta}) = \frac{\bar{\eta}^2}{2} - \frac{\bar{\eta}^3}{15} \quad (3.46)$$

$$\bar{f}'(\bar{\eta}) = \bar{\eta} - \frac{\bar{\eta}^2}{5} \tag{3.47}$$

TABLE 3. 5 Values of $\bar{\eta}$, $\bar{f}(\bar{\eta})$ and $\bar{f}'(\bar{\eta})$ for Sisko fluid

$\bar{\eta}$	$\bar{f}(\bar{\eta})$	$\bar{f}'(\bar{\eta})$
0	0	0
0.1	0.005	0.09802
0.2	0.01947	0.19208
0.3	0.0432	0.2818
0.4	0.0757	0.36382
0.5	0.11675	0.4505
0.6	0.1657	0.52872
0.7	0.222	0.60298
0.8	0.2862	0.67328
0.9	0.35688	0.73962
1	0.434	0.802

Thus, $a = \left(\frac{1}{\bar{f}'(2.5)}\right)^{\frac{3}{2}} = 0.7155$

$\eta = a^{\alpha_1} \bar{\eta} \Rightarrow \eta = 1.11805 \bar{\eta}$ and $f = a^{\alpha_2} \bar{f} \Rightarrow f = 0.8944 \bar{f}$ and $f' = 0.7999 \bar{f}'$

The calculated values of η , $f(\eta)$ and $f'(\eta)$ are given in Table 3.6.

TABLE 3. 6 Values of η , $f(\eta)$ and $f'(\eta)$ for Sisko fluid

η	$f(\eta)$	$f'(\eta)$
0	0	0
0.1118	0.004	0.0784
0.2236	0.0174	0.1536
0.3354	0.0386	0.2257
0.4472	0.0677	0.2946
0.559	0.1044	0.36035
0.6078	0.1482	0.4229
0.7826	0.1985	0.48232
0.8944	0.2559	0.53855
1.006	0.3191	0.5916
1.11805	0.3875	0.64151

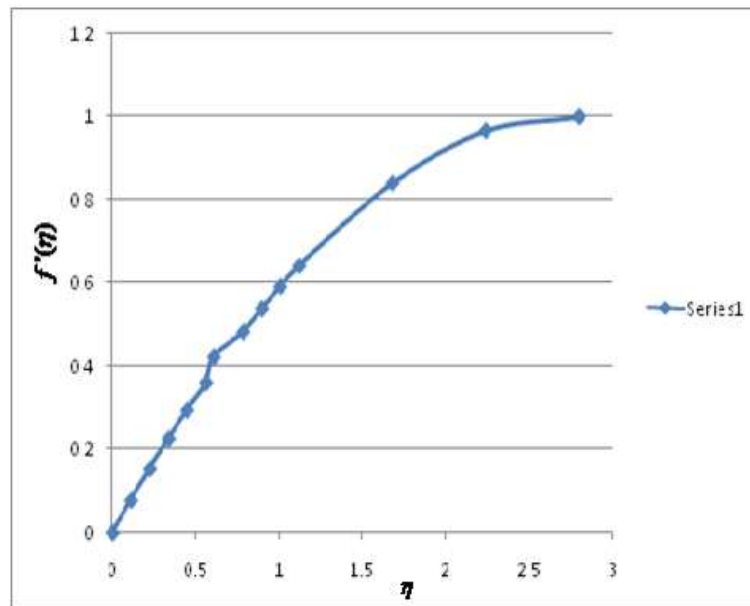


FIGURE 3. 3 Velocity profile for Sisko fluid model

3.7 Conclusion

Two different applications of similarity methods are explored to solve 2-D boundary value problem of Sisko-fluid for steady flow. In order to derive similarity solution, the Generalized Dimensional Analysis technique with two-parameter group transformation is utilised. The achieved similarity equations are extremely non-linear ordinary differential equations with boundary conditions. The same are converted to the initial value problem (IVP) by applying another similarity technique namely linear group of transformation technique. Simple Taylor's series technique provides the numerical solution of the resulting IVP. The benefit of this similarity technique methodology is that after turning it into IVP, extremely non-linear BVP can be solved through easy numerical technique such as Taylor's series method.

CHAPTER-4

Similarity Analysis of Non-Newtonian Powell-Eyring and Prandtl-Eyring Fluid Boundary Value Problem

4.1 Introduction

The heat transfer process is called convection which happens in the presence of bulk fluid movement through a fluid. Convection is generally categorized in two ways based on how the fluid movement begins. One is the natural convection or free convection, launched as a result of the impact of buoyancy. Warmer fluid rises in the natural convection process and cooler fluid falls. The second form is forced convection, in this process external force like a fan or pump is forced to flow the fluid. It is very interesting to study the issue of heat transfer because of its broad applications in various industrial tools and some household appliances.

In the presence of viscous dissipation and inner heat generation/absorption, Prasad [51] explored the issue of thermal transfer of an incompressible, viscous non-Newtonian fluid over a non-isothermal stretching sheet. Munir et al. [52] evaluated the forced convective heat transfer over a nonlinearly stretching sheet in a steady boundary layer flow of Sisko fluid for two cases, i.e. the sheet with variable temperature and the sheet with variable heat flux. Hayat et al. [53] addressed the influences of convective heat and mass transfer over an inclined exponential stretching surface in the flow of Powell-Eyring fluid. Lin and Lin [54] had examined heat transfer in free and forced convection mode, by implementing a similarity technique for any Prandtl number for isothermal or uniform-flux surfaces. Timol and Surati [55] studied numerically the transfer of heat in two modes, free and forced convection, for boundary layer flow of various non-Newtonian fluids past a wedge.

We noted that most of the similarity work was performed for distinct non-Newtonian models either by assuming the similarity variable or by assumed transformation group in heat transfer subject. Jain et al. [56] obtained the similarity transformations by applying the general group-theoretic technique for two dimensional steady, incompressible, laminar boundary layer flow, for natural convection heat transfer mode of Sutterby non-Newtonian fluid past a vertical isothermal plate.

In this chapter, investigation and comparison of two non-Newtonian fluid models is carried out. This is done using one-parameter deductive group-theoretic method by deriving similarity variable and by obtaining invariant group transformation. The system of governing equations is converted into a system of non-linear ordinary differential equations. These equations are solved by MATLAB BVP4C solver and the results are represented graphically.

4.2 One-parameter deductive group-theoretic method

In 1968, Moran and Gaggioli [11] presented a systematic group formulation for similarity analysis. A formulation is nothing but modifications of Moran's methods with a very general class of transformation groups of the form

$$G: \begin{cases} s: \{\bar{x}_i = f_i(x_1, x_2, \dots, x_m; a), i = 1, \dots, m, m \geq 2 \\ \bar{y}_j = f_j(y_j; a), j = 1, \dots, n, n \geq 1 \end{cases} \quad (4.1)$$

4.2.1 Deductive Group

In the same year, Moran et al [57] modified their method and proposed a group in the form

$$G: \begin{cases} s: \{\bar{x}_i = \mathcal{P}^{x_i}(a)x_i + \mathcal{Q}^{x_i}(a), i = 1, \dots, m; m \geq 2 \\ \bar{y}_j = \mathcal{P}^{y_j}(a)y_j + \mathcal{Q}^{y_j}(a), j = 1, \dots, n; n \geq 1 \end{cases} \quad (4.2)$$

where, $\mathcal{P}^{x_i}, \mathcal{Q}^{x_i}, \mathcal{P}^{y_j}, \mathcal{Q}^{y_j}$ are real valued differentiable functions in real argument a .

4.2.2 Construction of the similarity representation

Utilizing invariant principle on partial differential equations and auxiliary conditions, constraints on the functions f of G are observed. Thus, transformation groups (if exist) have been identified whose f 's satisfies invariance criteria for equation and auxiliary condition.

4.2.3 Determination of the absolute invariants

In chapter 2, group generator in two-dimensional space (x, y) were determined in the form

$$\mathcal{U}f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \quad (4.3)$$

and invariant conditions in following form

$$\mathcal{U}f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0 \quad (4.4)$$

Same way for groups of transformations, $\bar{x}_i = f_i(x_1, x_2, \dots, x_m; a), i = 1, \dots, m; m \geq 2$,

in n -dimensional space. The group generator is written as,

$$X = \sum \xi_i(x_1, x_2, \dots, x_m) \frac{\partial}{\partial x_i}, \text{ where } \xi_i(x_1, x_2, \dots, x_m) = \left[\frac{\partial f_i(x_1, x_2, \dots, x_m; a)}{\partial a} \right]_{a=a_0}$$

So, invariant conditions in n -dimensional is obtained in the form as

$$\xi_1(x_1, x_2, \dots, x_m) \frac{\partial f}{\partial x_1} + \xi_2(x_1, x_2, \dots, x_m) \frac{\partial f}{\partial x_2} + \dots + \xi_m(x_1, x_2, \dots, x_m) \frac{\partial f}{\partial x_m} = 0 \quad (4.5)$$

The above equation can be solved by solving the following system of equations:

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \dots = \frac{dx_m}{\xi_m} \quad (4.6)$$

We get $(m - 1)$ independent solutions of the above equations. That means for group of transformations, of one-parameter in m -variables has $(m - 1)$ independent invariants $\eta_1, \eta_2, \dots, \eta_{m-1}$. If we consider whole group G of m independent and n dependent variables then there exist $(m + n - 1)$ functionally independent absolute invariants.

4.3 Boundary value problem for Powell-Eyring and Prandtl-Eyring Models

Here, laminar, two-dimensional, steady, incompressible boundary layer flow over a flat plate is considered for analysis. Non-Newtonian fluid with shear stress τ_{yx} is related by arbitrary function with shear strain in the following form:[55]

$$F\left(\tau_{yx}, \frac{\partial u}{\partial x}\right) = 0 \quad (4.7)$$

The basic governing equations of forced convection flow with conditions on boundary are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.8)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{1}{\rho} \frac{\partial}{\partial y} (\tau_{yx}) \quad (4.9)$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = a \frac{\partial^2 \theta}{\partial y^2} \quad (4.10)$$

$$u(0) = 0, \quad v(0) = 0, \quad \theta(0) = 0, \quad u(\infty) = U(x), \quad \theta(\infty) = \theta_W \quad (4.11)$$

Defining stream function $\psi(x, y)$, which satisfies continuity equation (4.8) is as follows:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (4.12)$$

The equations are converted in dimensionless form by defining non-dimensional terms as follows

$$\left. \begin{aligned} u' &= \frac{u}{U_0}, \quad v' = \frac{v}{U_0} \sqrt{\frac{Re}{3}}, \quad x' = \frac{x}{L}, \quad y' = \frac{y}{L} \sqrt{\frac{Re}{3}}, \\ U' &= \frac{U}{U_0}, \quad \tau'_{y'x'} = \frac{\tau_{yx}}{\rho U_0^2} \sqrt{\frac{Re}{3}}, \quad Re = \frac{\rho U_0 L}{\mu}, \\ pr &= \frac{U_0 L}{a Re} \end{aligned} \right\} \quad (4.13)$$

Re is the Reynolds number, pr represents the Prandtl number, thermal diffusivity is a , l is the reference length, U is the free-stream velocity, ρ is the density.

Applying the above non-dimensional quantities and stream function, the following equations (for simplicity symbol without des “'” are used in following equations) are obtained.

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U' \frac{dU'}{dx'} + \frac{1}{\rho} \frac{\partial}{\partial y'} (\tau'_{yx}) \quad (4.14)$$

$$\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} = \frac{1}{3pr} \frac{\partial^2 \theta}{\partial y'^2} \quad (4.15)$$

with the stress–strain relation

$$F \left(\tau'_{yx}, \frac{\partial^2 \psi}{\partial y'^2} \right) = 0 \quad (4.16)$$

Subject to the boundary conditions;

$$\left. \begin{aligned} y = 0 &\Rightarrow \frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0, \theta = 0 \\ y = \infty &\Rightarrow \frac{\partial \psi}{\partial y} = U(x), \theta \Rightarrow \theta_W \end{aligned} \right\} \quad (4.17)$$

4.4 Application of one-parameter group-theoretic method

Here, one-parameter group-theoretic method is applied on the system of governing partial differential equations. So, first, the following group of transformations with ‘ a ’ as one-parameter and with real-valued differentiable functions \mathcal{P}^S and Q^S in their real argument ‘ a ’ are defined.

$$G: \begin{cases} \bar{x} = \mathcal{P}^x(a)x + Q^x(a) \\ \bar{y} = \mathcal{P}^y(a)y + Q^y(a) \\ \bar{\psi} = \mathcal{P}^\psi(a)\psi + Q^\psi(a) \\ \bar{\tau} = \mathcal{P}^\tau(a)\tau + Q^\tau(a) \\ \bar{\theta} = \mathcal{P}^\theta(a)\theta + Q^\theta(a) \\ \bar{\theta}_w = \mathcal{P}^{\theta_w}(a)\theta_w + Q^{\theta_w}(a) \\ \bar{U} = \mathcal{P}^U(a)U + Q^U(a) \end{cases} \quad (4.18)$$

Applying invariance principle on equations in (4.14) - (4.16) under a group of transformations defined by G in equation (4.18), the following are obtained.

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y} \partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - \bar{U} \frac{d\bar{U}}{d\bar{x}} - \frac{\partial}{\partial \bar{y}} \{\bar{\tau}_{yx}\} = \mathcal{L}(a) \left\{ \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - U \frac{dU}{dx} - \frac{\partial}{\partial y} (\tau_{yx}) \right\} \quad (4.19)$$

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial \bar{\theta}}{\partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial \bar{\theta}}{\partial \bar{y}} - \frac{1}{3pr} \frac{\partial^2 \bar{\theta}}{\partial \bar{y}^2} = \mathcal{M}(a) \left\{ \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} - \frac{1}{3pr} \frac{\partial^2 \theta}{\partial y^2} \right\} \quad (4.20)$$

$$F \left(\mathcal{P}^\tau \tau + Q^\tau, \frac{\mathcal{P}^\psi}{(\mathcal{P}^y)^2} \frac{\partial^2 \psi}{\partial y^2} \right) = 0 \quad (4.21)$$

The invariance of equations (4.19), (4.20), (4.21) gives:

$$\left(\frac{(\mathcal{P}^\psi)^2}{(\mathcal{P}^y)^2 \mathcal{P}^x} \right) = \frac{(\mathcal{P}^U)^2}{\mathcal{P}^x} = \frac{\mathcal{P}^\tau}{\mathcal{P}^y} = \mathcal{L}(a) \quad (4.22)$$

$$\frac{\mathcal{P}^\psi \mathcal{P}^\theta}{\mathcal{P}^x \mathcal{P}^y} = \frac{\mathcal{P}^\theta}{(\mathcal{P}^y)^2} = \mathcal{M}(a) \quad (4.23)$$

$$\mathcal{P}^\tau = \frac{\mathcal{P}^\psi}{(\mathcal{P}^y)^2} = 1 \quad (4.24)$$

The invariance of boundary conditions in (4.11) gives:

$$Q^y = Q^\theta = Q^{\theta_w} = Q^U = Q^\tau = 0 \quad (4.25)$$

Solving (4.22) to (4.24) equations the following relations are established.

$$\mathcal{P}^\psi = (\mathcal{P}^y)^2, \mathcal{P}^x = (\mathcal{P}^y)^3, \mathcal{P}^U = \mathcal{P}^y, \mathcal{P}^\theta = \mathcal{P}^{\theta_w}, \mathcal{P}^\tau = 1 \quad (4.26)$$

Thus, the one-parameter group G , which transforms the differential equation with the auxiliary conditions is obtained in the following form

$$G: \begin{cases} \bar{x} = (\mathcal{P}^y)^3(a)x + Q^x(a) \\ \bar{y} = \mathcal{P}^y(a)y \\ \bar{\psi} = (\mathcal{P}^y)^2(a)\psi + Q^\psi(a) \\ \bar{\tau} = \tau \\ \bar{\theta} = \mathcal{P}^\theta(a)\theta \\ \bar{\theta}_W = \mathcal{P}^\theta(a)\theta_W \\ \bar{U} = \mathcal{P}^y(a)U \end{cases} \quad (4.27)$$

4.5 Derivation of Absolute Invariants

Now, the analysis is proceeded to obtain a complete set of absolute invariants so that the original problem is transformed into an ordinary differential equation in a similarity variable via group-theoretic method. The application of a basic theorem in group theory by Moran and Gajjoli [12] States that: A function g is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation:

$$\sum_{i=1}^7 (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0 \quad (4.28)$$

where, $S_i = x, y; \psi, \tau, U, \theta, \theta_W$

$$\alpha_i = \left(\frac{\partial \mathcal{P}^{S_i}}{\partial a} \right) (a_0), \quad \beta_i = \left(\frac{\partial Q^{S_i}}{\partial a} \right) (a_0), \quad i = 1, 2, \dots, 7$$

where, a_0 denotes the value of which yields the identity element of the group. By considering $x_1 = x, x_2 = y, y_1 = \psi, y_2 = \tau, y_3 = U, y_4 = \theta, y_5 = \theta_W$ and using the definitions of $\alpha_i, \beta_i; i = 1, 2, \dots, 7$. the following is obtained.

$$\begin{aligned} & (\alpha_1 x_1 + \beta_1) \frac{\partial g}{\partial x_1} + (\alpha_2 x_2 + \beta_2) \frac{\partial g}{\partial x_2} + (\alpha_3 y_1 + \beta_3) \frac{\partial g}{\partial y_1} + (\alpha_4 y_2 + \beta_4) \frac{\partial g}{\partial y_2} + \\ & (\alpha_5 y_3 + \beta_5) \frac{\partial g}{\partial y_3} + (\alpha_6 y_4 + \beta_6) \frac{\partial g}{\partial y_4} + (\alpha_7 y_5 + \beta_7) \frac{\partial g}{\partial y_5} = 0 \end{aligned} \quad (4.29)$$

On solving this and using the relations between α_i 's & β_i 's from Equations (4.25) and (4.26), the similarity variables are obtained as follows

$$\eta = \frac{y}{(x+\lambda)^{\frac{1}{3}}} \quad (\text{where, } \lambda = \frac{\beta_1}{\alpha_1})$$

$$F_1(\eta) = \frac{\psi + \gamma}{(x+\lambda)^{\frac{2}{3}}} \quad (\text{where, } \gamma = \frac{\beta_3}{\alpha_1})$$

The reduction to an Ordinary Differential equation

$$F_2(\eta) = \frac{U}{(x+\lambda)^{\frac{1}{3}}}, \quad F_3(\eta) = \tau_{yx}, \quad F_4(\eta) = \frac{\theta}{(x+\lambda)^m} \quad (m = \frac{\alpha_6}{\alpha_1}),$$

$$F_5(\eta) = \frac{\theta_W}{(x+\lambda)^m} \quad (\text{By taking } U = (x + \lambda)^{\frac{1}{3}})$$

4.6 The reduction to an Ordinary Differential equation

The similarity transformations (4.27) maps equations (4.19) - (4.21) into the following non-linear ordinary differential equations using above similarity variables.

$$(F_1')^2(\eta) - 2F_1(\eta)F_1''(\eta) = 1 + 3F_3'(\eta) \quad (4.30)$$

$$F_4''(\eta) - pr(3mF_4(\eta)F_1'(\eta) - 2F_1(\eta)F_4'(\eta)) = 0 \quad (4.31)$$

$$F(F_3, F_1'') = 0 \quad (4.32)$$

with boundary conditions

$$\eta = 0 \Rightarrow F_1(\eta) = 0, F_1'(\eta) = 0, F_4(\eta) = 0 \quad (4.33)$$

$$\eta = \infty \Rightarrow F_1'(\eta) = 1, F_4(\eta) = 1 \quad (4.34)$$

4.6.1 Powell-Eyring Model

The stress-strain relationship for Powell-Eyring model is:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} + \frac{1}{B} \sinh^{-1} \left(\frac{1}{C} \frac{\partial u}{\partial y} \right) \quad (4.35)$$

where B and C are characteristic constants of the model. Introducing the non-dimensional quantities given by (4.13) and stream function ψ into equation (4.35) and simplifying, the above model becomes

$$\frac{\rho U_0^2}{\sqrt{\frac{Re}{3}}} F_3(\eta) = \frac{\mu U_0}{L} \sqrt{\frac{Re}{3}} F_1''(\eta) + \frac{1}{B} \sinh^{-1} \left(\frac{1}{C} \frac{U_0}{L} \sqrt{\frac{Re}{3}} F_1''(\eta) \right) \quad (4.36)$$

Differentiating it the following is obtained

$$F_1'''(\eta) = \frac{3\sqrt{(1+\beta(F_1''(\eta))^2)} F_3'(\eta)}{3\alpha + \sqrt{1+\beta(F_1''(\eta))^2}} \quad (4.37)$$

$$(\text{By taking } \alpha = \frac{1}{3\mu BC}, \beta = \frac{\rho U_0^3}{3\mu LC^2})$$

Above equation (4.30) - (4.32) reduced into

$$\begin{aligned} & \sqrt{(1 + \beta(F_1''(\eta))^2)[(F_1')^2(\eta) - 2F_1(\eta)F_1''(\eta) - 1]} \\ & = [3\alpha + \sqrt{(1 + \beta(F_1''(\eta))^2)}]F_1'''(\eta) \end{aligned} \quad (4.38)$$

$$F_4''(\eta) - pr(3mF_4(\eta)F_1'(\eta) - 2F_1(\eta)F_4'(\eta)) = 0 \quad (4.39)$$

with boundary conditions

$$\eta = 0 \Rightarrow F_1(\eta) = 0, F_1'(\eta) = 0, F_4(\eta) = 0 \quad (4.40)$$

$$\eta = \infty \Rightarrow F_1'(\eta) = 1, F_4(\eta) = 1 \quad (4.41)$$

4.6.2 Prandtl-Eyring Model

Mathematically, Prandtl-Eyring model is written as

$$\tau_{yx} = \mathcal{B} \sinh^{-1}\left(\frac{1}{\mathcal{C}} \frac{\partial u}{\partial y}\right) \quad (4.42)$$

where \mathcal{B} and \mathcal{C} are characteristic constants of the model. Introducing the non-dimensional quantities given by (4.13) and stream function ψ into equation (4.42) and simplifying, the above model becomes

$$\frac{\rho U_0^2}{\sqrt{\frac{Re}{3}}} F_3(\eta) = \mathcal{B} \sinh^{-1}\left(\frac{1}{\mathcal{C}} \frac{U_0}{L} \sqrt{Re/3} F_1''(\eta)\right) \quad (4.43)$$

Differentiating it the following is obtained

$$F_1'''(\eta) = \frac{\sqrt{(1 + \beta'(F_1''(\eta))^2) F_3'(\eta)}}{\alpha'} \quad (4.44)$$

$$\left(\beta' = \frac{\rho U_0^3}{3\mu L \mathcal{C}^2}, \alpha' = \frac{\mathcal{B}}{3\mu \mathcal{C}}\right)$$

Above equation (4.30) - (4.32) are reduced into

$$\sqrt{(1 + \beta'(F_1''(\eta))^2)[(F_1')^2(\eta) - 2F_1(\eta)F_1''(\eta) - 1]} = 3\alpha' F_1'''(\eta) \quad (4.45)$$

$$F_4''(\eta) - pr(3mF_4(\eta)F_1'(\eta) - 2F_1(\eta)F_4'(\eta)) = 0 \quad (4.46)$$

$$\eta = 0 \Rightarrow F_1(\eta) = 0, F_1'(\eta) = 0, F_4(\eta) = 0 \quad (4.47)$$

$$\eta = \infty \Rightarrow F_1'(\eta) = 1, F_4(\eta) = 1 \quad (4.48)$$

($F_1 = f$ and $F_4 = g$ in graph)

4.7 Graphical presentation

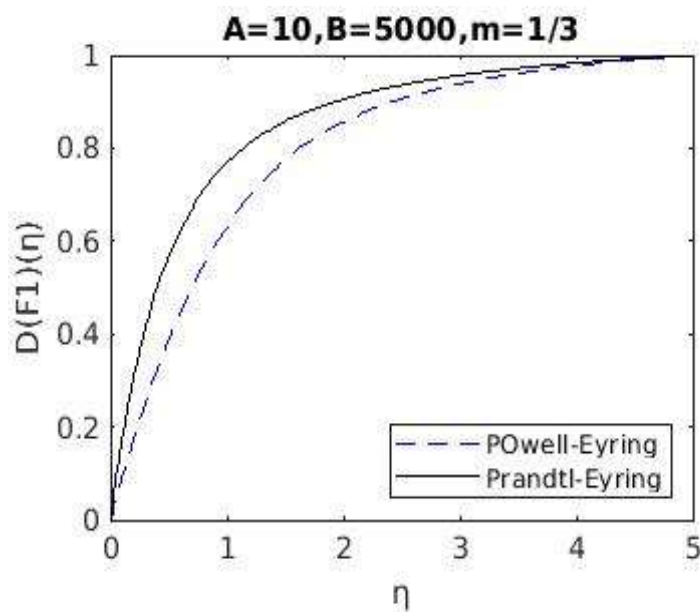


FIGURE 4. 1 Velocity profile for Powell-Eyring and Prandtl-Eyring model

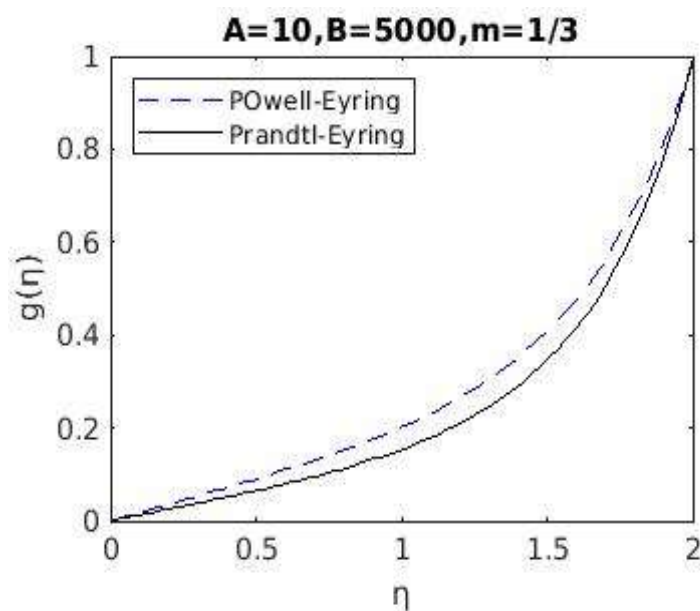


FIGURE 4. 2 Temperature profile for Powell-Eyring and Prandtl-Eyring model

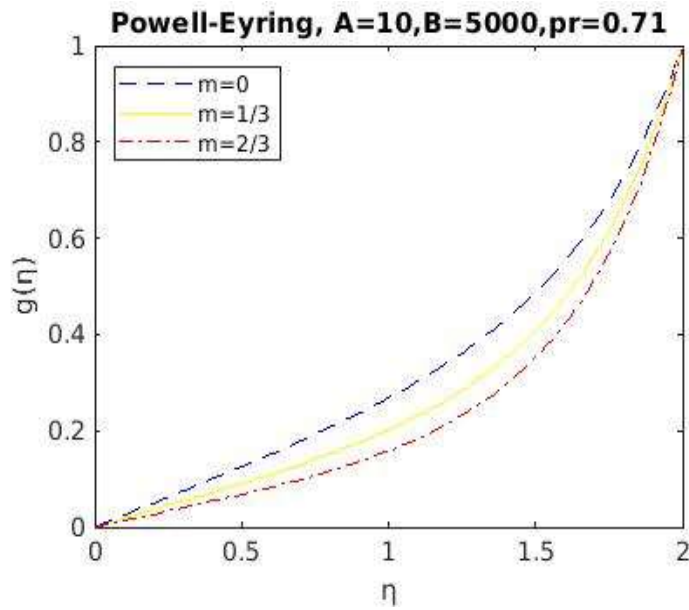


FIGURE 4. 3 Temperature profile for Powell-Eyring model for different values of m .

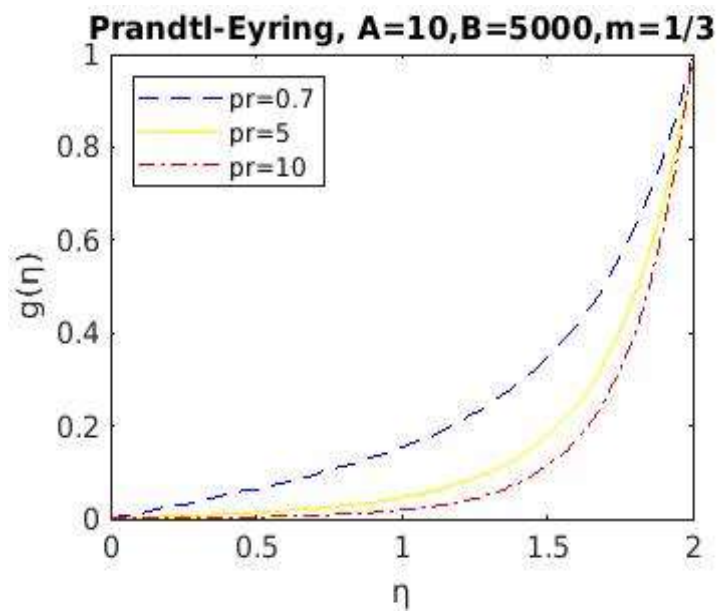


FIGURE 4. 4 Temperature profile for Prandtl-Eyring model for different Prandtl number

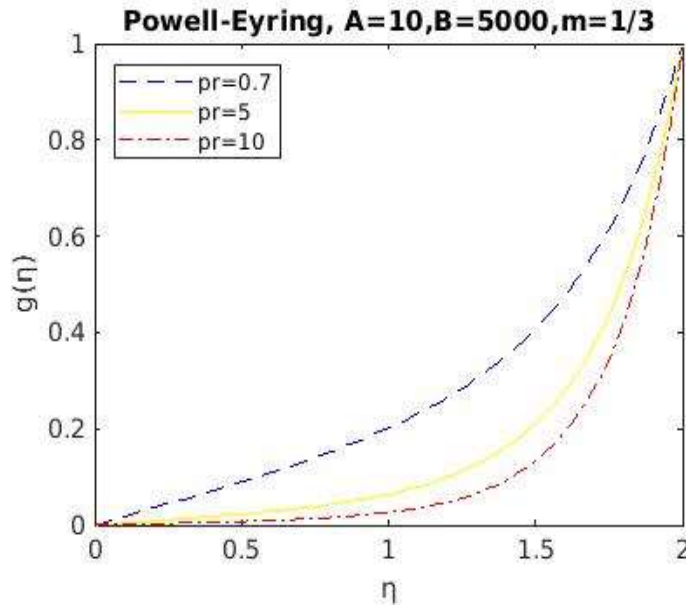


FIGURE 4. 5 Temperature profile for Powell-Eyring model for different Prandtl number

4.8 Conclusion

In this chapter, two different non-Newtonian fluid models for flow over a flat plate of forced convection using the one-parameter group-theoretic technique is investigated. Similarity solutions for Powell-Eyring and Prandtl-Eyring model are derived. The non-linear partial differential equations are transformed into non-linear ordinary differential equations. Comparisons of both the models for the different physical parameters are done for velocity and temperature profile. Effect of Prandtl number is shown, for both the model, Powell-Eyring and Prandtl-Eyring, on the temperature profile. As the Prandtl number increases there is a rapid decrease in temperature. The temperature profile for different values of m for the Powell-Eyring model is shown graphically. The effect of temperature is more on Powell-Eyring fluid than the Prandtl-Eyring model for constant heat flux. The velocity distribution in Prandtl-Eyring fluid is quite higher than the Powell- Eyring fluid.

CHAPTER-5

Similarity Analysis of Newtonian Nanofluid Boundary Value Problem

5.1 Introduction

In the previous chapter, the one-parameter group-theoretic method was applied to eliminate one independent variable from governing equations. For elimination of two or more than two independent variables from partial differential equations, in order to convert it into ordinary differential equations, the same procedure has to be repeated two or more times. The method thus loses its powerful characteristic of simplicity and becomes tedious. To overcome this limitation, the method of group with multi-parameter is explained and applied by Manohar [32] and then by Ames [14]. The multi-parameter group-theoretic technique is useful to solve many engineering boundary value problems. Here, important basic theorems for invariance analysis for multi-parameter deductive group-theoretic techniques given by Morgan [10] and Eisenhart [33] are provided.

5.1.1 Multi parameter Group-theoretic method

Consider the r - parameter continuous transformations group of the form

$$G: \{\bar{z}^i = f^i(z^1, z^2, \dots, z^m; a_1, a_2, \dots, a_r), i = 1, 2, \dots, m. r \leq m, m \geq 2 \quad (5.1)$$

The generator of group G is defined by

$$X_l = \sum_{i=1}^m \xi_l^i(z^1, z^2, \dots, z^m) \frac{\partial}{\partial z^i}, l = 1, 2, \dots, r \quad (5.2)$$

The matrix and its elements are defined by

$$\xi = [\xi_l^i], \text{ where } \xi_l^i = \frac{\partial f^i}{\partial a_l}(z^1, z^2, \dots, z^m; a^0), l = 1, 2, \dots, r \quad (5.3)$$

$a^0 = (a_1^0, a_2^0, \dots, a_r^0)$ is the identity element of the group.

Theorem 5.1.1: A function $F(z^1, z^2, \dots, z^m)$ is an absolute invariant of group G defined as above in equation (5.1) if and only if $X_i F = 0$ (5.4)

Theorem 5.1.2: The group possesses exactly $(m - r)$ functionally independent absolute invariants, where $r \leq m$ is the rank of the matrix $= [\xi_i^j]$.

Theorem 5.1.3: If $F_j(z^1, z^2, \dots, z^m)$ is the set of functionally independent solutions of equation (5.4) and if $F(z^1, z^2, \dots, z^m)$ any other solution of equation (5.4), then ,

$F = \Delta(F_1, F_2, \dots, F_{m-r})$, where Δ is a differentiable function.

The deductive group of r -parameter

$$G: \begin{cases} s: \{\bar{x}_i = \mathcal{P}^{x_i}(a_1, a_2, \dots, a_r)x_i + \mathcal{Q}^{x_i}(a_1, a_2, \dots, a_r), i = 1, \dots, m; m \geq 2 \\ \bar{y}_j = \mathcal{P}^{y_j}(a_1, a_2, \dots, a_r)y_j + \mathcal{Q}^{y_j}(a_1, a_2, \dots, a_r), j = 1, \dots, n; n \geq 1 \end{cases} \quad (5.5)$$

possesses $(m + n - r)$ functionally independent absolute invariants. Here we have only the $(m - r)$ functionally independent absolute invariants, of subgroup s of group G denoted by

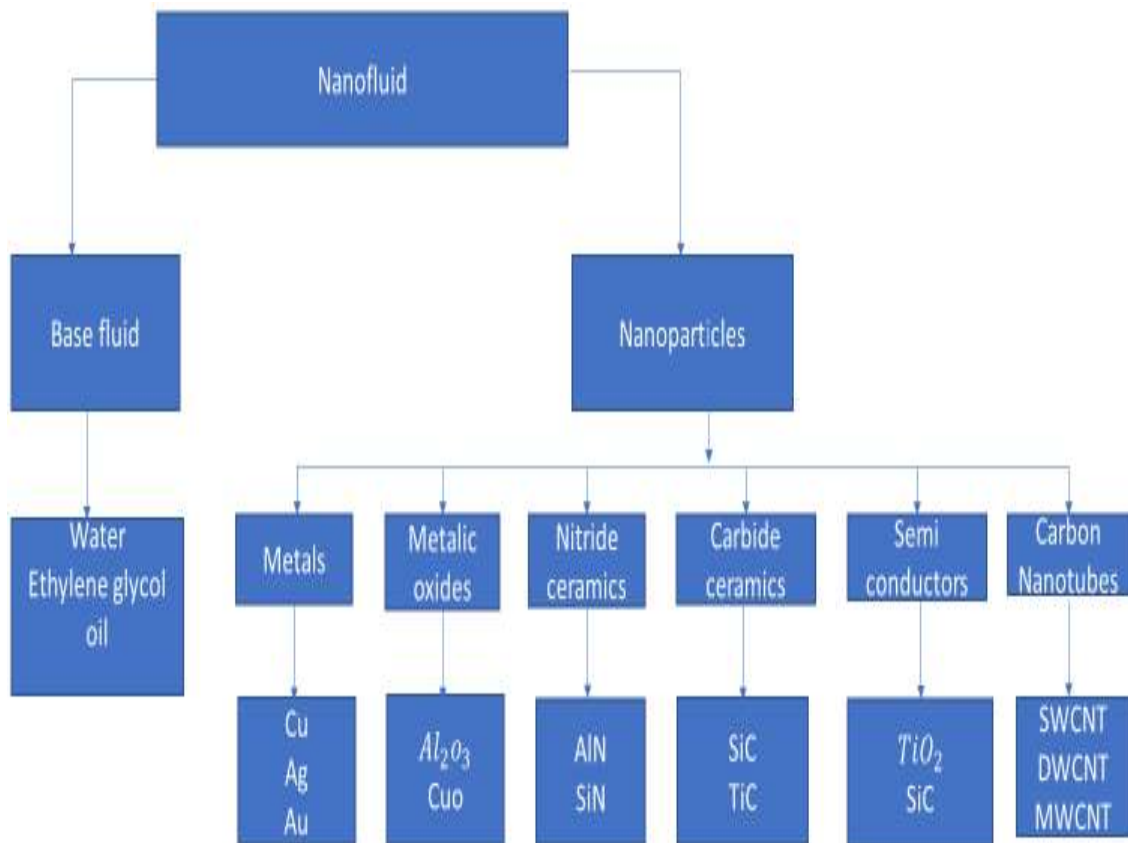
$\eta_j(x_1, x_2, \dots, x_m) = \eta_j(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m), j = 1, \dots, m - r$ and additional n absolute invariants of group G as

$$g_k(y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_m) = g_k(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_m), k = 1, 2, \dots, n$$

which satisfy the condition that the Jacobian $\frac{\partial(g_1, g_2, \dots, g_n)}{\partial(y_1, y_2, \dots, y_n)} \neq 0$.

5.1.2 Nanofluid

A nanofluid is a suspension of base fluid (conventional fluid) and nanoparticles which are of nanometre-size. Nanofluid is also called next-generation heat transfer fluid because of its characteristics of enhancing heat transfer. Generally, the size of the nanoparticles is less than 100nm.



5.1.3 Application of nanofluid

Nanofluid is useful in

- Automobiles as coolants, brake fluid and as gear lubrication.
- Refrigeration equipment.
- Industrial cooling.
- Solar devices.
- Cancer drug etc...

Choi [25] was the first who introduced the word nanofluid. The study of nanofluid attracted researchers because of its cooling and heat transfer enhancement properties in a different field from engineering to medicine. Hayat et al. [58] investigated Magnetohydrodynamic (MHD) three-dimensional flow of nanofluid in the presence of thermophoresis and Brownian motion effects. Using the Homotopy analysis method Zhao et al. [59] investigated three-dimensional nanofluid flow and heat transfer for the flat surface stretching continuously in both x-and y-directions.

The analysis of the three-dimensional boundary layer flow over a stretching surface in the ambient fluid was started by Wang [29]. Gupta et al. [26] presented different kinds of applications and challenges of nanofluid. Nadeem et al. [60] investigated nanofluid flow over an exponentially stretching sheet using different types of nanoparticles. There are various models in the literature to study nanofluid flow. One of them is homogeneous which is proposed by Choi [25]. For this model, a solid volume fraction is less than 0.3. This model can be extended for the conversion of conventional conservation equations of pure fluid to nanofluid.

To derive a similarity solution, the two-parameter group-theoretic method is applied for the nanofluid flow problem. There are many researchers who have worked on the two-parameter group-theoretic method. Using this systematic approach many authors investigated different fluid flow problems successfully [[17], [61], [62], [63]]. M.A.A Hamad [64] presented HAMAD formulation using this generalized group theory method.

Most of the work done to study nanofluid flow properties is by assuming similarity variables. In this chapter, a complete set of similarity variables is derived and then using these similarity variables, set of partial differential equations given in governing equations is converted into ordinary differential equations.

5.2 Boundary value problem for Newtonian nanofluid

Here, the steady, laminar, incompressible and three-dimensional flow of nanofluid is investigated, in which base fluid is water and nanoparticles are spherical copper particles. Homogeneous flow is considered over continuously stretching surface with the velocities $u_w = ax$ and $v_w = by$, respectively.

The surface temperature is T_w and the ambient temperature is T_∞ , where T_w and T_∞ are two constants, with $T_w > T_\infty$.

Figure 5.1 shows physical model of co-ordinate system.

Here (u, v, w) are the velocity components along the axes (X, Y, Z) respectively.

a and b are positive constants.

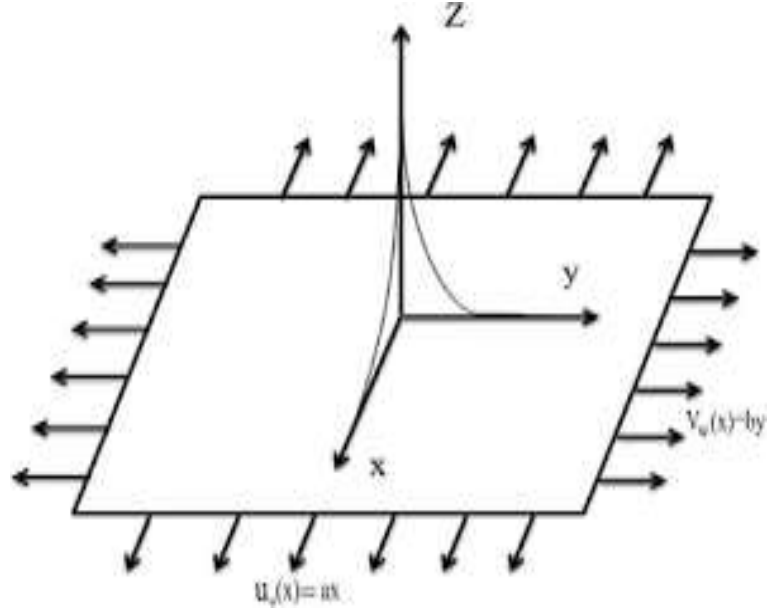


FIGURE 5. 1 Physical model and co-ordinate system

Using the mathematical model for nanofluids proposed by Choi [25], the governing equations are given by (refer to [59])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5.6)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial z^2} \quad (5.7)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 v}{\partial z^2} \quad (5.8)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_{nf} \frac{\partial^2 T}{\partial z^2} \quad (5.9)$$

Boundary conditions:

$$u(x, y, 0) = u_w = ax, \quad v(x, y, 0) = v_w = by, \quad w(x, y, 0) = 0 \quad (5.10)$$

$$T(x, y, 0) = T_w, \quad u(x, y, \infty) = 0, \quad v(x, y, \infty) = 0, \quad T(x, y, \infty) = T_\infty \quad (5.11)$$

$$\text{Take } \theta = \frac{T - T_\infty}{T_w - T_\infty} \quad (5.12)$$

T is the temperature of the nanofluid μ_{nf} , ρ_{nf} , and α_{nf} are the effective viscosity of the nanofluid, the effective density of the nanofluid, and the effective thermal diffusivity of the nanofluid, respectively, which are defined as follows:

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}, \quad \rho_{nf} = (1-\phi)\rho_f + \phi\rho_s, \quad \alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}}$$

$$(\rho c_p)_{nf} = (1 - \Phi)(\rho c_p)_f + \Phi(\rho c_p)_s, \frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\Phi(k_f - k_s)}{(k_s + 2k_f) + \Phi(k_f - k_s)} \quad (5.13)$$

where Φ is the solid volume fraction of the nanofluid, ρ is the density, k is the thermal conductivity, ρc_p is the heat capacitance. Subscript nf is for nanofluid, s for solid particles, f for base fluid. Formula for μ_{nf} is given by Brinkman [3], and k_{nf} is by Maxwell-Garnett's Model [65].

5.3 Application of Generalized group-theoretic method

The method used in this chapter is the generalized group-theoretic method. Under this general group of transformation, the three independent variables will be reduced by two. At the same time, the boundary value type partial differential equations (5.6) - (5.9) which have three independent variables x, y and z , transform into boundary value type ordinary differential equations with only one-independent variable. This is called similarity equation. Following is the group of transformations of two-parameters (a_1, a_2) in the form of

$$G : \bar{s} = \mathcal{P}^s(a_1, a_2)s + \mathcal{Q}^s(a_1, a_2) \quad (5.14)$$

where s stands for x, y, z, u, v, w, θ

\mathcal{P}^s and \mathcal{Q}^s are real-valued and at least differentiable in their real arguments (a_1, a_2) .

5.3.1 Derivation of absolute invariants

Derivatives of the transformation are obtained from G using chain rule.

$$\bar{s}_i = \frac{\mathcal{P}^s}{\mathcal{P}^i} s_i, \quad \bar{s}_{ij} = \frac{\mathcal{P}^s}{\mathcal{P}^i \mathcal{P}^j} s_{ij} \quad (5.15)$$

where i and j stands for x, y, z and s stands for u, v, w, θ .

Equations (5.6) to (5.9) remain invariant under group of transformations defined by G in equation (5.14) and derivatives in equation (5.15).

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = H(a_1, a_2) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (5.16)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} - \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} = I(a_1, a_2) \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial z^2} \right) + J(a_1, a_2) \quad (5.17)$$

$$\bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} - \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} = L(a_1, a_2) \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} - \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 v}{\partial z^2} \right) + M(a_1, a_2) \quad (5.18)$$

$$\bar{u} \frac{\partial \bar{\theta}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{\theta}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{\theta}}{\partial \bar{z}} - \alpha_{nf} \frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2} = N(a_1, a_2) \left(u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} - \alpha_{nf} \frac{\partial^2 \theta}{\partial z^2} \right) + P(a_1, a_2) \quad (5.19)$$

The invariance of above equations implies that

$$J(a_1, a_2) = M(a_1, a_2) = P(a_1, a_2) = 0$$

$$\text{This is satisfied if } Q^u = Q^v = Q^w = 0 \text{ and} \quad (5.20)$$

$$H(a_1, a_2) = \frac{p^u}{p^x} = \frac{p^v}{p^y} = \frac{p^w}{p^z} \quad (5.21)$$

$$I(a_1, a_2) = \frac{(p^u)^2}{p^x} = \frac{p^u p^v}{p^y} = \frac{p^u p^w}{p^z} = \frac{p^u}{(p^z)^2} \quad (5.22)$$

$$L(a_1, a_2) = \frac{(p^v)^2}{p^y} = \frac{p^u p^v}{p^x} = \frac{p^v p^w}{p^z} = \frac{p^v}{(p^z)^2} \quad (5.23)$$

$$N(a_1, a_2) = \frac{p^u p^\theta}{p^x} = \frac{p^v p^\theta}{p^y} = \frac{p^w p^\theta}{(p^z)^2} = \frac{p^\theta}{(p^z)^2} \quad (5.24)$$

So, from equation (5.20) to (5.24) with boundary conditions (5.10) to (5.11) following relations are obtained.

$$p^u = p^x, \quad p^v = p^y, \quad p^w = p^z = p^\theta = 1 \quad (5.25)$$

$$Q^u = Q^x = Q^v = Q^y = Q^w = Q^z = Q^\theta = 0 \quad (5.26)$$

Thus, a two-parameter group transformation is obtained of the form

$$G: \begin{cases} \bar{x} = \mathcal{P}^x x \\ \bar{y} = \mathcal{P}^y y \\ \bar{z} = z \\ \bar{u} = \mathcal{P}^x u \\ \bar{v} = \mathcal{P}^y v \\ \bar{w} = w \\ \bar{\theta} = \theta \end{cases} \quad (5.27)$$

Now our study continues to acquire a full set of absolute invariants in order to transform the original problem into the system of ordinary differential equations through group-theoretical technique in a similarity variable. The implementation of a fundamental theorem in group theory [12] states that:

A function g_j is an absolute invariant of a two-parameter group if it satisfies the following first-order linear differential equation:

$$\sum_i (\alpha_i S_i + \alpha_{i+1}) \frac{\partial g_j}{\partial S_i} = 0 \quad (5.28)$$

$$\sum_i (\beta_i S_i + \beta_{i+1}) \frac{\partial g_j}{\partial S_i} = 0 \quad (5.29)$$

($i = 1, 3, 5, 7, 9, 11, 13$) where, $S_i = x, y, z, u, v, w, \theta$ and

$$\alpha_i = \frac{\partial \mathcal{P}^{S_i}}{\partial a_1} \Big|_{(a_1^0, a_2^0)}, \quad \alpha_{i+1} = \frac{\partial \mathcal{Q}^{S_i}}{\partial a_1} \Big|_{(a_1^0, a_2^0)},$$

$$\beta_i = \frac{\partial \mathcal{P}^{S_i}}{\partial a_2} \Big|_{(a_1^0, a_2^0)}, \quad \beta_{i+1} = \frac{\partial \mathcal{Q}^{S_i}}{\partial a_2} \Big|_{(a_1^0, a_2^0)}$$

where, (a_1^0, a_2^0) denotes the value, which yields the identity element of the group.

Derivation of independent absolute invariant:

Now independent absolute invariants are obtained.

From first order differential equations in (5.28) and (5.29) the following is derived.

$$(\alpha_1 x) \frac{\partial \eta}{\partial x} + (\alpha_3 y) \frac{\partial \eta}{\partial y} + (\alpha_5 z) \frac{\partial \eta}{\partial z} = 0 \quad (5.30)$$

$$(\beta_1 x) \frac{\partial \eta}{\partial x} + (\beta_3 y) \frac{\partial \eta}{\partial y} + (\beta_5 z) \frac{\partial \eta}{\partial z} = 0 \quad (5.31)$$

Here, $\alpha_2 = \alpha_4 = \alpha_6 = 0$ since $Q^x = Q^y = Q^z = 0$

By eliminating $\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$ from equation (5.30) and (5.31) the following is obtained.

$$\lambda_{1_3} x \frac{\partial \eta}{\partial x} + \lambda_{5_3} z \frac{\partial \eta}{\partial z} = 0 \quad (5.32)$$

$$-\lambda_{1_3} y \frac{\partial \eta}{\partial y} + \lambda_{5_1} z \frac{\partial \eta}{\partial z} = 0 \quad (5.33)$$

where $\lambda_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$

According to the basic theorem of group theory, system of equation has one and only one solution if its coefficient matrix has a rank two. The matrix has a rank two whenever at least one of its two by two sub matrices has a non-vanishing determinant. So, the following cases come into picture:

Case (i): $\lambda_{5_3} \neq 0, \lambda_{1_3} \neq 0, \lambda_{5_1} = 0$

Using the definitions of α_i 's and β_i 's and equations (5.25), (5.26) and (5.27)

$$\lambda_{5_1} = \alpha_5 \beta_1 - \alpha_1 \beta_5 = 0 \text{ (because } \alpha_5 = \beta_5 = 0)$$

$$\frac{\partial \eta}{\partial y} = 0 \quad (5.34)$$

η is a function of only x and z .

$\eta = \gamma_1 z$, where γ_1 is an arbitrary constant.

Case (ii): $\lambda_{5_3} = 0, \lambda_{5_1} \neq 0, \lambda_{1_3} \neq 0$

But $\lambda_{5_1} = 0 (\alpha_1 = \alpha_5)$

Here the rank of the coefficient matrix is one. So, this case is not possible.

Case (iii): $\lambda_{5_3} \neq 0, \lambda_{1_3} = 0, \lambda_{5_1} \neq 0$

Here also, rank is one so this case is not possible.

So, from all cases

$\eta = \gamma_1 z$, here $\gamma_1 = \sqrt{\frac{\alpha}{\vartheta_f}}$ is selected

$$\eta = \sqrt{\frac{\alpha}{\vartheta_f}} z, \quad (5.35)$$

ϑ_f is the kinematic viscosity of the base fluid.

Similarly, the absolute invariants of the dependent variables u, v, w, θ are obtained from the group transformations.

Derivation of the absolute invariants of the dependent variable u

$$(\alpha_1 x) \frac{\partial g_1}{\partial x} + (\alpha_3 y) \frac{\partial g_1}{\partial y} + (\alpha_7 u) \frac{\partial g_1}{\partial u} = 0 \quad (5.36)$$

$$(\beta_1 x) \frac{\partial g_1}{\partial x} + (\beta_3 y) \frac{\partial g_1}{\partial y} + (\beta_7 u) \frac{\partial g_1}{\partial u} = 0 \quad (5.37)$$

Eliminating $\frac{\partial g_1}{\partial x}, \frac{\partial g_1}{\partial y}$

$$(\lambda_{3_1} y) \frac{\partial g_1}{\partial y} + (\lambda_{7_1} u) \frac{\partial g_1}{\partial u} = 0 \quad (5.38)$$

$$(-\lambda_{3_1} x) \frac{\partial g_1}{\partial x} + (\lambda_{7_3} u) \frac{\partial g_1}{\partial u} = 0 \quad (5.39)$$

Case (i): $\lambda_{3_1} \neq 0, \lambda_{7_1} = 0, \lambda_{7_3} \neq 0$ (Because $\alpha_1 = \alpha_7, \beta_1 = \beta_7$)

Using the definitions of α_i 's and β_i 's, so $\lambda_{7_1} = 0$

$$\frac{\partial g_1}{\partial y} = 0$$

$$g_1(\eta) = g_1(x, u)$$

$$g_1(\eta) = \frac{u}{x}$$

$$u = xg_1(\eta)$$

Case (ii): $\lambda_{3_1} = 0, \lambda_{7_1} \neq 0, \lambda_{7_3} \neq 0$

But $\lambda_{7_1} = 0$ (Because $\alpha_1 = \alpha_7, \beta_1 = \beta_7$)

Here rank of the coefficient matrix is one so this case is not possible.

Case (iii): $\lambda_{3_1} \neq 0, \lambda_{7_1} \neq 0, \lambda_{7_3} = 0$

But $\lambda_{7_1} = 0$ (Because $\alpha_1 = \alpha_7, \beta_1 = \beta_7$)

Here also, rank is one so this case is not possible.

Thus, $u = xg_1(\eta)$ is obtained.

(5.40)

Derivation of the absolute invariants of the dependent variable v

$$(\alpha_1 x) \frac{\partial g_2}{\partial x} + (\alpha_3 y) \frac{\partial g_2}{\partial y} + (\alpha_9 v) \frac{\partial g_2}{\partial v} = 0 \quad (5.41)$$

$$(\beta_1 x) \frac{\partial g_2}{\partial x} + (\beta_3 y) \frac{\partial g_2}{\partial y} + (\beta_9 v) \frac{\partial g_2}{\partial v} = 0 \quad (5.42)$$

Eliminating $\frac{\partial g_2}{\partial x}, \frac{\partial g_2}{\partial y}$ from equations (5.41) and (5.42)

$$(\lambda_{3_1} y) \frac{\partial g_2}{\partial y} + (\lambda_{9_1} v) \frac{\partial g_2}{\partial v} = 0 \quad (5.43)$$

$$(-\lambda_{3_1} x) \frac{\partial g_2}{\partial x} + (\lambda_{9_3} v) \frac{\partial g_2}{\partial v} = 0 \quad (5.44)$$

Case (i): $\lambda_{9_1} \neq 0, \lambda_{3_1} \neq 0, \lambda_{9_3} = 0$

Using the definitions of α_i 's and β_i 's, $\lambda_{9_3} = 0$

$$\frac{\partial g_2}{\partial x} = 0$$

$$g_2(\eta) = g_2(v, y)$$

$$g_2(\eta) = \frac{v}{y}$$

$$v = yg_2(\eta)$$

Case (ii): $\lambda_{9_1} = 0, \lambda_{3_1} \neq 0, \lambda_{9_3} \neq 0$

But $\lambda_{9_3} = 0$ ($\alpha_3 = \alpha_9$)

Here rank of the coefficient matrix is one so this case is not possible.

Case (iii): $\lambda_{9_1} \neq 0, \lambda_{3_1} = 0, \lambda_{9_3} \neq 0$

But $\lambda_{9_3} = 0$ ($\alpha_3 = \alpha_9$)

Here also, rank is one so this case is not possible.

So, from all cases,

$$v = yg_2(\eta) \quad (5.45)$$

is thus asserted.

Derivation of the absolute invariants of the dependent variable w, θ

$$(\alpha_1 x) \frac{\partial g_3}{\partial x} + (\alpha_3 y) \frac{\partial g_3}{\partial y} + (\alpha_{11} w) \frac{\partial g_3}{\partial w} = 0 \quad (5.46)$$

$$(\beta_1 x) \frac{\partial g_3}{\partial x} + (\beta_3 y) \frac{\partial g_3}{\partial y} + (\beta_{11} w) \frac{\partial g_3}{\partial w} = 0 \quad (5.47)$$

Eliminating $\frac{\partial g_3}{\partial x}, \frac{\partial g_3}{\partial y}$

$$(\lambda_{3_1} y) \frac{\partial g_3}{\partial y} + (\lambda_{11_1} w) \frac{\partial g_3}{\partial w} = 0 \quad (5.48)$$

$$(-\lambda_{3_1} x) \frac{\partial g_3}{\partial x} + (\lambda_{11_3} w) \frac{\partial g_3}{\partial w} = 0 \quad (5.49)$$

Here, $\lambda_{11_1} = 0$ ($\alpha_{11} = 0, \beta_{11} = 0$)

Only possible case for rank two is $\lambda_{3_1} \neq 0, \lambda_{11_3} \neq 0, \lambda_{11_1} = 0$

$$\frac{\partial g_3}{\partial y} = 0$$

$$g_3(\eta) = g_3(w, x)$$

$g_3(\eta) = \gamma_2 w$ is obtained.

$\gamma_2 = -\sqrt{\frac{\vartheta_f}{a}}$ is chosen.

$$\text{So, now } g_3(\eta) = -\sqrt{\frac{\vartheta_f}{a}} w \quad (5.50)$$

Similarly, $g_4(\eta) = \Lambda \theta$, if $\Lambda = 1$, is chosen

$$g_4(\eta) = \theta \quad (5.51)$$

Thus, the following absolute invariants are obtained:

$$\left. \begin{aligned} \eta &= \sqrt{\frac{a}{\vartheta_f}} z \\ g_1(\eta) &= \frac{u}{x} \\ g_2(\eta) &= \frac{v}{y} \\ g_3(\eta) &= -\sqrt{\frac{\vartheta_f}{a}} w \\ g_4(\eta) &= \theta \end{aligned} \right\} \quad (5.52)$$

5.3.2 The reduction to an ordinary differential equation

Differentiating equations in (5.52) with respect to x, y, z the following is obtained:

$$\frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial y} = 0, \quad \frac{\partial \eta}{\partial z} = \sqrt{\frac{a}{\vartheta_f}} \quad (5.53)$$

$$\frac{\partial u}{\partial x} = g_1(\eta), \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = \sqrt{\frac{a}{\vartheta_f}} x g_1'(\eta), \quad \frac{\partial^2 u}{\partial z^2} = \frac{a}{\vartheta_f} x g_1''(\eta) \quad (5.54)$$

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = g_2(\eta), \quad \frac{\partial v}{\partial z} = \sqrt{\frac{a}{\vartheta_f}} y g_2'(\eta), \quad \frac{\partial^2 v}{\partial z^2} = \frac{a}{\vartheta_f} y g_2''(\eta) \quad (5.55)$$

$$\frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = -\sqrt{\frac{a}{\vartheta_f}} \sqrt{\frac{\vartheta_f}{a}} g_3'(\eta) = -g_3'(\eta) \quad (5.56)$$

$$\frac{\partial \theta}{\partial x} = 0, \quad \frac{\partial \theta}{\partial y} = 0, \quad \frac{\partial \theta}{\partial z} = \sqrt{\frac{a}{\vartheta_f}} g_4'(\eta), \quad \frac{\partial^2 \theta}{\partial z^2} = \frac{a}{\vartheta_f} g_4''(\eta) \quad (5.57)$$

Applying above differentiations, equation (5.6) - (5.9) are converted into the following form

$$g_1(\eta) + g_2(\eta) - g_3'(\eta) = 0 \quad (5.58)$$

$$(g_1(\eta))^2 - g_1'(\eta)g_3(\eta) = \frac{a}{\vartheta_f} g_1''(\eta)\vartheta_{nf} \quad (5.59)$$

$$(g_2(\eta))^2 - g_2'(\eta)g_3(\eta) = \frac{a}{\vartheta_f} g_2''(\eta)\vartheta_{nf} \quad (5.60)$$

$$g_4'(\eta)g_3(\eta) = -\frac{a}{\vartheta_f} \alpha_{nf} g_4''(\eta) \quad (5.61)$$

with boundary conditions:

$$g_1(0) = a, \quad g_2(0) = b, \quad g_3(0) = 0, \quad g_4(0) = 1,$$

$$g_1(\infty) = 0, \quad g_2(\infty) = 0, \quad g_4(\infty) = 0 \quad (5.62)$$

$$\varepsilon_1 = \frac{1}{(1-\phi)^{2.5} \left\{ (1-\phi) + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f} \right\}} = \frac{\vartheta_{nf}}{\vartheta_f} \quad (5.63)$$

$$\varepsilon_2 = \frac{\frac{k_{nf}}{k_f}}{(1-\phi) + \phi \frac{\rho_s}{\rho_f}}, \quad \frac{\varepsilon_2}{pr} = \frac{\alpha_{nf}}{\vartheta_f}, \quad \text{where } pr = \frac{\vartheta_f}{\alpha_f} \quad (5.64)$$

Hence,

$$g'_3(\eta) - (g_1(\eta) + g_2(\eta)) = 0 \quad (5.65)$$

$$a\varepsilon_1 g''_1(\eta) + g'_1(\eta)g_3(\eta) - (g_1(\eta))^2 = 0 \quad (5.66)$$

$$a\varepsilon_1 g''_2(\eta) + g'_2(\eta)g_3(\eta) - (g_2(\eta))^2 = 0 \quad (5.67)$$

$$g'_4(\eta)g_3(\eta) + \frac{a\varepsilon_2}{pr} g''_4(\eta) = 0 \quad (5.68)$$

with boundary conditions:

$$\begin{aligned} g_1(0) = a, \quad g_2(0) = b, \quad g_3(0) = 0, \quad g_4(0) = 1, \\ g_1(\infty) = 0, \quad g_2(\infty) = 0, \quad g_4(\infty) = 0 \end{aligned} \quad (5.69)$$

5.4 Numerical Solution

Thermal conductivity of nanofluid depends on the following parameters:

- Particle volume fraction
- Particle material
- Particle size
- Particle shape
- Temperature
- Base fluid properties

Out of the above list nanoparticle volume fraction is considered for analysis of thermal conductivity of nanofluid in this thesis.

TABLE 5. 1 Physical properties

Physical properties	Base fluid(water)	Nanoparticles(cu)
c_p (J/kg K)	4179	385
ρ (kg/m ³)	997.1	8933
k (W/mK)	0.613	400

Values of parameters are as follows for following tables.

$$a = 2, b = 1, pr = 1$$

$$\text{For } \phi = 0, \varepsilon_1 = 1, \varepsilon_2 = 1$$

$$\text{For } \phi = 0.1, \varepsilon_1 = 0.7246, \varepsilon_2 = 1.3553$$

$$\text{For } \phi = 0.2, \varepsilon_1 = 0.6740, \varepsilon_2 = 1.8089$$

TABLE 5. 2 Values of velocity profile g_1 for different nanoparticle volume fraction

η	g_1 -values (for $\phi = 0$)	g_1 -values (for $\phi = 0.1$)	g_1 -values (for $\phi = 0.2$)
0	1	1	1
0.25	0.84131041	0.8175255	0.81170205
0.5	0.70875504	0.6700234	0.66071
0.75	0.59781376	0.550484	0.53931
1	0.50466858	0.4532017	0.4412495
2	0.25504	0.20928789	0.1992617
3	0.1209028	0.091806	0.0857307
4	0.04475469	0.03176585	0.029167
5	0	0	0

TABLE 5. 3 Values of velocity profile g_2 for different nanoparticle volume fraction

η	g_2 -values (for $\phi = 0$)	g_2 -values (for $\phi = 0.1$)	g_2 -values (for $\phi = 0.2$)
0	2	2	2
0.25	1.60465339	1.5468235	1.53284638
0.5	1.300117632	1.2126957	1.1920748
0.75	1.0619673	0.961434239	0.938225
1	0.873138	0.769132664	0.7455767
2	0.4133	0.331773678	0.314386
3	0.190825	0.14180708	0.13183
4	0.0700832	0.048705989	0.04452
5	0	0	0

TABLE 5. 4 Values of temperature profile g_4 for different nanoparticle volume fraction

η	g_4 -values (for $\phi = 0$)	g_4 -values (for $\phi = 0.1$)	g_4 -values (for $\phi = 0.2$)
0	1	1	1
0.25	0.9023868	0.918637	0.927455
0.5	0.8073524	0.838828	0.85594
0.75	0.7167648	0.761682	0.786211
1	0.631828	0.687926	0.718757
2	0.356814	0.432779	0.47638
3	0.177166	0.240692	0.279477
4	0.0665021	0.100715	0.123106
5	0	0	0

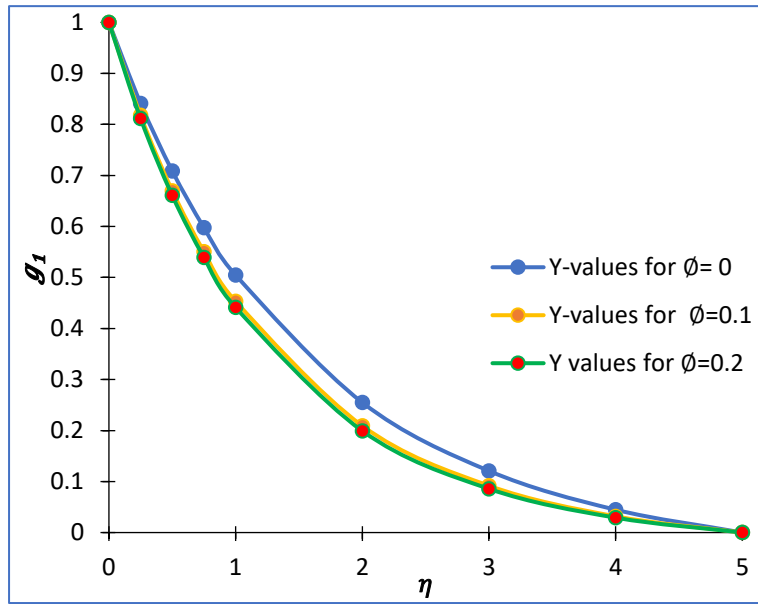


FIGURE 5. 2 X-directional velocity profile for different values of ϕ

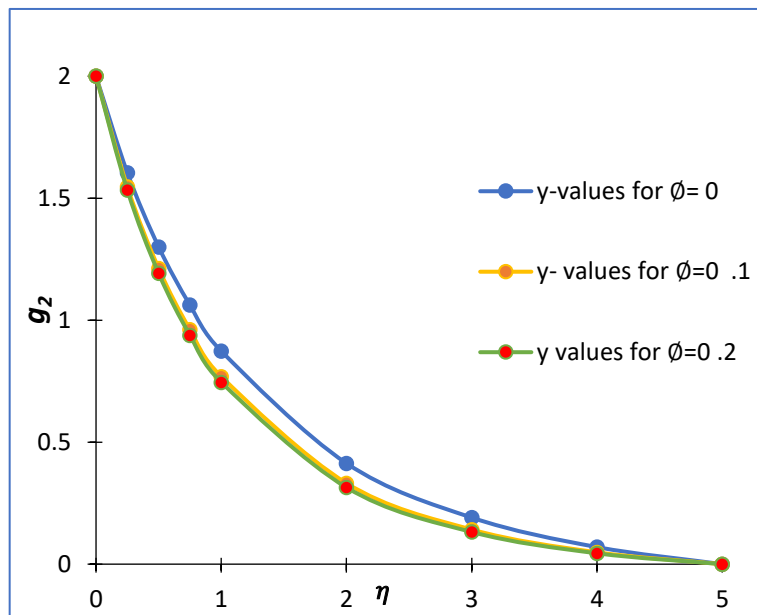


FIGURE 5. 3 Y-directional velocity profile for different values of ϕ

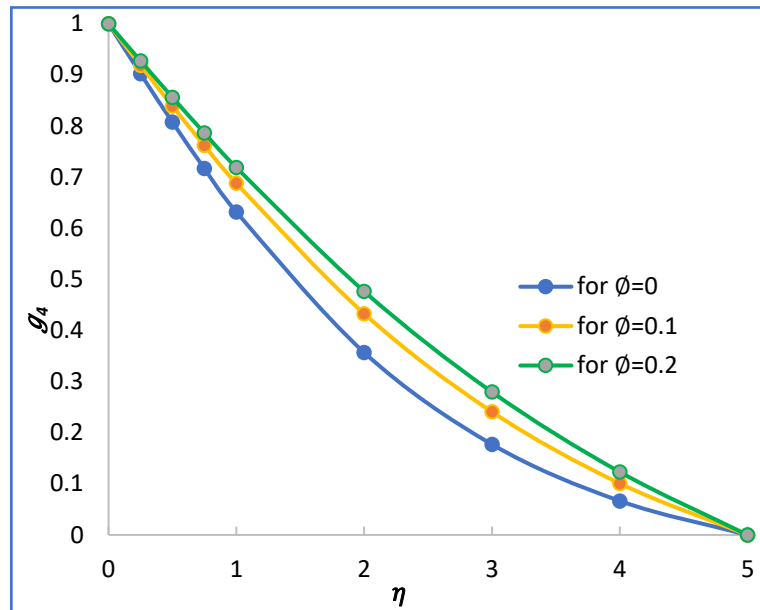


FIGURE 5. 4 Temperature profile for different values of ϕ

5.5 Conclusion

Deductive two-parameter group-theory technique is implemented to transform partial differential equations into ordinary differential equations for achieving similarity solution. Using powerful mathematical ODE solver MAPLE, the similarity equations are solved.

Here, the effect of the nanoparticle quantity on the fluid flow is investigated. We know that for pure fluid nanoparticle quantity $\phi = 0$ and for nanofluid it is $\phi \geq 0$. So, characteristics of pure fluid is compared with nanofluid for distinct quantities of volume fraction of nanoparticles. The impact of the fractional quantity of nanoparticles is examined on the velocity and temperature distributions.

Figure 5.2 and 5.3 and Table 5.1 and 5.2 show that the velocity profiles g_1 and g_2 are continuously decreasing as η rises, for all values of nanoparticle quantity ϕ . The downward trend becomes slower as ϕ rises. Figure 5.4 and Table 5.3 shows temperature profiles for distinct values of ϕ . Typically, same nature of velocity profiles and temperature profiles g_4 are observed. As η increases, temperature profile g_4 decreases. Contrary nature of the velocity profile for various values of ϕ is observed. As ϕ increases, the temperature profile g_4 rises.

Thus, it is concluded that the thermal conductivity of nanofluids is greater than that of pure fluid and that g_4 rises as ϕ rises.

CHAPTER-6

Similarity Analysis of Non-Newtonian Power-law Nanofluid Boundary Value Problem

6.1 Introduction

In chapter 5, analysis of Newtonian nanofluid flow boundary value problem was carried out. Similarity solutions were derived by applying two-parameter group-theoretic method. In this chapter the work is further extended for non-Newtonian MHD nanofluid flow over linearly stretching sheet in three dimensions. Brownian motion and thermophoresis effect on power-law model is considered for the same. Nonlinear system of partial differential equations with convective boundary conditions in governing equations is converted into the system of ordinary differential equations using deductive two-parameter group-theoretic similarity technique. The system consists of ordinary differential equations with given boundary conditions that are solved using MATLAB BVP4C coding. The influence of different physical parameters like flow consistency index, magnetic parameter, Biot number, thermophoresis parameter, stretching ratio parameter, Brownian motion parameter, Lewis number, Prandtl number on concentration, temperature and velocity are investigated with graphical representation.

6.1.1 Application of flow over stretching surface

The following are the application of flow over stretching surface:

- Paper production
- Plastic sheets production
- Rubber sheets production
- Glass fiber production
- Wire drawing

- Hot rolling etc... [66]

6.1.2 Importance of magnetic field

The existence of a magnetic field in fluid flow is of great importance because the magnetic particles mixture with fluid is used in many appliances such as MHD pump, nuclear reactors, MHD generator.

The magnetic nanofluid consist of both magnetic and liquid properties. Such materials have intriguing applications like optical modulators, magneto-optical wavelength filters, nonlinear optical components, optical switches, optical gratings etc. Magnetic particles play a key role in the construction of loudspeakers as sealing materials and in sink float separation. Magneto nanofluids are useful to guide the particles up the bloodstream to a tumour with magnets. The magnetic nanoparticles are considered to be more adhesive than non-malignant cells to tumour cells. This type of particles absorbs more power than microparticles in alternating current magnetic fields which is tolerable in humans i.e. for cancer therapy [67].

6.1.3 Literature review

A fluid is defined by its viscosity behaviour. Non-Newtonian behaviour of fluid is due to very long chains, coils or a platelet type of molecular structure of the fluids. Examples of non-Newtonian fluids are the mud encountered in drilling oil wells, the paper pulp suspensions of paper industry, the molten rayon, molten plastics, high polymers, pastes, thick suspensions, emulsions, oils and lubricating greases, substances of food industries, slurries, blood etc. Because of different properties of non-Newtonian fluids, one cannot propose a single model.

So, in the literature there are various models of non-Newtonian fluids available, proposed by scientists working in this area. Among them, the power-law model is the easiest model, that describes the most common existing nature of fluids like shear thinning and shear thickening. For $n > 1$ the fluid is described as dilatant, $n < 1$ pseudo- plastic and when $n = 1$ it is simply the Newtonian fluid [[34], [68]].

The impact of thermophoresis and Brownian motion on Powell-Eyring nanofluid model over a linearly stretching sheet is investigated by Hayat et al. using the series solution method [66]. Hayat et al. analysed second-grade nanofluid over the exponentially stretching surface using similarity method. The obtained transformed similarity equations are solved by applying the Homotopy analysis technique [58]. Nadeem et al. investigated the flow of nanofluid over an exponentially stretching surface using different types of nanoparticles [60]. Zhao et al. studied the effect of nanoparticle volume fraction on various parameters for three-dimensional nanofluid flow over a stretching sheet [59].

Effects of Brownian motion and thermophoresis on two-dimensional non-Newtonian power-law model of MHD nanofluid flow is examined by Masood Khan and W. A. K. They examined the nanofluid flow over a non-linear stretching sheet with zero nanoparticle mass flux boundary condition. They first applied similarity transformation and then used the shooting method to solve the converted system of ordinary differential equation numerically [67].

Two-dimensional Non-Newtonian Sisko fluid model is studied by M. Khan et al. They examined the influence of different physical parameters on nanofluid flow with convective boundary conditions using the Homotopy analysis technique [69]. Stanford Shateyi had analyzed three dimensional Newtonian nanofluid flow over linearly stretching sheet under the Magnetic field in porous media. This was analyzed in the presence of Brownian motion and thermophoresis with convective boundary conditions. They also applied spectral relaxation method to solve transformed governing equation numerically [70].

M. Khan et al. analysed heterogeneous-homogeneous chemical reactions for Sisko fluid flow in three dimensions past a bidirectional stretching sheet. Heat transfer analysis is done for Cattaneo-Christov heat flux model. They presented the results for the impact of different physical parameters on fluid flow for non-integer value of flow consistency index [71]. MHD two-dimensional Sisko nanofluid flow over a nonlinearly stretching sheet under radiation effect and chemical reactions is examined by B.C. Prasanna kumara et al. They observed more effectiveness of nonlinear radiation than linear thermal radiation [72].

Most of the similarity analysis is done on a nanofluid flow by assuming similarity variables. In this chapter, similarity variables are deduced systematically. Two independent variables from the governing equation are reduced by using deductive two-parameter group-theoretical method.

Moran and Gajjoli had developed deductive group formalism for similarity analysis and also included auxiliary conditions for boundary layer problems ([11], [12]). Many researchers solved different types of fluid flow problem using deductive group-theoretic similarity method to reduce independent variables ([16], [17], [61], [67], [73]).

Recently, Shukla et al. analysed nanofluid flow in three dimensions over linearly stretching sheet for Newtonian fluid model using two-parameter deductive group-theoretic similarity technique [74]. S. M. Hussain et al. analysed the influence of thermal radiation for three different types of nanofluid on viscous dissipative boundary layer flow. The flow is over a permeable exponentially stretching sheet under the magnetic field for two-dimensional Newtonian fluid flow using similarity method [75].

From the literature review, it is observed that most of the work done on the Newtonian fluid model is in three dimensions and little work on two-dimensional Non-Newtonian nanofluid.

Most of the work done to study MHD nanofluid flow, is by assuming similarity variables in two-dimension for non-Newtonian fluid models. This chapter considers three-dimensional MHD nanofluid flow with convective boundary conditions for non-Newtonian fluid model. The non-Newtonian model considered is power-law model. A complete set of similarity variables is derived as a part of this thesis. The same is then utilized to convert set of partial differential equations given in governing equations into ordinary differential equations.

6.2 Boundary value problem for MHD non-Newtonian nanofluid power-law model

The present research is based on the following assumptions:

- Flow is steady, laminar and three dimensional.
- Power-law nanofluid model is considered.
- Nanofluid is incompressible.
- The flat surface is stretching continuously in both x-and y-directions with the velocities $u_w = ax$ and $v_w = by$, respectively. The ambient temperature and concentration far away from the surface of the sheet T_∞ and C_∞ are assumed to be uniform.

- A hot fluid with temperature T_f is utilized to heat up or cool down the surface of the sheet by convective heat transfer mode, which provides the heat transfer coefficient h_f and convective mass transfer coefficient h_s .
- Convective concentration of fluid is C_f .
- A uniform magnetic field B is applied normally to the stretched sheet and the induced magnetic field is neglected by assuming very small Reynolds number.

The continuity, momentum, energy and concentration equations governing by boundary layer theory are as follows [52]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\lambda}{\rho} \frac{\partial}{\partial z} \left(-\frac{\partial u}{\partial z} \right)^n - \frac{\sigma B^2}{\rho} u \quad (6.2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \frac{\lambda}{\rho} \frac{\partial}{\partial z} \left[\left(-\frac{\partial u}{\partial z} \right)^{n-1} \frac{\partial v}{\partial z} \right] - \frac{\sigma B^2}{\rho} v \quad (6.3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2} + \tau_1 \left[D_B \left(\frac{\partial T}{\partial z} \frac{\partial C}{\partial z} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial z} \right)^2 \right] \quad (6.4)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D_B \left(\frac{\partial^2 C}{\partial z^2} \right) + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial z^2} \quad (6.5)$$

Values at boundary are given by:

$$z = 0 \Rightarrow u = u_w = ax, \quad v = v_w = by, \quad w = 0,$$

$$-k \frac{\partial T}{\partial z} = h_f (T_f - T), \quad -D_B \frac{\partial C}{\partial z} = h_s (C_f - C),$$

$$z = \infty \Rightarrow u = 0, \quad v = 0, \quad w = 0, \quad T = T_\infty, \quad C = C_\infty \quad (6.6)$$

Here u, v, w are velocity in the directions with respect to X, Y and Z respectively. Symbol T for the fluid temperature and C for fluid concentration, ρ is the fluid density, τ_1 indicates the heat capacitance ratio, Thermophoresis diffusion coefficient is D_T and the Brownian diffusion coefficient is D_B , $\lambda (> 0)$ the rheological constant, flow index is n , electrical conductivity of the fluid is σ , thermal diffusivity is α .

6.3 Application of Generalised two-parameter group-theoretic method

The method used in this chapter is generalized group-theoretic method. Under this general group of transformation, the three independent variables will be reduced by two. The boundary value type partial differential equations (6.1) - (6.5) which has three independent variables x, y and z is transformed into boundary value type ordinary differential equations with only one-independent variable. This is called similarity equation. Following is the group of transformations of two-parameters (a_1, a_2) in the form of

$$G: \bar{s} = \mathcal{P}^s(a_1, a_2)s + Q^s(a_1, a_2) \quad (6.7)$$

where s stands for x, y, z, u, v, w, T, C

\mathcal{P}^s and Q^s are real valued and at least differentiable in their real arguments (a_1, a_2) .

Derivatives of the transformations are obtained from G using chain rule.

$$\bar{s}_i = \frac{\mathcal{P}^s}{\mathcal{P}^i} s_i, \bar{s}_{ij} = \frac{\mathcal{P}^s}{\mathcal{P}^i \mathcal{P}^j} s_{ij} \quad (6.8)$$

where i and j stands for x, y, z and s stands for u, v, w, T, C .

Equations (6.1) to (6.5) remain invariant under group of transformations defined by G in equation (6.7) and derivatives in equation (6.8).

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = H(a_1, a_2) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (6.9)$$

$$\begin{aligned} & \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\lambda}{\rho} \frac{\partial}{\partial \bar{z}} \left(-\frac{\partial \bar{u}}{\partial \bar{z}} \right)^n + \frac{\sigma B^2}{\rho} \bar{u} \\ & = I(a_1, a_2) \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\lambda}{\rho} \frac{\partial}{\partial z} \left(-\frac{\partial u}{\partial z} \right)^n + \frac{\sigma B^2}{\rho} u \right) + J(a_1, a_2) \end{aligned} \quad (6.10)$$

$$\begin{aligned} & \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} - \frac{\lambda}{\rho} \frac{\partial}{\partial \bar{z}} \left[\left(-\frac{\partial \bar{u}}{\partial \bar{z}} \right)^{n-1} \frac{\partial \bar{v}}{\partial \bar{z}} \right] + \frac{\sigma B^2}{\rho} \bar{v} \\ & = L(a_1, a_2) \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} - \frac{\lambda}{\rho} \frac{\partial}{\partial z} \left[\left(-\frac{\partial u}{\partial z} \right)^{n-1} \frac{\partial v}{\partial z} \right] + \frac{\sigma B^2}{\rho} v \right) + E(a_1, a_2) \end{aligned} \quad (6.11)$$

$$\begin{aligned} & \bar{u} \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{T}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{T}}{\partial \bar{z}} - \alpha_{nf} \frac{\partial^2 \bar{T}}{\partial \bar{z}^2} - \tau_1 \left[D_B \left(\frac{\partial \bar{T}}{\partial \bar{z}} \frac{\partial \bar{C}}{\partial \bar{z}} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial \bar{T}}{\partial \bar{z}} \right)^2 \right] \\ & = N(a_1, a_2) \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} - \alpha_{nf} \frac{\partial^2 T}{\partial z^2} - \tau_1 \left[D_B \left(\frac{\partial T}{\partial z} \frac{\partial C}{\partial z} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial z} \right)^2 \right] \right) + R(a_1, a_2) \end{aligned} \quad (6.12)$$

$$\begin{aligned} & \bar{u} \frac{\partial \bar{C}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{C}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{C}}{\partial \bar{z}} - D_B \left(\frac{\partial^2 \bar{C}}{\partial \bar{z}^2} \right) - \frac{D_T}{T_\infty} \frac{\partial^2 \bar{T}}{\partial \bar{z}^2} = \\ & O(a_1, a_2) \left(u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} - D_B \left(\frac{\partial^2 C}{\partial z^2} \right) - \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial z^2} \right) + Q(a_1, a_2) \end{aligned} \quad (6.13)$$

The invariance of above equations (6.9) to (6.13) implies that

$$J(a_1, a_2) = E(a_1, a_2) = R(a_1, a_2) = Q(a_1, a_2) = 0 \quad (6.14)$$

This is satisfied if taken as $Q^u = Q^v = Q^w = 0$ and

$$H(a_1, a_2) = \frac{p^u}{p^x} = \frac{p^v}{p^y} = \frac{p^w}{p^z} \quad (6.15)$$

$$I(a_1, a_2) = \frac{(p^u)^2}{p^x} = \frac{p^u p^v}{p^y} = \frac{p^u p^w}{p^z} = \frac{(p^u)^n}{(p^z)^{n+1}} = p^u \quad (6.16)$$

$$L(a_1, a_2) = \frac{(p^v)^2}{p^y} = \frac{p^u p^v}{p^x} = \frac{p^v p^w}{p^z} = \frac{(p^u)^{n-1} p^v}{(p^z)^{n+1}} = p^v \quad (6.17)$$

$$N(a_1, a_2) = \frac{p^u p^T}{p^x} = \frac{p^v p^T}{p^y} = \frac{p^w p^T}{p^z} = \frac{p^T}{(p^z)^2} = \frac{p^T p^C}{(p^z)^2} = \left(\frac{p^T}{p^z} \right)^2 \quad (6.18)$$

$$O(a_1, a_2) = \frac{p^u p^C}{p^x} = \frac{p^v p^C}{p^y} = \frac{p^w p^C}{p^z} = \frac{p^C}{(p^z)^2} = \left(\frac{p^T}{p^z} \right)^2 \quad (6.19)$$

So, from equation (6.15) to (6.19) with boundary conditions (6.6) the following relation is established.

$$p^u = p^x, \quad p^v = p^y, \quad p^w = p^z = (p^x)^{\frac{n-1}{n+1}}, \quad p^C = 1, \quad (6.20)$$

$$Q^u = Q^x = Q^v = Q^y = Q^w = Q^z = Q^T = Q^C = 0 \quad (6.21)$$

Thus, a two-parameter group transformation of the following form is obtained

$$G: \begin{cases} \bar{x} = \mathcal{P}^x x \\ \bar{y} = \mathcal{P}^y y \\ \bar{z} = (\mathcal{P}^x)^{\frac{n-1}{n+1}} z \\ \bar{u} = \mathcal{P}^x u \\ \bar{v} = \mathcal{P}^y v \\ \bar{w} = (\mathcal{P}^x)^{\frac{n-1}{n+1}} w \\ \bar{T} = T, \bar{C} = C \end{cases} \quad (6.22)$$

6.4 Derivation of Absolute Invariants

The objective is to derive proper absolute invariants in a manner so as to transform the set of partial differential equations into ordinary differential equations. Moran and Gaggioli's theorem say that a g_j function gives the absolute invariant for group transformations of two-parameters if it satisfies the following linear differential equation of first-order:

$$\begin{aligned}
 &(\alpha_1 x + \alpha_2) \frac{\partial g}{\partial x} + (\alpha_3 y + \alpha_4) \frac{\partial g}{\partial y} + (\alpha_5 z + \alpha_6) \frac{\partial g}{\partial z} + (\alpha_7 u + \alpha_8) \frac{\partial g}{\partial u} + \\
 &(\alpha_9 v + \alpha_{10}) \frac{\partial g}{\partial v} + (\alpha_{11} w + \alpha_{12}) \frac{\partial g}{\partial w} + (\alpha_{13} T + \alpha_{14}) \frac{\partial g}{\partial T} + (\alpha_{15} C + \alpha_{16}) \frac{\partial g}{\partial C} = 0 \quad (6.23)
 \end{aligned}$$

$$\begin{aligned}
 &(\beta_1 x + \beta_2) \frac{\partial g}{\partial x} + (\beta_3 y + \beta_4) \frac{\partial g}{\partial y} + (\beta_5 z + \beta_6) \frac{\partial g}{\partial z} + (\beta_7 u + \beta_8) \frac{\partial g}{\partial u} + \\
 &(\beta_9 v + \beta_{10}) \frac{\partial g}{\partial v} + (\beta_{11} w + \beta_{12}) \frac{\partial g}{\partial w} + (\beta_{13} T + \beta_{14}) \frac{\partial g}{\partial T} + (\beta_{15} C + \beta_{16}) \frac{\partial g}{\partial C} = 0 \quad (6.24)
 \end{aligned}$$

where

$$\begin{aligned}
 \alpha_i &= \frac{\partial \mathcal{P}^{s_i}}{\partial a_1} \Big|_{(a_1^0, a_2^0)}, \quad \alpha_{i+1} = \frac{\partial \mathcal{Q}^{s_i}}{\partial a_1} \Big|_{(a_1^0, a_2^0)} \\
 \beta_i &= \frac{\partial \mathcal{P}^{s_i}}{\partial a_2} \Big|_{(a_1^0, a_2^0)}, \quad \beta_{i+1} = \frac{\partial \mathcal{Q}^{s_i}}{\partial a_2} \Big|_{(a_1^0, a_2^0)} \quad (6.25)
 \end{aligned}$$

$$(i = 1, 3, 5, 7, 9, 11, 13, 15)$$

The identity element is (a_1^0, a_2^0) of the group G .

6.4.1 Derivation of independent absolute invariant

Now independent absolute invariants are to be obtained.

From first order differential equations in (6.23) and (6.24) the following is obtained

$$(\alpha_1 x) \frac{\partial \eta}{\partial x} + (\alpha_3 y) \frac{\partial \eta}{\partial y} + (\alpha_5 z) \frac{\partial \eta}{\partial z} = 0 \quad (6.26)$$

$$(\beta_1 x) \frac{\partial \eta}{\partial x} + (\beta_3 y) \frac{\partial \eta}{\partial y} + (\beta_5 z) \frac{\partial \eta}{\partial z} = 0 \quad (6.27)$$

Here, $\alpha_2 = \alpha_4 = \alpha_6 = 0$ since $Q^x = Q^y = Q^z = 0$

By eliminating $\frac{\partial \eta}{\partial y}, \frac{\partial \eta}{\partial x}$ from equation (6.26) and (6.27)

$$\lambda_{1_3} x \frac{\partial \eta}{\partial x} + \lambda_{5_3} z \frac{\partial \eta}{\partial z} = 0 \quad (6.28)$$

$$-\lambda_{13}y \frac{\partial \eta}{\partial y} + \lambda_{51}z \frac{\partial \eta}{\partial z} = 0 \quad (6.29)$$

where $\lambda_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$

According to the basic theorem of group theory, above equations have one and only one solution if the coefficient matrix has a rank two. The matrix has a rank two whenever at least one of its two by two submatrices has a non-vanishing determinant. So, the following cases are considered.

Case (i): $\lambda_{53} \neq 0, \lambda_{13} \neq 0, \lambda_{51} = 0$

Using the definitions of α_i 's and β_i 's from (6.25) and using equations (6.26) - (6.29) the following is obtained:

$$\lambda_{51} = \alpha_5 \beta_1 - \alpha_1 \beta_5 = 0 \quad (\text{Because } \alpha_5 = \frac{n-1}{n+1} \alpha_1, \beta_5 = \frac{n-1}{n+1} \beta_1)$$

$$\frac{\partial \eta}{\partial y} = 0$$

η is a function of only x and z .

$$\eta = c_1 z(x)^{\frac{1-n}{1+n}}, \text{ where } c_1 \text{ is an arbitrary constant.}$$

Case (ii): $\lambda_{53} = 0, \lambda_{51} \neq 0, \lambda_{13} \neq 0$

$$\text{But we have } \lambda_{51} = 0 \quad (\alpha_5 = \frac{n-1}{n+1} \alpha_1, \beta_5 = \frac{n-1}{n+1} \beta_1)$$

Here the rank of the coefficient matrix is one so this case is not possible.

Case (iii): $\lambda_{53} \neq 0, \lambda_{13} = 0, \lambda_{51} \neq 0$

Here also, rank is one so this case is not possible.

So, from all cases

$$\eta = c_1 z(x)^{\frac{1-n}{1+n}} \quad (6.30)$$

Similarly, the absolute invariants of the dependent variables u, v, w, T, C are obtained from the group transformation.

Derivation of dependent absolute invariant for variable u :

$$(\alpha_1 x) \frac{\partial F_1}{\partial x} + (\alpha_3 y) \frac{\partial F_1}{\partial y} + (\alpha_7 u) \frac{\partial F_1}{\partial u} = 0 \quad (6.31)$$

$$(\beta_1 x) \frac{\partial F_1}{\partial x} + (\beta_3 y) \frac{\partial F_1}{\partial y} + (\beta_7 u) \frac{\partial F_1}{\partial u} = 0 \quad (6.32)$$

Eliminating $\frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y}$

$$(\lambda_{31}y) \frac{\partial F_1}{\partial y} + (\lambda_{71}u) \frac{\partial F_1}{\partial u} = 0 \quad (6.33)$$

$$(-\lambda_{31}x) \frac{\partial F_1}{\partial x} + (\lambda_{73}u) \frac{\partial F_1}{\partial u} = 0 \quad (6.34)$$

Case (i): $\lambda_{31} \neq 0, \lambda_{71} = 0, \lambda_{73} \neq 0$ (Because $\alpha_1 = \alpha_7, \beta_1 = \beta_7$)

Using the definitions of α_i 's and β_i 's from (6.25) we have $\lambda_{71} = 0$

$$\frac{\partial F_1}{\partial y} = 0 \quad (6.35)$$

$$F_1(\eta) = F_1(x, u) \quad (6.36)$$

$$c_2 F_1(\eta) = \frac{u}{x} \quad (6.37)$$

$$u = c_2 x F_1(\eta) \quad (6.38)$$

Case (ii): $\lambda_{31} = 0, \lambda_{71} \neq 0, \lambda_{73} \neq 0$

But we have $\lambda_{71} = 0$ (Because $\alpha_1 = \alpha_7, \beta_1 = \beta_7$)

Here rank of the coefficient matrix is one so this case is not possible.

Case (iii): $\lambda_{31} \neq 0, \lambda_{71} \neq 0, \lambda_{73} = 0$

But we have $\lambda_{71} = 0$ (Because $\alpha_1 = \alpha_7, \beta_1 = \beta_7$)

Here also, rank is one so this case is not possible.

$$\text{Thus, we get } u = c_2 x F_1(\eta) \quad (6.39)$$

Derivation of dependent absolute invariant for variable v :

$$(\alpha_1 x) \frac{\partial F_2}{\partial x} + (\alpha_3 y) \frac{\partial F_2}{\partial y} + (\alpha_9 v) \frac{\partial F_2}{\partial v} = 0 \quad (6.40)$$

$$(\beta_1 x) \frac{\partial F_2}{\partial x} + (\beta_3 y) \frac{\partial F_2}{\partial y} + (\beta_9 v) \frac{\partial F_2}{\partial v} = 0 \quad (6.41)$$

Eliminating $\frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial y}$

$$(\lambda_{31}y) \frac{\partial F_2}{\partial y} + (\lambda_{91}v) \frac{\partial F_2}{\partial v} = 0 \quad (6.42)$$

$$(-\lambda_{31}x) \frac{\partial F_2}{\partial x} + (\lambda_{93}v) \frac{\partial F_2}{\partial v} = 0 \quad (6.43)$$

Case (i): $\lambda_{91} \neq 0, \lambda_{31} \neq 0, \lambda_{93} = 0$

Using the definitions of α_i 's and β_i 's from (20), (21) and (22), we have $\lambda_{93} = 0$

$$\frac{\partial F_2}{\partial x} = 0$$

$$F_2(\eta) = F_2(v, y)$$

$$c_3 F_2(\eta) = \frac{v}{y}$$

$$v = c_3 y F_2(\eta) \quad (6.44)$$

Case (ii): $\lambda_{91} = 0, \lambda_{31} \neq 0, \lambda_{93} \neq 0$

But, $\lambda_{93} = 0$ ($\alpha_3 = \alpha_9$)

Here rank of the coefficient matrix is one so this case is not possible.

Case (iii): $\lambda_{91} \neq 0, \lambda_{31} = 0, \lambda_{93} \neq 0$

But, $\lambda_{93} = 0$ ($\alpha_3 = \alpha_9$)

Here also, rank is one so this case is not possible.

So, from all cases, we get

$$v = c_3 y F_2(\eta) \quad (6.45)$$

Derivation of dependent absolute invariant for variable w, T, C :

$$(\alpha_1 x) \frac{\partial F_3}{\partial x} + (\alpha_3 y) \frac{\partial F_3}{\partial y} + (\alpha_{11} w) \frac{\partial F_3}{\partial w} = 0 \quad (6.46)$$

$$(\beta_1 x) \frac{\partial F_3}{\partial x} + (\beta_3 y) \frac{\partial F_3}{\partial y} + (\beta_{11} w) \frac{\partial F_3}{\partial w} = 0 \quad (6.47)$$

Eliminating $\frac{\partial F_3}{\partial x}, \frac{\partial F_3}{\partial y}$

$$(\lambda_{31} y) \frac{\partial F_3}{\partial y} + (\lambda_{111} w) \frac{\partial F_3}{\partial w} = 0 \quad (6.48)$$

$$(-\lambda_{31} x) \frac{\partial F_3}{\partial x} + (\lambda_{113} w) \frac{\partial F_3}{\partial w} = 0 \quad (6.49)$$

Here, $\lambda_{111} = 0$ ($\alpha_{11} = \frac{n-1}{n+1} \alpha_1, \beta_{11} = \frac{n-1}{n+1} \beta_1$)

Only possible case for rank two is $\lambda_{31} \neq 0, \lambda_{113} \neq 0, \lambda_{111} = 0$

$$\frac{\partial F_3}{\partial y} = 0$$

$$c_4 F_3(\eta) = F_3(w, x)$$

$$\text{Hence, } w = c_4 F_3(\eta)(x)^{\frac{1-n}{1+n}} \quad (6.50)$$

Similarly,

$$\pi_4(\eta) = c_5 \theta \quad (c_5 = 1 \text{ and } \theta = \frac{T-T_\infty}{T_f-T_\infty} \text{ taken})$$

$$\pi_4(\eta) = \theta \quad (6.51)$$

$$\pi_5(\eta) = c_6 \emptyset \quad (c_6 = 1 \text{ and } \emptyset = \frac{C-C_\infty}{C_f-C_\infty} \text{ taken})$$

$$\pi_5(\eta) = \emptyset \quad (6.52)$$

Thus, the following absolute invariants are obtained:

$$\left\{ \begin{array}{l} F_1(\eta) = \frac{u}{c_2 x} \\ F_2(\eta) = \frac{v}{c_3 y} \\ F_3(\eta) = \frac{w}{c_4 (x)^{\frac{n-1}{1+n}}} \\ \pi_4(\eta) = \theta = \frac{T-T_\infty}{T_f-T_\infty} \\ \pi_5(\eta) = \emptyset = \frac{C-C_\infty}{C_f-C_\infty} \end{array} \right. \quad (6.53)$$

6.4.2 The reduction to an ordinary differential equation

Differentiating equations in (6.53) with respect to x, y, z the following is deduced

$$\frac{\partial \eta}{\partial x} = \frac{1-n}{1+n} \frac{\eta}{x}, \quad \frac{\partial \eta}{\partial y} = 0, \quad \frac{\partial \eta}{\partial z} = c_1 (x)^{\frac{1-n}{1+n}} \quad (6.54)$$

$$\frac{\partial u}{\partial x} = a \left[F_1(\eta) + \frac{1-n}{1+n} \eta F_1'(\eta) \right], \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = a c_1 (x)^{\frac{2}{1+n}} F_1'(\eta) \quad (6.55)$$

$$\frac{\partial^2 u}{\partial z^2} = a c_1^2 (x)^{\frac{3-n}{1+n}} F_1''(\eta) \quad (6.56)$$

$$\frac{\partial v}{\partial x} = b y \frac{1-n}{1+n} \frac{\eta}{x} F_2'(\eta), \quad \frac{\partial v}{\partial y} = b F_2(\eta), \quad \frac{\partial v}{\partial z} = b y F_2'(\eta) c_1 (x)^{\frac{1-n}{1+n}} \quad (6.57)$$

$$\frac{\partial^2 v}{\partial z^2} = b c_1^2 y (x)^{\frac{2(1-n)}{1+n}} F_2''(\eta) \quad (6.58)$$

$$\frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = c_1 c_4 F_3'(\eta) \quad (6.59)$$

$$\frac{\partial T}{\partial x} = (T_f - T_\infty) \frac{1-n}{1+n} \frac{\eta}{x} \pi_4'(\eta), \quad \frac{\partial T}{\partial y} = 0, \quad \frac{\partial T}{\partial z} = (T_f - T_\infty) \pi_4'(\eta) c_1 (x)^{\frac{1-n}{1+n}},$$

$$\frac{\partial^2 T}{\partial z^2} = (T_f - T_\infty) c_1^2(x) \frac{2(1-n)}{1+n} F_4''(\eta) \quad (6.60)$$

$$\text{Choose } c_1 = \left(\frac{a^{2-n}}{\lambda} \right)^{\frac{1}{n+1}}, \quad c_4 = -a \left(\frac{a^{n-2}}{\rho} \right)^{\frac{1}{n+1}}, \quad pr = \frac{\rho c_p u_w x}{k} (Re)^{\frac{-2}{n+1}},$$

$$Re = \frac{(u_w)^{2-n} x^n \rho}{\lambda}, \quad Bi_1 = \frac{h_f}{k} x (Re)^{\frac{-1}{n+1}}, \quad Bi_2 = \frac{k}{D_B} x (Re_b)^{\frac{-1}{n+1}}, \quad \frac{\sigma B^2}{\rho} = M$$

$$N_t = \tau_1 D_T \frac{(T_f - T_\infty)}{\alpha T_\infty}, \quad N_b = \tau D_B \frac{(C_f - C_\infty)}{\alpha}, \quad Le = \frac{\alpha}{D_B}, \quad c_1 c_4 = -a \quad (6.61)$$

Re is the local Reynolds number, pr represents the generalized Prandtl number and Bi_1 and Bi_2 the generalized Biot numbers, Le represents the Lewis number, N_b represents the Brownian motion parameter, N_t represents thermophoresis parameter.

Applying above differentiations and equations (6.54) - (6.61) in equation (6.1) - (6.5), equations are converted into following form

$$aF_1 + bF_2 - aF_3' + \frac{1-n}{1+n} a \eta F_1' = 0 \quad (6.62)$$

$$a(F_1)^2 - aF_1'F_3 + \frac{1-n}{1+n} a \eta F_1'F_1 - n a (-F_1')^{n-1} F_1'' + MF_1 = 0 \quad (6.63)$$

$$b(F_2)^2 - a F_2' F_3 + \frac{1-n}{1+n} a \eta F_2' F_1 + a(n-1)(-F_1')^{n-2} F_2' F_1'' + MF_2 - a (-F_1')^{n-1} F_2'' = 0 \quad (6.64)$$

$$pr \pi_4' F_3 - pr \frac{1-n}{1+n} \eta F_1 \pi_4' + \pi_4'' + N_b \pi_4' \pi_5' + N_t (\pi_4')^2 = 0 \quad (6.65)$$

$$\pi_5'' + \frac{N_t}{N_b} \pi_4'' + pr Le \pi_5' F_3' - pr Le \frac{1-n}{1+n} \eta F_1 \pi_5' = 0 \quad (6.66)$$

with boundary conditions:

$$F_1(0) = 1, F_2(0) = 1, F_3(0) = 0, \pi_4'(0) = -Bi_1(1 - \pi_4(0)), \pi_5'(0) = -Bi_2(1 - \pi_5(0)), \\ F_1(\infty) = 0, F_2(\infty) = 0, F_3(\infty) = 0, \pi_4(\infty) = 0, \pi_5(\infty) = 0 \quad (6.67)$$

To reduce, one equation chosen as below.

$$F_1 = g_1', \quad F_2 = g_2', \quad F_3 = \frac{2n}{1+n} g_1 + \frac{b}{a} g_2 + \frac{1-n}{1+n} \eta g_1' \quad (6.68)$$

Equations (6.62) - (6.66) are transformed as follows.

$$a(g_1')^2 - b g_1'' g_2 - \frac{2n}{1+n} a g_1'' g_1 - n a (-g_1'')^{n-1} g_1'' + M g_1' = 0 \quad (6.69)$$

$$b(g''_2)^2 - bg''_2g_2 - \frac{2n}{1+n}ag''_2g_1 - a(n-1)(-g''_1)^{n-2}g''_2g''_1 - a(-g''_1)^{n-1}g''_2 + Mg_2 = 0 \quad (6.70)$$

$$\pi''_4 + N_b\pi'_4\pi'_5 + N_t(\pi'_4)^2 + \frac{b}{a}pr\pi'_4g_2 + \frac{2n}{1+n}prg_1(\eta)\pi'_4 = 0 \quad (6.71)$$

$$\pi''_5 + \frac{N_t}{N_b}\pi''_4 + \frac{b}{a}Lepr\pi'_5g_2 + \frac{2n}{1+n}prLe g_1\pi'_5 = 0 \quad (6.72)$$

with boundary conditions:

$$g'_1(0) = 1, g'_2(0) = 1, \pi'_4(0) = -Bi_1(1 - \pi_4(0)), \pi'_5(0) = -Bi_2(1 - \pi_5(0))$$

$$g'_1(\infty) = 0, g'_2(\infty) = 0, g_3(\infty) = 0, g_4(\infty) = 0, g_5(\infty) = 0$$

6.5 Numerical Solution

For Bvp4c coding the above system of equations are converted in system of first order differential equations as follows:

Substitute y_i , for $i = 1, 2, \dots, 10$ for functions

$g_1, g'_1, g''_1, g_2, g'_2, g''_2, \pi_4, \pi'_4, \pi_5, \pi'_5$ respectively

$$y'_1 = y_2 \quad (6.73)$$

$$y'_2 = y_3 \quad (6.74)$$

$$y'_3 = \frac{(a(y_2)^2 - by_3y_4 - \frac{2n}{1+n}ay_1y_3 + My_2)}{na(-y_3)^{n-1}} \quad (6.75)$$

$$y'_4 = y_5 \quad (6.76)$$

$$y'_5 = y_6 \quad (6.77)$$

$$y'_6 = \frac{b(y_5)^2 - by_4y_6 - \frac{2n}{1+n}ay_1y_6 - a(n-1)(-y_3)^{n-2}y'_3y_6 + My_5}{a(-y_3)^{n-1}} \quad (6.78)$$

$$y'_7 = y_8 \quad (6.79)$$

$$y'_8 = -N_b y_8 y_{10} - N_t (y_8)^2 - \frac{b}{a} pr y_4 y_8 - \frac{2n}{1+n} pr y_1 y_8 \quad (6.80)$$

$$y'_9 = y_{10} \quad (6.81)$$

$$y'_{10} = -\frac{N_t}{N_b} y'_8 - \frac{b}{a} Le pr y_4 y_{10} - \frac{2n}{1+n} pr Le y_1 y_{10} \quad (6.82)$$

with boundary conditions

$$\begin{aligned} \eta = 0 &\Rightarrow y_1 = y_4 = 0, \quad y_2 = y_5 = 1, \quad y_8 = -Bi_1(1 - y_7(0)), \quad y_{10} = -Bi_2(1 - y_9(0)) \\ \eta = \infty &\Rightarrow y_1 = 0, \quad y_4 = 0, \quad y_7 = 0, \quad y_9 = 0 \end{aligned} \quad (6.83)$$

6.6 Graphical Representation

Highly nonlinear ordinary differential equations are obtained using similarity transformations for three-dimensional non-Newtonian MHD nanofluid flow, for a power-law fluid model over a linearly stretching sheet.

To solve these differential equations analytically is a very difficult task. So, here, bvp4c MATLAB coding is used to get a numerical solution to the given problem. Influence on velocity, temperature, and concentration is investigated under different physical parameters.

Figure 6.1 and 6.2 shows the effect of stretching parameter ratio (b/a) on concentration and temperature profile respectively. By increasing the stretching parameter ratio both the profile decreases.

Influence of thermal Biot number and concentration Biot number, on concentration and temperature profile are shown in Figure 6.3, 6.4, 6.5, 6.6. We observed an enhancement in the temperature and concentration profile for the increase in Biot numbers. Concentration Biot number does not affect the temperature profile much.

Lewis number is inversely proportional to Brownian diffusion coefficients. So, as Lewis number increases, Brownian diffusion decreases which further decreases nanoparticle concentration. So, by increasing the value of Lewis number, concentration boundary layer thickness decreases. This type of behaviour can be observed in Figure 6.7.

Figure 6.8 to 6.11 illustrate the behaviour of concentration and temperature profile for the impact of Brownian motion parameter and thermophoresis parameter. Both the profiles increase as the thermophoresis parameter increases. For large values of thermophoresis parameter N_t , thermophoresis forces are produced which increases temperature and concentration.

Increase in Brownian motion parameter increases temperature profile whereas decreases concentration profile. For very small nanoparticles Brownian motion is high in fluid and so the Brownian motion parameter is large. So, for large N_b , Brownian motion in nanofluid increases the kinetic energy of particles which further enhances the temperature.

Influence of Prandtl number on temperature and concentration profiles is shown in Figure 6.12 and 6.13. It is known that large Prandtl number has lower thermal diffusivity. So, by increasing the value of the Prandtl number, the thermal boundary layer thickness decreases. In the figure, temperature and concentration profile both decrease as Prandtl number increases.

Influence of magnetic parameter M on temperature and velocity profile is shown in Figure 6.14, 6.15 and 6.16. By increasing value of magnetic parameter M , thermal boundary layer thickness enhances and opposite behaviour is seen on velocity profiles. This happens because of magnetic field, Lorentz force is produced, which slows down the motion of fluid and thus increases temperature and concentration. Impact of flow index on two velocity profile g'_1 and g'_2 is shown in figure 6.17 to 6.20. For velocity g'_1 two different behaviours are observed, for power law index n . Near to sheet, velocity profile enhances as n increases while it decreases far from the sheet and the reverse the trend is observed for velocity g'_2 .

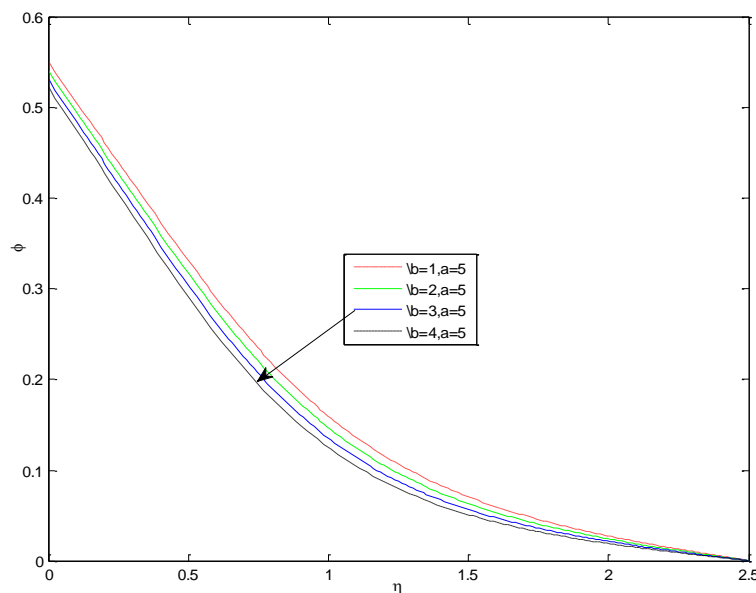


FIGURE 6. 1 Effect of stretching parameter ratio (b/a) on concentration profile for $a = 5, n = 1, M = 1, pr = 2, N_b = N_t = 0.2, Le = Bi_1 = Bi_2 = 1$

Graphical Representation

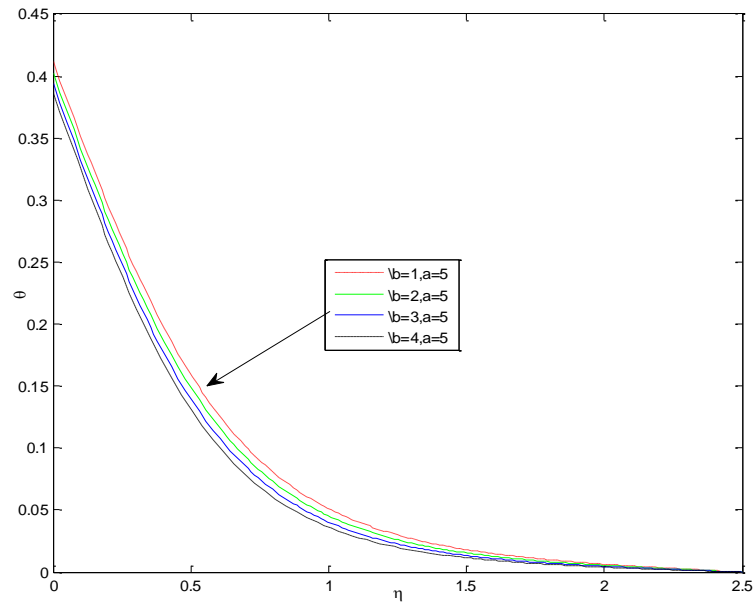


FIGURE 6. 2 Effect of stretching parameter (b/a) on temperature profile for $a = 5, n = 1, M = 1, pr = 2, N_b = N_t = 0.2, Le = Bi_1 = Bi_2 = 1$

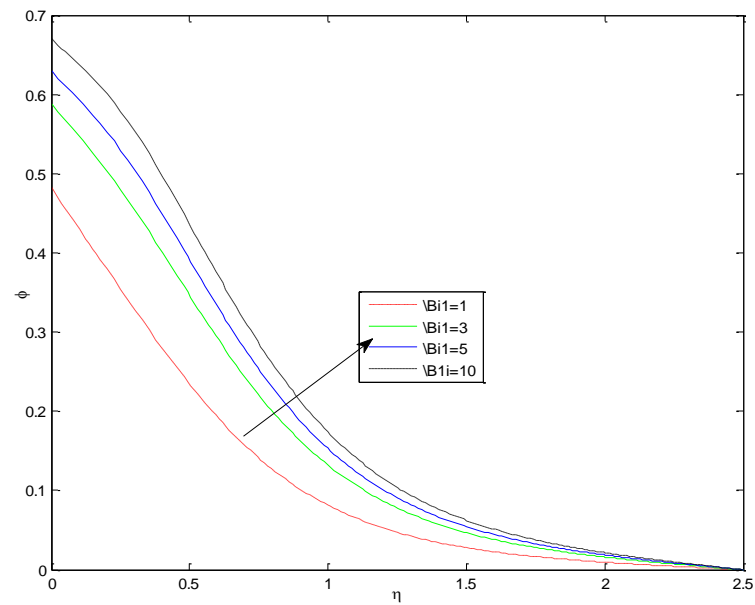


FIGURE 6. 3 Effect of Biot number Bi_1 on concentration profile for $a = 1, b = 2, n = 1, M = 1, pr = 2, N_b = N_t = 0.2, Le = 1, Bi_2 = 1$

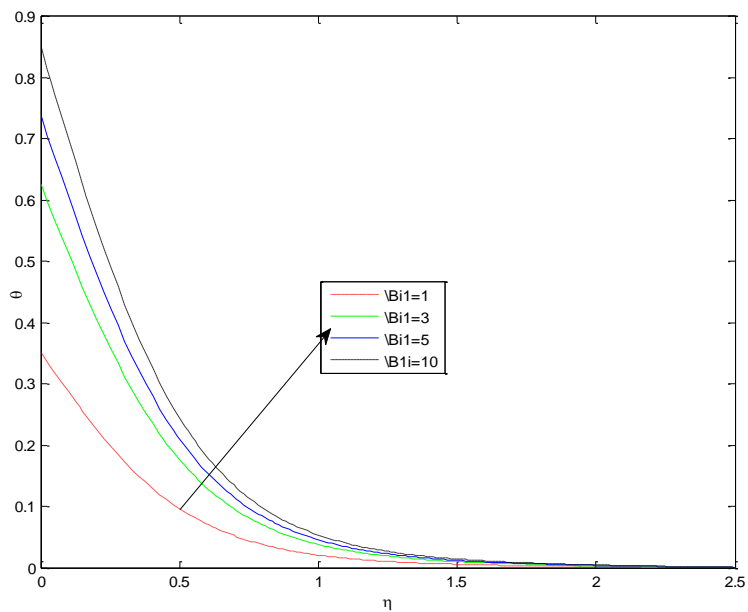


FIGURE 6. 4 Impact of Biot number Bi_1 on temperature profile for $a = 1, b = 2, n = 1, M = 1, pr = 2, N_b = 0.2, N_t = 0.2, Le = 1, Bi_2 = 1$

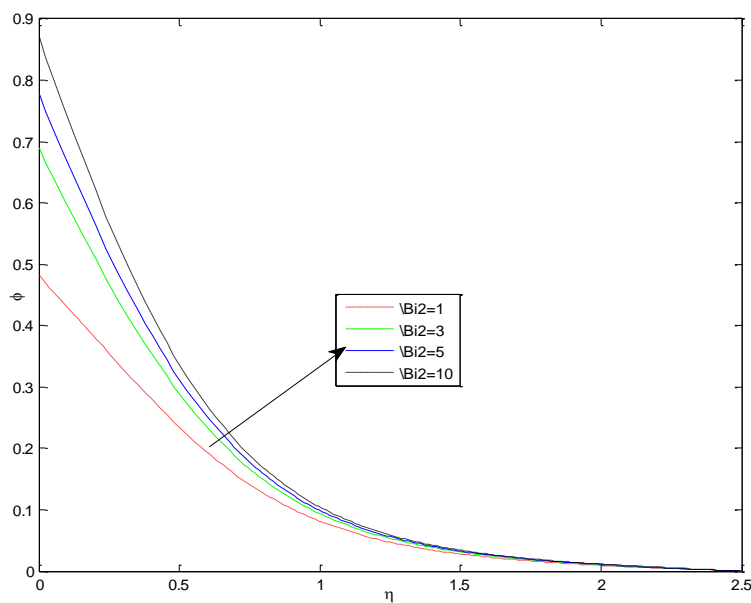


FIGURE 6. 5 Effect of Biot number Bi_2 on concentration profile for $a = 1, b = 2, n = 1, M = 1, pr = 2, N_b = 0.2, N_t = 0.2, Le = 1, Bi_1 = 1$

Graphical Representation

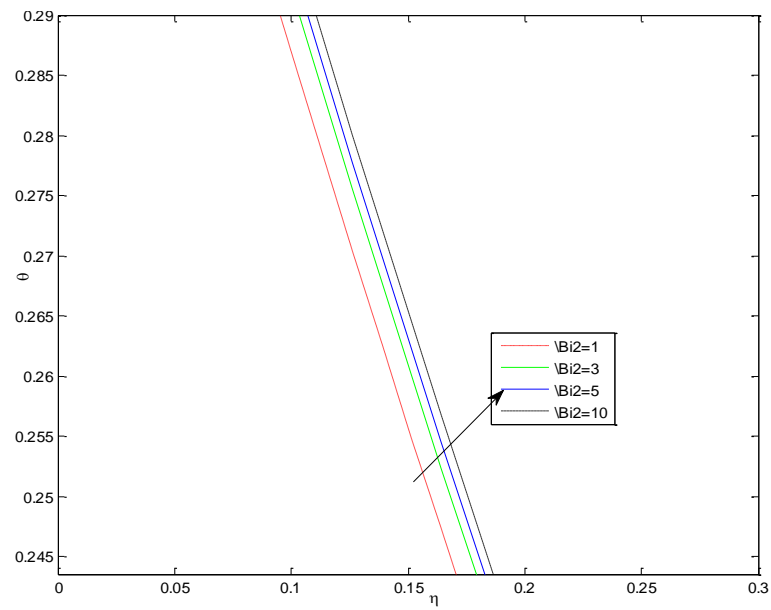


FIGURE 6. 6 Impact of Biot number Bi_2 on temperature for $\alpha = 1, b = 2, n = 1, M = 1, pr = 2, N_b = 0.2, N_t = 0.2, Le = 1, Bi_1 = 1$

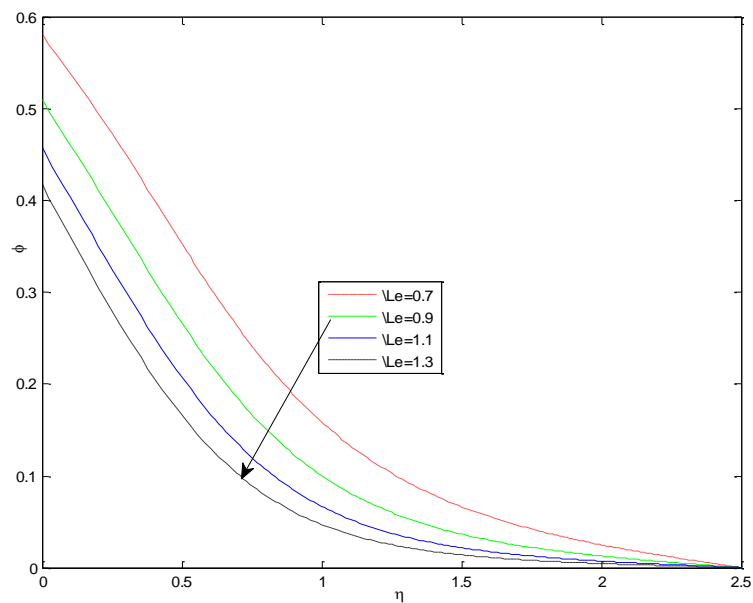


FIGURE 6. 7 Influence of Lewis number Le on concentration profile for $\alpha = 1, b = 2, n = 1, M = 1, pr = 2, N_b = N_t = 0.2, Bi_1 = Bi_2 = 1$

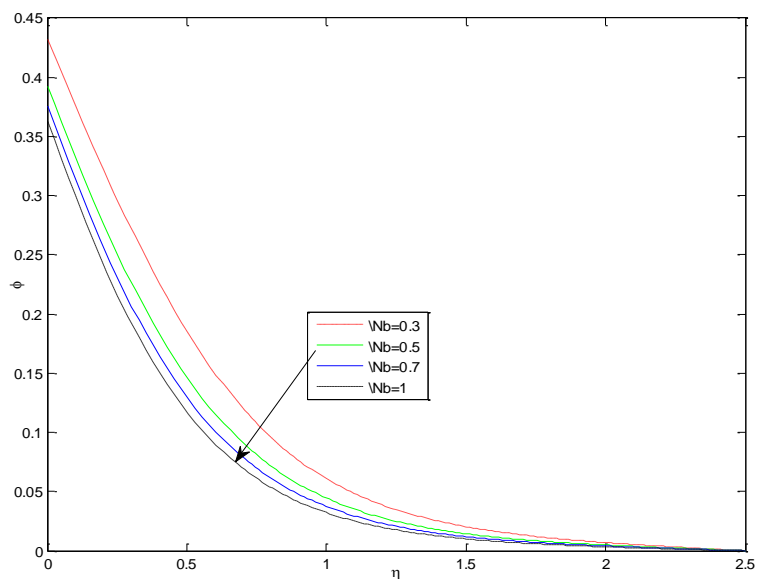


FIGURE 6. 8 Impact of Brownian motion parameter N_b on concentration profile for $a = 1, b = 2, n = 1, M = 1, pr = 2, N_t = 0.2, Le = Bi_1 = Bi_2 = 1$

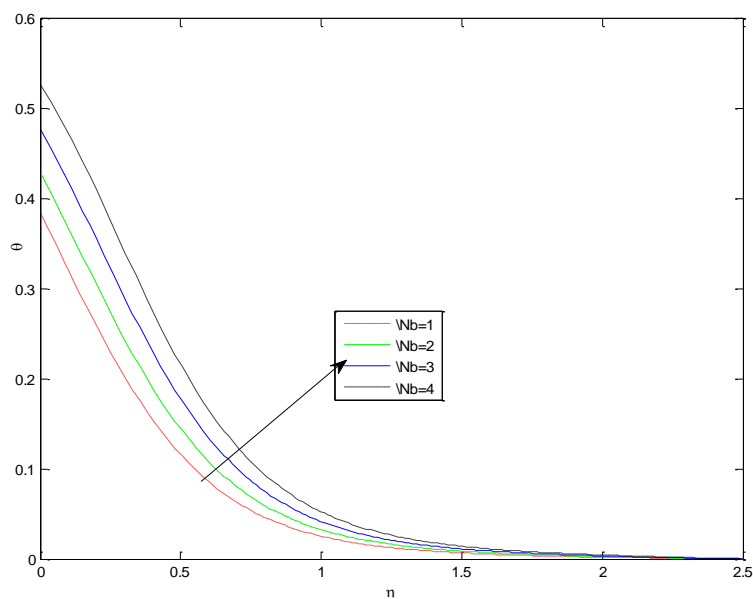


FIGURE 6. 9 Impact of Brownian motion parameter N_b on temperature profile for $a = 1, b = 2, n = 1, M = 1, pr = 2, N_t = 0.2, Le = Bi_1 = Bi_2 = 1$

Graphical Representation

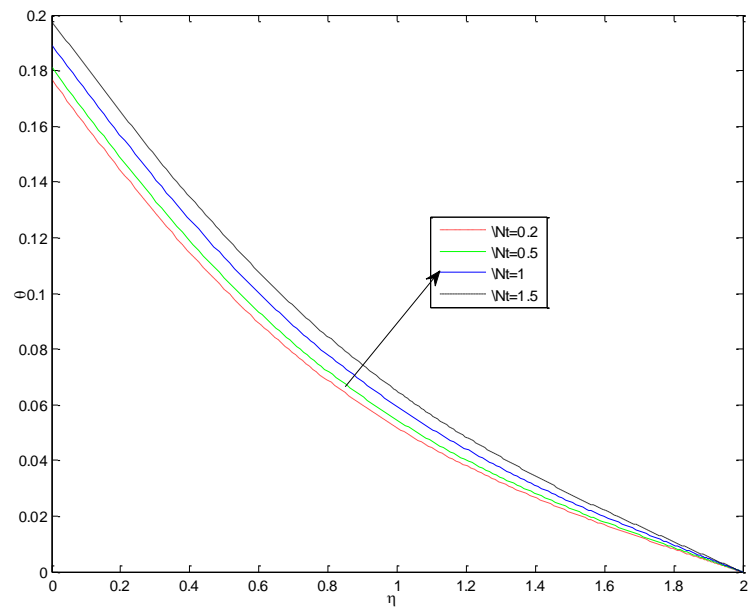


FIGURE 6. 10 Effect of thermophoresis parameter N_t on temperature profile for $a = 1, b = 2, n = 0.5, Le = 2, M = 1, N_b = 0.1, pr = 0.7, Bi_1 = Bi_2 = 0.2$

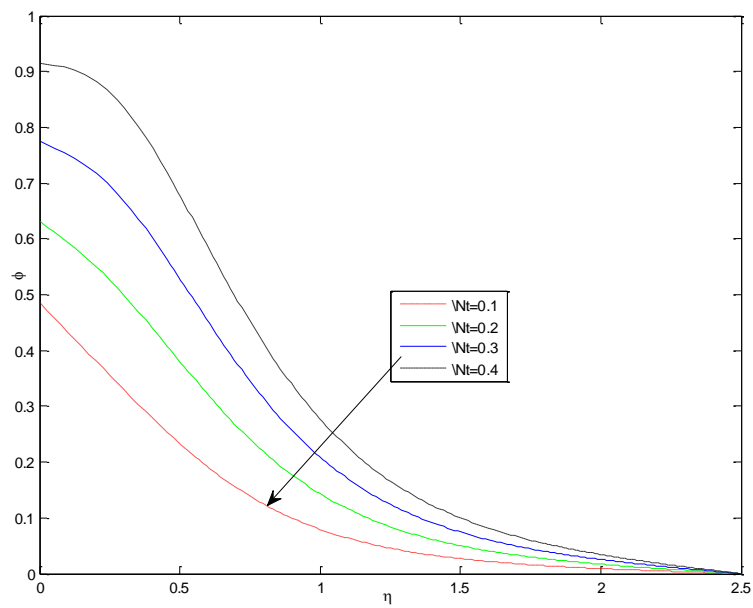


FIGURE 6. 11 Influence of thermophoresis parameter N_t on concentration profile for $a = 1, b = 2, n = 1, M = 1, pr = 2, N_b = 0.1, Le = Bi_1 = Bi_2 = 1$

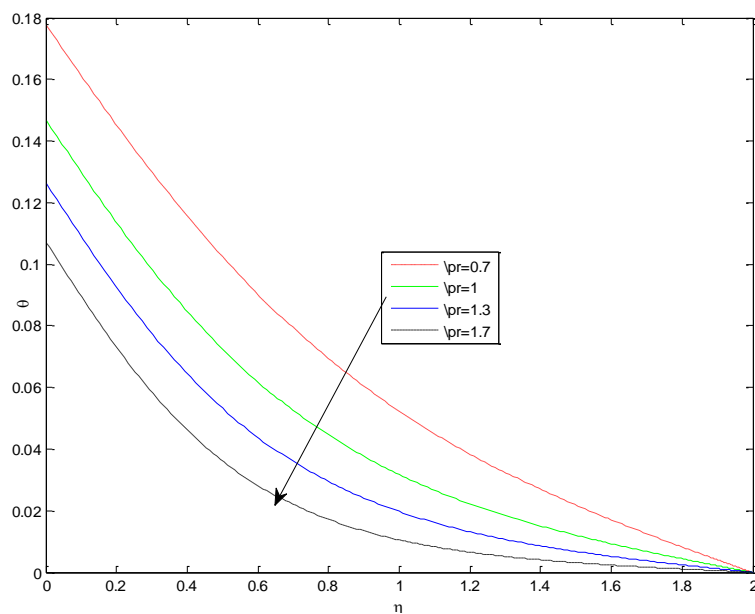


FIGURE 6.12 Impact of Prandtl number on temperature profile for $a = 1, b = 2, n = 0.5, Le = 2, M = 1, N_b = 0.2, N_t = 0.2, Bi_1 = Bi_2 = 0.2$

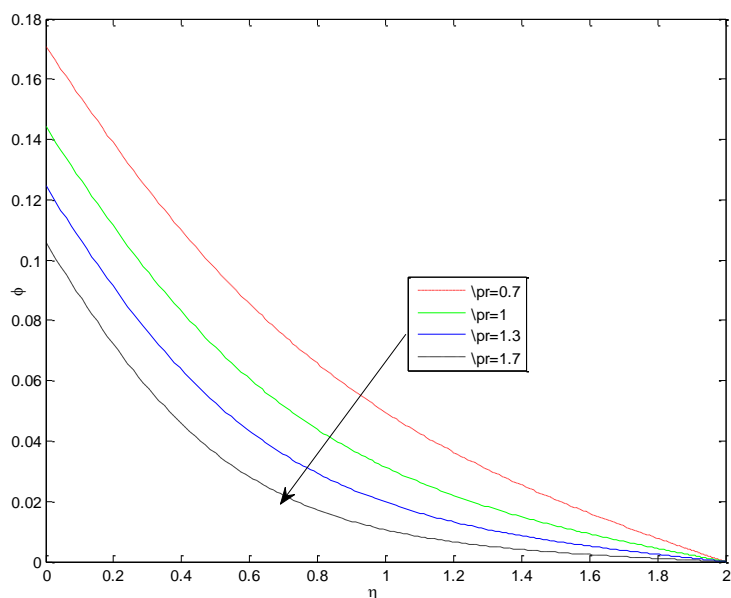


FIGURE 6.13 Effect of Prandtl number on concentration profile for $a = 1, b = 2, n = 0.5, M = 1, N_b = 0.2, N_t = 0.2, Le = 2, Bi_1 = Bi_2 = 0.2$

Graphical Representation

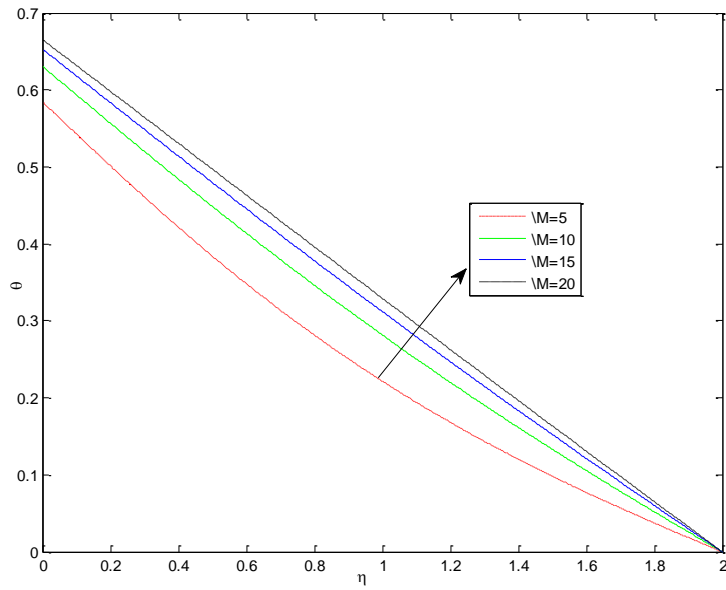


FIGURE 6. 14 Influence on temperature of magnetic parameter for M for $a = 2, b = 1, n = 0.5, M = 5, pr = 1, N_b = N_t = 0.2, Le = 2, Bi_1 = Bi_2 = 1$

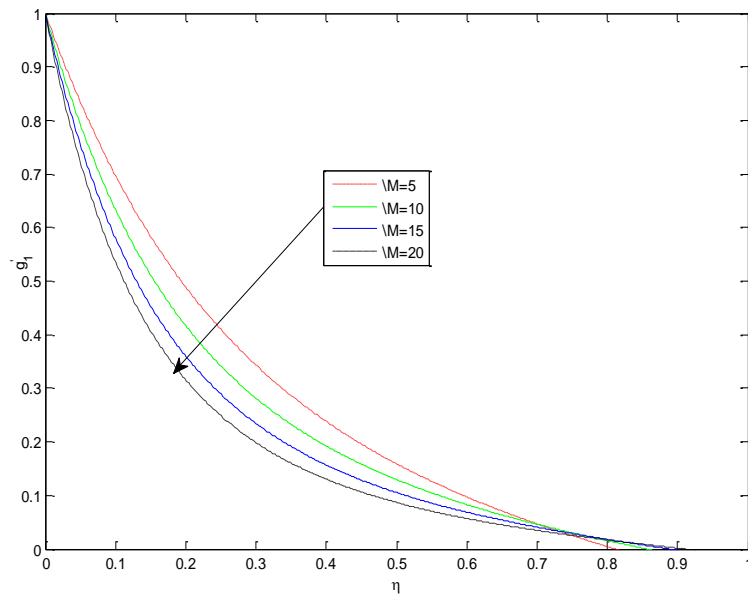


FIGURE 6. 15 Influence on velocity g'_1 of magnetic parameter M for $a = 2, b = 1, n = 0.5, pr = 1, N_b = N_t = 0.2, Le = 2, Bi_1 = Bi_2 = 1$

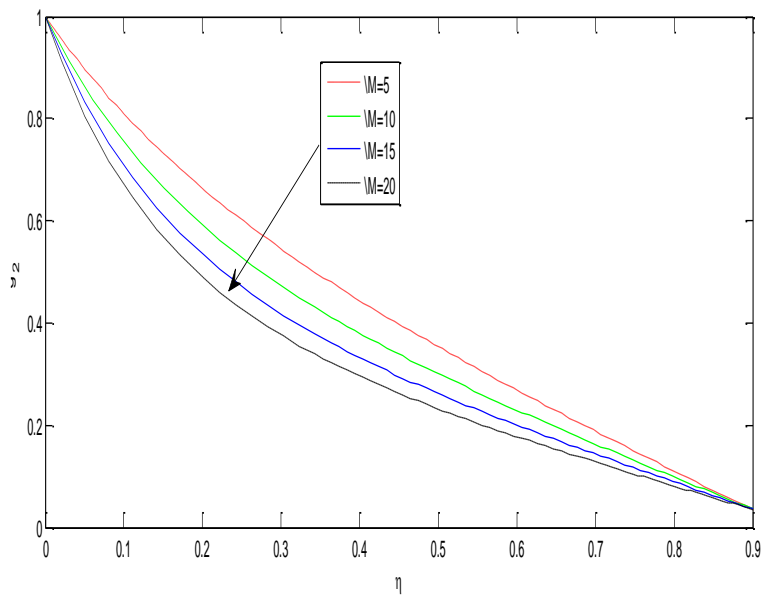


FIGURE 6. 16 Effect of magnetic parameter M on velocity g'_2 for $a = 1, b = 1, n = 0.5, pr = 1, N_b = N_t = 0.2, Le = 1, Bi_1 = Bi_2 = 0.2$

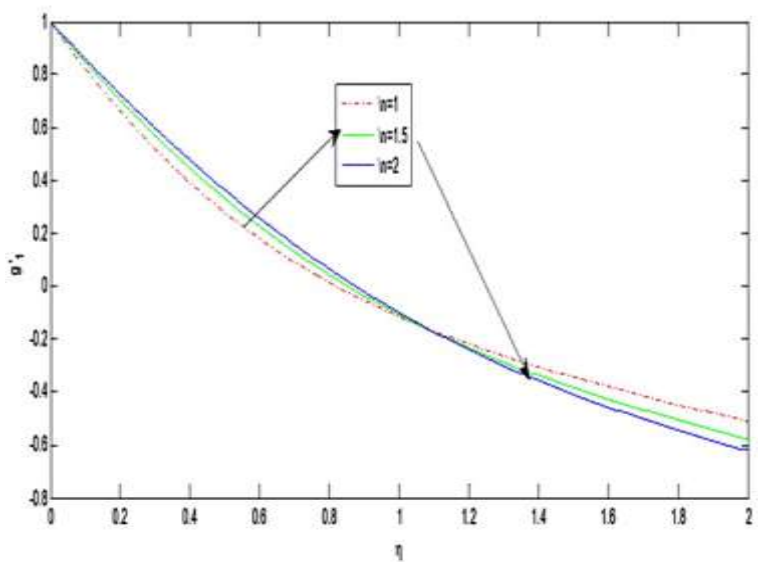


FIGURE 6. 17 Effect of power law index n greater than or equal to one on velocity g'_1 for $a = 2, b = 1, M = 1, pr = 1, N_b = 0.2, N_t = 0.2,$

Graphical Representation

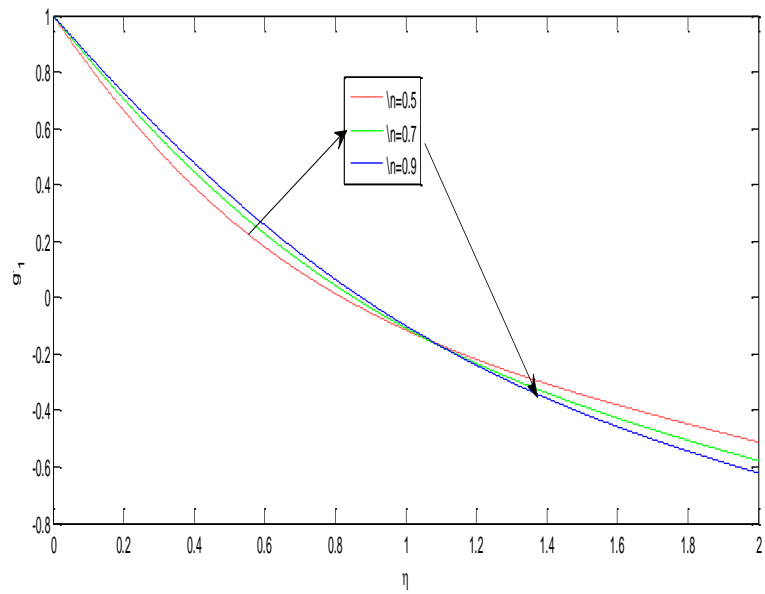


FIGURE 6. 18 Effect of power law index n less than one on velocity g'_1 for $a = 2, b = 1, M = 1, pr = 1, N_b = N_t = 0.2, Le = 2, Bi_1 = Bi_2 = 1$

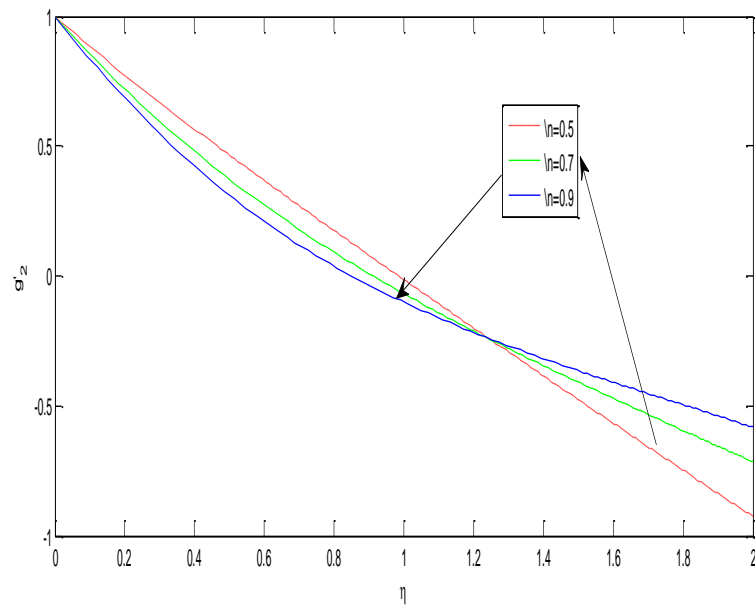


FIGURE 6. 19 Influence on velocity g'_2 of flow index n less than one for $a = 2, b = 1, M = 1, pr = 1, N_b = N_t = 0.2, Bi_1 = Bi_2 = 1$

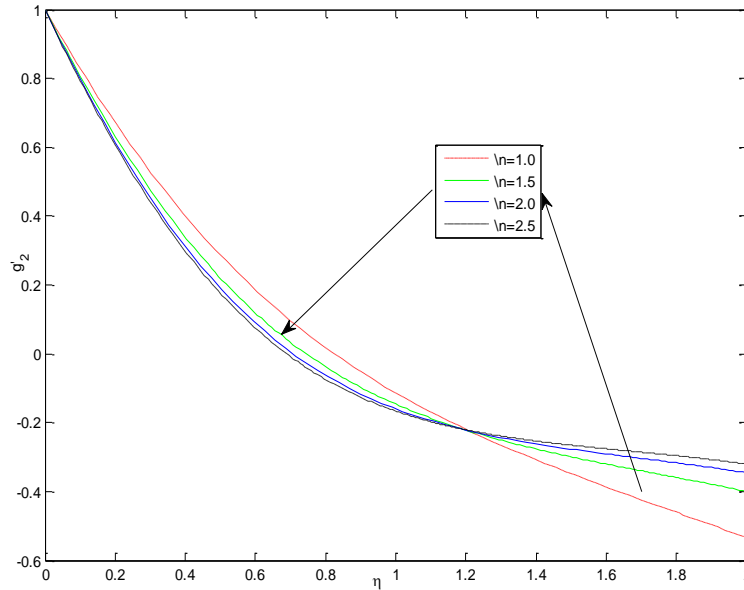


FIGURE 6. 20 Influence on velocity g'_2 of flow index n greater than or equal to one for $a = 2, b = 1, M = 1, pr = 1, N_b = 0.2, N_t = 0.2, Le = 2, Bi_1 = Bi_2 = 1$

6.7 Conclusion

- Nanofluid concentration and temperature both decrease as stretching ratio parameter increases.
- For increase in thermal Biot number and concentration Biot number, nanofluid concentration and temperature both increases. Concentration Biot number does not have much influence on temperature of nanofluid.
- Nanofluid concentration decreases for large Lewis number.
- Nanofluid concentration and temperature both increases by increasing thermophoresis parameter N_t
- Nanofluid temperature increases by increasing Brownian motion parameter whereas fluid concentration decreases as Brownian motion parameter increases.
- Large value of Magnetic parameter enhances the fluid temperature whereas the velocity diminishes.
- The velocity g'_1 and the velocity g'_2 have two different behaviour for power-law index n .

CHAPTER-7

Non-Similarity Analysis of Non-Newtonian Powell-Eyring Fluid Boundary Value Problem

7.1 Introduction

The boundary value problems raised in engineering can be solved by using two different techniques namely similarity and non-similarity. The similarity techniques used in the previous chapters make the boundary value problems easier by converting governing equations, which are partial differential equations, into ordinary differential equations. Invariant principle plays an important role in similarity methods. The similarity solution does not exist in certain kinds of physical situation for different flow problems. For these types of cases, in literature, non-similarity techniques give generalized solution.

There are various reasons for non-similarity to occur in boundary layer. For example, stream-wise variations in the free-stream velocity, surface mass transfer, transverse curvature, stream-wise variations in surface temperature, surface heat flux, volume heat generation, etc. (Sparrow et al. 1970 [21], Sparrow and Yu 1971 [22], Mehrdad Massoudi 2001 [76]). For Newtonian fluids, it is well known that the similarity solutions exist for the class of bodies known as the Falkner-Skan problems, which includes many practical geometries. Some authors (Hansen and Na (1968) [8], T.Y. Na (1994) [77] and V.S. Patil et al. (2015) [18]) observed in their research that the similarity solution exists only for the flow past a 90-degree wedge for non-Newtonian fluid model characterized by composite and implicit types of stress-strain relationship.

Even though very useful information is revealed from the similarity solution which gives information on the various physical parameters on the boundary layer characteristics, it is of limited engineering value. For practical purposes, bodies other than 90-degree wedge will most likely be encountered by not finding a solution by similarity methods.

This points to the need, for a general formulation and solution technique which can solve any problem of boundary layer flow of non-Newtonian fluids. The various schemes available for arriving to a solution to the above-mentioned problem, found in literature are: Finite difference scheme, Homotopy analysis method, local similarity method, local non-similarity method etc. But generally, from these methods, local similarity method is used frequently.

7.2 Local similarity method

In local similarity approach, the non-similar type boundary value problems are firstly transformed, by applying appropriate transformations, then the non-similar type of terms are discarded and then finally the equations derived are solved locally and independently at different streamwise locations. The main advantage of this method is that, we can find locally independent solution, at a particular stream-wise location, without knowledge of the solution of upstream locations. Another attractive feature of this method is that, the converted governing equations can be considered as ordinary differential equations similar to those obtained in similarity flow problems. So, the equations can be treat using well-known methods.

The main drawback of this method is uncertainty in the accuracy of results. This is because of the discarded terms from the equation and there is no way to adjust the effect of deleted terms on obtained results.

Thus, this approach of local similarity is based on the assumption that non-similar terms are so small that their values are near to zero, hence can be neglected. Therefore, governing partial differential equations are converted in the ordinary differential equations. So, results obtained, from the scheme of local similarity, are not accurate and valid for a small value of variables. Uncertainty in the accuracy is due to the negligence of non-similar terms which are near to zero, but are not zero actually and ideally must take into account.

7.3 Local Non-Similarity method

To overcome the drawback of local similarity, Sparrow et al. ([21], [22]) had introduced a new method called local non-similarity, by maintaining the same attractive aspects of local similarity method.

This feature being that, autonomous solution can be found at any streamwise position without information about upstream position. The differential equations thus obtained by this method can be treated as ordinary differential equations.

In the local non-similarity method, first differentiating the original governing equations with respect to the non-similar variable, auxiliary equations are obtained. Then new dependent variables, for the derivatives with respect to the non-similar variable ξ , are introduced. Therefore, the obtained equations are to be considered as the auxiliary equations of given original equations. Now, from this auxiliary equation, non-similar terms are deleted, considering them as very small. This is very similar to the way it is done in local similarity method. The main advantage is that, non-similar terms are not neglected from original equations but they are neglected from auxiliary equations. So, the results obtained from the local non-similarity method, are more accurate as compared to the ones obtained from the local similarity method.

7.4 Literature Review

If invariance principle of similarity is not satisfied by any of the governing equations or boundary conditions, then the problem is going to be of non-similarity type. One reason behind the non-existence of similarity solution is, non-linear relation of stress and strain in non-Newtonian fluid.

Hansen and Na [8] found a similarity solution of non-Newtonian Powell-Eyring fluid model using linear transformations, which forms group. The probability of similarity solution only for the flow over a 90-degree wedge was noticed. T.Y. Na [77] analysed the two-dimensional Reiner-Philippoff non-Newtonian fluid model. He found a similarity solution for a boundary layer flow over a 90-degree wedge and non-similarity solution on any body shape over boundary layer, flow using the method of finite difference. The similarity solution is recently achieved by H. Shukla et al. [61] for forced convection flow of Powell-Eyring and Prandtl-Eyring model. Here, boundary layer flow over a 90-degree wedge with free-stream velocity U in power-law form with power $1/3$ is considered. Thus, it is concluded that the similarity types of solution exist only for the value of $m = 1/3$, when free-stream velocity is in a power-law form $U = x^m$ for non-Newtonian fluid flow problem.

Main reasons for boundary layer non-similarity are surface mass transfer, stream-wise variations in the free-stream velocity, stream-wise variations in surface temperature, surface heat flux, transverse curvature, volume heat generation, etc. (Sparrow et al. [21], Sparrow and Yu [22], Mehrdad Massoudi [76]). There are different methods like Homotopy analysis method, Finite difference scheme, etc. applied by researchers to find non-similar solutions.

Sparrow et al. [21] had launched the method of local non-similarity and applied it on various non-similar velocity boundary value problems. Thermal boundary value problems are assessed utilizing the local non-similarity technique by Sparrow and Yu. [22]. Numerous scientists prefer to apply the local non-similarity method on because of its straightforwardness in concept and computation. There are two alluring highlights of the strategy, one is independence of local solutions from upstream information and second is obtained equations can be treated as ordinary differential equations as in similarity solutions.

Mehrdad Massoudi [76] had examined similarity, local-similarity, and local non-similarity technique to examine non-Newtonian power law model. He had considered the free-stream velocity, the injection velocity, and the surface temperature, as varying functions of the streamwise coordinate x for the flow over a porous wedge. M. Isomen et al. [78] carried out similarity and non-similarity analysis to contemplate the effect of buoyancy force on velocity and temperature, for the steady incompressible flow of fluid over an impermeable wedge. As a result, ordinary equations which are solved utilizing Runge-Kutta Gill with Shooting technique are obtained.

Governing equations of Casson non-Newtonian fluid flow are transformed in non-similar form by A. Subba Rao et al. [79] and utilized Keller-Box implicit difference method to solve the non-similar equations. The MHD Powell-Eyring fluid flow over a vertical plate in a porous medium, is examined numerically by applying the implicit finite difference Keller-Box method, by M. Sudhakar Reddy [80]. E.W. Mureithi and D.P. Mason [81] obtained non-similarity solution for a forced–free convection boundary layer flow over a horizontal plate with power-law variations in the freestream velocity and wall temperature, using local non-similarity method. T. C. Chiam [82] showed the proficiency of local non-similarity method using the two-equation model and three-equation model, by examining the MHD boundary layer flow over continuously moving flat plate.

Boundary Value Problem for Powell-Eyring model

Lok Yian Yian & Norsarahaida Amin [83] had applied the local non-similarity method to examine laminar free convection boundary layer flow, over a vertical flat plate with an exponential variation in surface temperature. Impact of various physical parameters over forced convective Hiemenz flow in porous media, is examined by M. Isomen et al. [78] by applying the local non-similarity method.

M. B. Akgul and M. Pakdemirli [84] had obtained the local non-similarity solution for the flow of an electrically conducting fluid over a Microcantilever-Based Sensor by considering mass transfer and chemical reaction at the sensor surface.

Abdullah et al. [85] contemplated unsteady mixed convection in the stagnation flow on a heated vertical surface, embedded in a nanofluid-saturated porous medium. The governing system of nonlinear partial differential equations is transformed using the Sparrow-Quack-Boerner local non-similarity technique. The obtained system is considered as a system of ordinary differential equations.

So, from the review of past work done on non-similarity, still more work is needed to be done in the area of simple method of local non-similarity to analyse different flow model in various physical situations for non-similar variations in boundary layer. In this thesis, two dimensional steady incompressible, laminar flow over a flat plate for non-Newtonian fluid model namely Powell-Eyring fluid model is considered. By applying local non-similarity method, the non-similar solution is found. Using the same, effects of the different physical parameters on velocity and temperature profile are examined. The aim of this chapter is to investigate the boundary layer flow whose non-similarity is caused by variations in the free-stream velocity by considering shapes other than a 90-degree wedge. This is done by entering the expression of the free-stream velocity into a power law form x^m .

7.5 Boundary Value Problem for Powell-Eyring model

The governing equations of 2-D, laminar, steady, incompressible flow over a flat plate are, continuity, momentum and energy in Cartesian form as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{1}{\rho} \frac{\partial}{\partial y} (\tau_{yx}) \quad (7.2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (7.3)$$

Subject to the boundary conditions:

$$y = 0: u = 0, v = 0, T = T_w \quad (7.4)$$

$$y = \infty: u = U(x), T = T_\infty \quad (7.5)$$

Here, velocities in the x, y directions are u, v respectively. Also, T - the fluid temperature, ρ - the fluid density, α - the thermal diffusivity, T_w - surface temperature, T_∞ - ambient temperature, U - free-stream velocity.

$\psi(x, y)$ is a stream function which satisfies continuity equation (7.1) and reduces one dependent variable.

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \quad (7.6)$$

Using stream function, equations (7.1) to (7.5) transformed as follows:

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U \frac{dU}{dx} + \frac{1}{\rho} \frac{\partial}{\partial y} (\tau_{yx}) \quad (7.7)$$

$$\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (7.8)$$

with boundary conditions:

$$y = 0 \Rightarrow \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} = 0, T = T_w \quad (7.9)$$

$$y = \infty \Rightarrow \frac{\partial \psi}{\partial y} = U(x), T = T_\infty \quad (7.10)$$

Powell-Eyring model is written as

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} + \frac{1}{B} \sinh^{-1} \left(\frac{1}{C} \frac{\partial u}{\partial y} \right) \quad (7.11)$$

where μ is dynamic viscosity, B and C are Powell-Eyring fluid parameters.

$$\sinh^{-1} \left(\frac{1}{C} \frac{\partial u}{\partial y} \right) \cong \frac{1}{C} \frac{\partial u}{\partial y} - \frac{1}{6} \left(\frac{1}{C} \frac{\partial u}{\partial y} \right)^3 \text{ for } \left| \frac{1}{C} \frac{\partial u}{\partial y} \right| \leq 1 \quad (7.12)$$

Substituting (7.11) and (7.12) in equation (7.7), the following is obtained:

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U \frac{dU}{dx} + \frac{1}{\rho} \left(\mu + \frac{1}{BC} \right) \frac{\partial^3 \psi}{\partial y^3} - \frac{1}{2\rho BC^3} \left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 \frac{\partial^3 \psi}{\partial y^3}. \quad (7.13)$$

7.6 Transformation of the Governing equation

Now, defining new variables ξ and η (M. B. Akgul and M. Pakdemirli [84]).

$$\xi = x^{\frac{1-m}{2}}, \quad \eta = \frac{y\sqrt{U}}{\sqrt{x\vartheta U}}, \quad f(\xi, \eta) = \frac{\psi}{\sqrt{x\vartheta U}}, \quad U = x^m, \quad \theta(\xi, \eta) = \frac{T-T_\infty}{T_w-T_\infty} \quad (7.14)$$

where variable η is chosen as a similarity variable if the boundary layer is similar, which depends on variable x and y . Variable ξ depends only on a variable x .

Applying above transformations converted equations (7.13), (7.8), (7.9) and (7.10) are converted in the following form.

$$(1+A)f'''' - AB\xi^{\frac{2(3m-1)}{1-m}}(f'')^2 f'''' + m(1-(f')^2) + \frac{m+1}{2}ff''$$

$$= \frac{1-m}{2}\xi[f' \frac{\partial^2 f}{\partial \xi \partial \eta} - f'' \frac{\partial f}{\partial \xi}] \quad (7.15)$$

$$\frac{1}{pr}\theta'' + \frac{m+1}{2}f\theta' = \frac{m-1}{2}\xi[\theta' \frac{\partial f}{\partial \xi} - f' \frac{\partial \theta}{\partial \xi}] \quad (7.16)$$

with boundary conditions:

$$f(\xi, 0) = 0, \quad f'(\xi, 0) = 0, \quad f'(\xi, \infty) = 1, \quad \theta(\xi, 0) = 1, \quad \theta(\xi, \infty) = 0 \quad (7.17)$$

where, prime on the functions denotes the differentiation with respect to variable η and

$$A = \frac{1}{\mu BC}, \quad B = \frac{1}{2\vartheta C^2}, \quad pr = \frac{\vartheta}{\alpha}.$$

7.7 Application of Local similarity method

According to local similarity method, the terms of RHS of equation (7.15) and (7.16) are discarded. It is true when ξ is very small, near to zero. And if ξ is not very small then the derivatives with respect to ξ are very small.

$$(1+A)f'''' - AB\xi^{\frac{2(3m-1)}{1-m}}(f'')^2 f'''' + m(1-(f')^2) + \frac{m+1}{2}ff'' = 0 \quad (7.18)$$

$$\frac{1}{pr}\theta'' + \frac{m+1}{2}f\theta' = 0 \quad (7.19)$$

with boundary conditions:

$$f(\xi, 0) = 0, \quad f'(\xi, 0) = 0, \quad f'(\xi, \infty) = 1, \quad \theta(\xi, 0) = 1, \quad \theta(\xi, \infty) = 0 \quad (7.20)$$

For $m = \frac{1}{3}$ above equations (7.18) - (7.20) give similarity solution.

Equation (7.18) and (7.19) can be treat as ODE (By considering ξ as a constant parameter [21]).

It should be noted that the solution corresponding to a given particular ξ is independent of the solution at any other ξ . As a result, the accuracy of the results obtained becomes uncertain as ξ increases. In fact, it is quite poor at large value of ξ .

7.8 Application of Local Non-similarity method

Sparrow et al. ([21], [22]) introduced the so-called method of local non-similarity.

Differentiating the original governing equations with respect to a variable ξ , auxillary equations are obtained. These obtained auxillary equations are considered as auxiliary equations of original equation. Then considering the variable ξ in this partial differential equation to be a constant, the system reduces to a system of ordinary differential equation.

Take $g(\xi, \eta) = \frac{\partial f(\xi, \eta)}{\partial \xi}$, $h(\xi, \eta) = \frac{\partial \theta(\xi, \eta)}{\partial \xi}$ in equation (7.15) - (7.17) with boundary conditions. Equations are transformed as follows:

$$\begin{aligned} & (1 + A)g''' - AB\xi^{\frac{2(3m-1)}{1-m}}(f'')^2g''' - AB\frac{2(3m-1)}{1-m}\xi^{\frac{2(3m-1)}{1-m}-1}(f'')^2f''' \\ & - 2AB\xi^{\frac{2(3m-1)}{1-m}}f''f'''g'' - 2mf'g' + \frac{m+1}{2}gf'' + \frac{m+1}{2}fg'' + \frac{m-1}{2}[f'g' - f''g] \\ & = \frac{m-1}{2}\xi\frac{\partial}{\partial \xi}[-f'g' + f''g]. \end{aligned} \tag{7.24}$$

$$\frac{1}{pr}h'' + \frac{m+1}{2}fh' + \frac{m+1}{2}g\theta' - \frac{m-1}{2}[\theta'g - f'h] = \frac{m-1}{2}\xi\frac{\partial}{\partial \xi}[\theta'g - f'h]. \tag{7.25}$$

with

$$\begin{aligned} & f(\xi, 0) = 0, \quad f'(\xi, 0) = 0, \quad f'(\xi, \infty) = 1, \quad \theta(\xi, 0) = 1, \quad \theta(\xi, \infty) = 0, \\ & g(\xi, 0) = 0, \quad g'(\xi, 0) = 0, \quad g'(\xi, \infty) = 0, \quad h(\xi, 0) = 0, \quad h(\xi, \infty) = 0. \end{aligned} \tag{7.26}$$

Equations (7.24) - (7.26) are auxiliary equations to the governing equations (7.21) - (7.22) with their boundary conditions in equation (7.23). Now, deleting the terms from the auxiliary equations (7.24) - (7.25) containing the differentiation with respect to stream-wise coordinate, from the right- hand side of equations as discussed by Sparrow and Yu [22].

With the above assumption, the momentum and energy boundary-layer equation (7.21) - (7.22) and its auxiliary equation (7.24) - (7.25) can be brought together with their boundary conditions as

$$(1 + A)f''' - AB\xi^{\frac{2(3m-1)}{1-m}}(f'')^2f''' + m(1 - (f')^2) + \frac{m+1}{2}ff'' - \frac{1-m}{2}\xi[f'g' - f''g] = 0. \quad (7.27)$$

$$(1 + A)g''' - AB\xi^{\frac{2(3m-1)}{1-m}}(f'')^2g''' - AB\frac{2(3m-1)}{1-m}\xi^{\frac{2(3m-1)}{1-m}-1}(f'')^2f''' - 2mf'g' - 2AB\xi^{\frac{2(3m-1)}{1-m}}f''f'''g'' + \frac{m+1}{2}gf'' + \frac{m+1}{2}fg'' + \frac{m-1}{2}[f'g' - f''g] = 0. \quad (7.28)$$

$$\frac{1}{pr}\theta'' + \frac{m+1}{2}f\theta' - \frac{m-1}{2}\xi[\theta'g - f'h] = 0. \quad (7.29)$$

$$\frac{1}{pr}h'' + \frac{m+1}{2}fh' + \frac{m+1}{2}g\theta' - \frac{m-1}{2}[\theta'g - f'h] = 0. \quad (7.30)$$

$$f(\xi, 0) = 0, \quad f'(\xi, 0) = 0, \quad f'(\xi, \infty) = 1, \quad \theta(\xi, 0) = 1, \quad \theta(\xi, \infty) = 0, \\ g(\xi, 0) = 0, \quad g'(\xi, 0) = 0, \quad g'(\xi, \infty) = 0, \quad h(\xi, 0) = 0, \quad h(\xi, \infty) = 0. \quad (7.31)$$

By considering ξ as a constant parameter, equations (7.28) to (7.30) may be treated as a system of ordinary differential equation.

7.9 Results and discussion

The system of ordinary differential equations, (7.28) to (7.30) is tackled numerically utilizing MATLAB bvp4c solver. Results are portrayed graphically for various physical parameters of the flow model. Figure 7.1 demonstrates the impact of different Prandtl numbers on the temperature profile. The figure indicates that, the temperature profile move towards the boundary when Prandtl number is increased. As Physical aspect of Prandtl number, this is attributed to the fact that a larger Prandtl number has a relatively lower thermal diffusivity causing a reduction in the thermal boundary layer thickness.

Figure 7.2 shows the impact of Powell-Eyring fluid parameter A on temperature profile for $m = 0$, means on a flat plate. From the observation of Figure 7.2, temperature θ enhances when Powell-Eyring fluid parameter A is increased.

The thermal boundary layer thickness is negligibly influenced by varying fluid parameter B at a wedge of 90 degree portrayed from Figure 7.3. From the zoom territory of Figure 7.3, it is observed that as B increases the thermal boundary layer thickness diminishes.

The impact of the stream-wise location on the temperature profile is also shown in Figure 7.4. Temperature profile upgrades with increment of ξ . Impact of the free-stream velocity power-law index m on temperature profile is shown in Figure 7.5. As the value of m increases, thermal boundary layer thickness decreases.

Figures 7.6 and 7.7 demonstrate the impact of the Powell-Eyring fluid parameters A and B , respectively, on the velocity profile. The velocity profile declines as A increments and velocity increases as B increases.

Figure 7.8 shows the velocity profile for different wedge angles. It is found that velocity increases with the increase of free-stream velocity power law index m . The effects of the stream-wise location on the velocity profile is also shown in Figure 7.9.

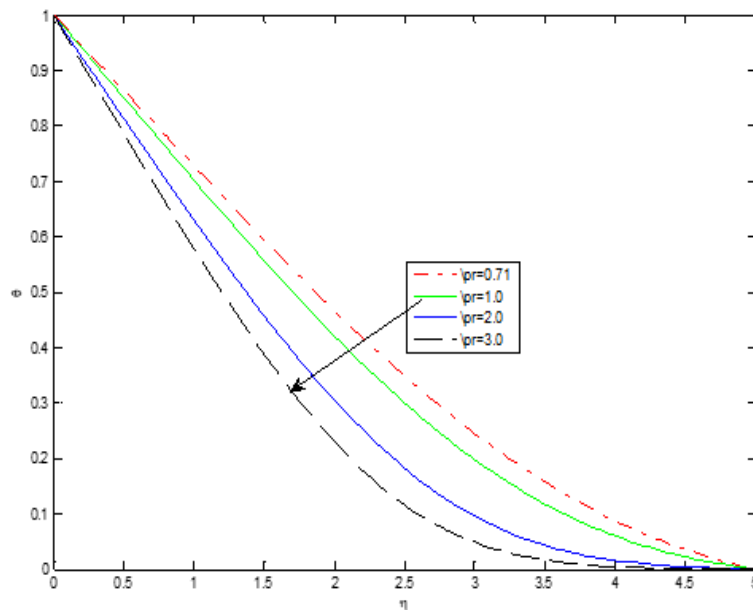


FIGURE 7. 1 Effect of Prandtl number on temperature profile for $A = 5, B = 0.1, m = 0, \xi = 0.1$

Results and discussion

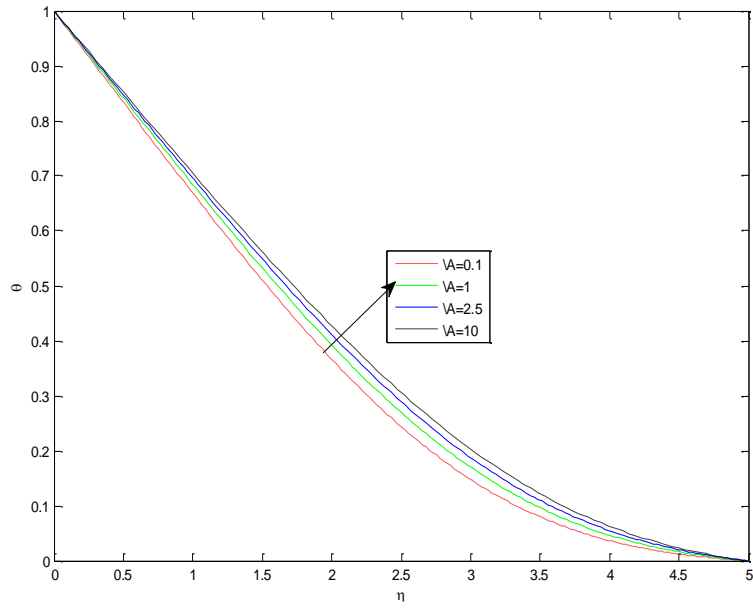


FIGURE 7. 2 Effect of fluid parameter A on temperature profile for $B = 0.1, pr = 1, m = 0, \xi = 0.1$

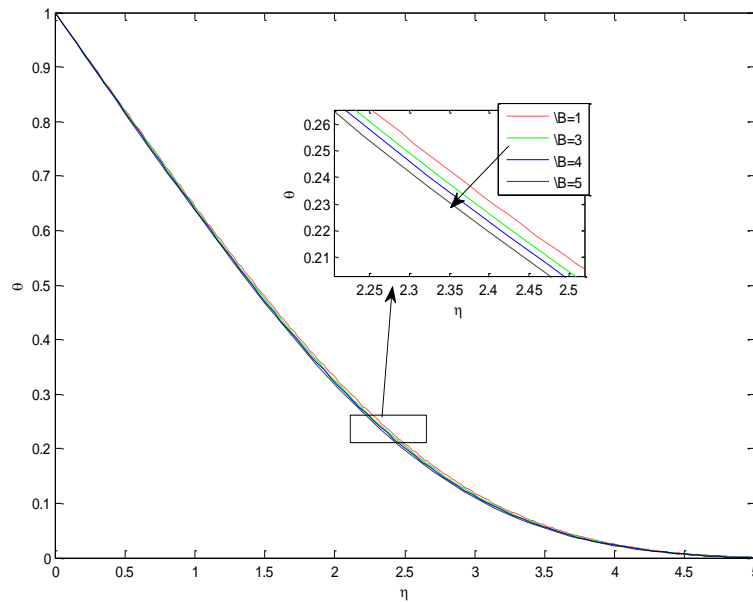


FIGURE 7. 3 Effect of fluid parameter B on temperature profile for $A = 5, pr = 1, m = 1/3, \xi = 0.1$

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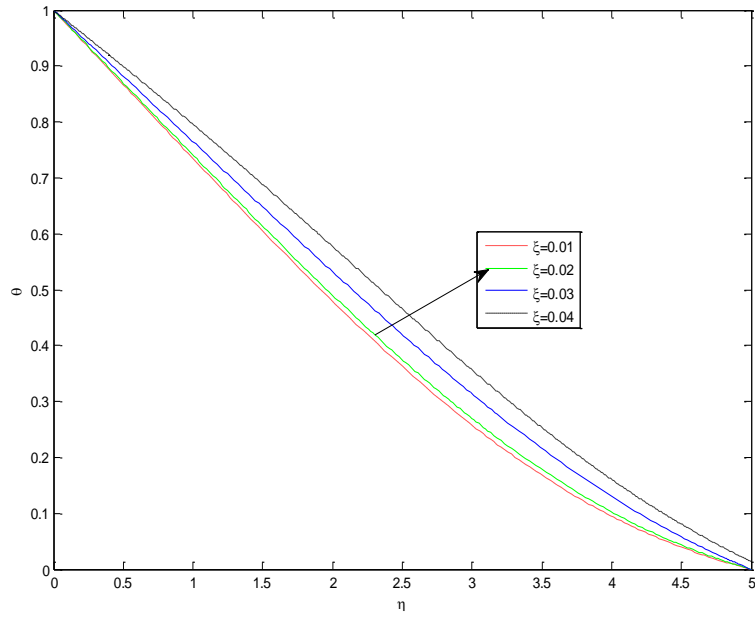


FIGURE 7. 4 Effect of parameter ξ on temperature profile for $A = 1, B = 0.1, pr = 0.7, m = 0$

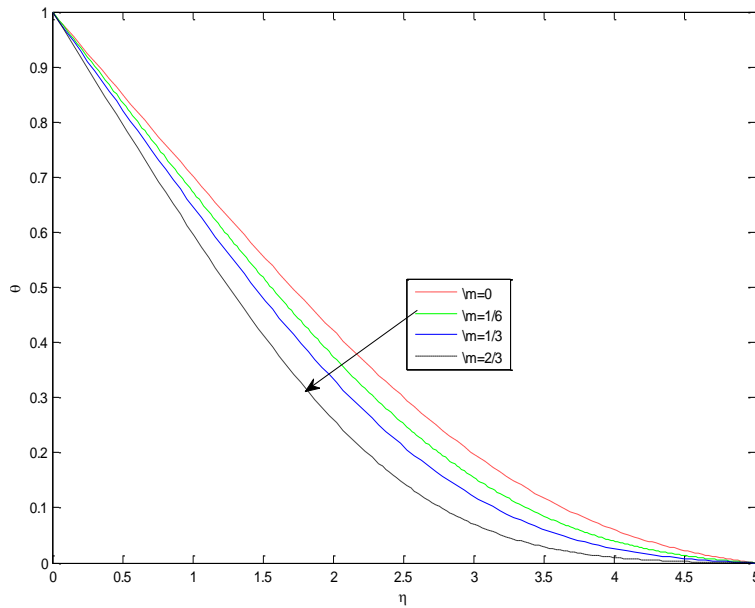


FIGURE 7. 5 Effect of parameter m on temperature profile for $A = 5, B = 1, pr = 1, \xi = 0.1$

Results and discussion

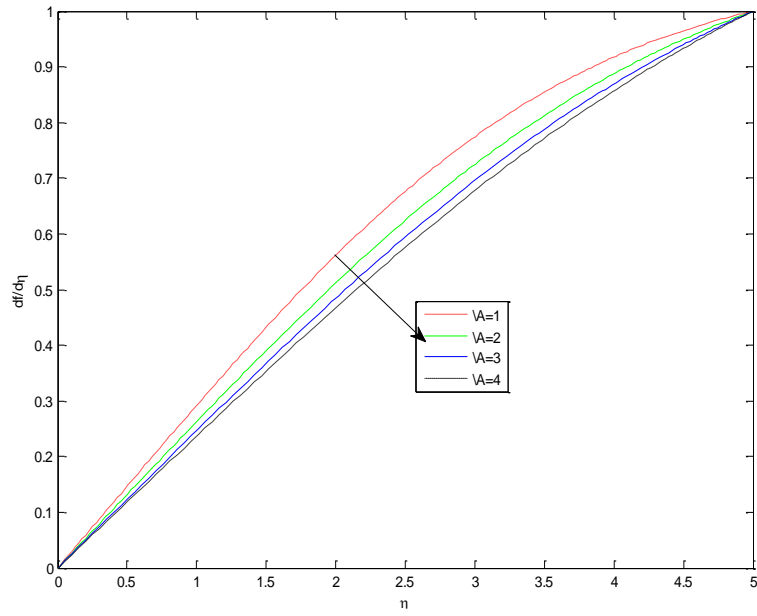


FIGURE 7. 6 Effect of fluid parameter A on velocity profile for $B = 0.1, pr = 1, m = 0, \xi = 0.1$

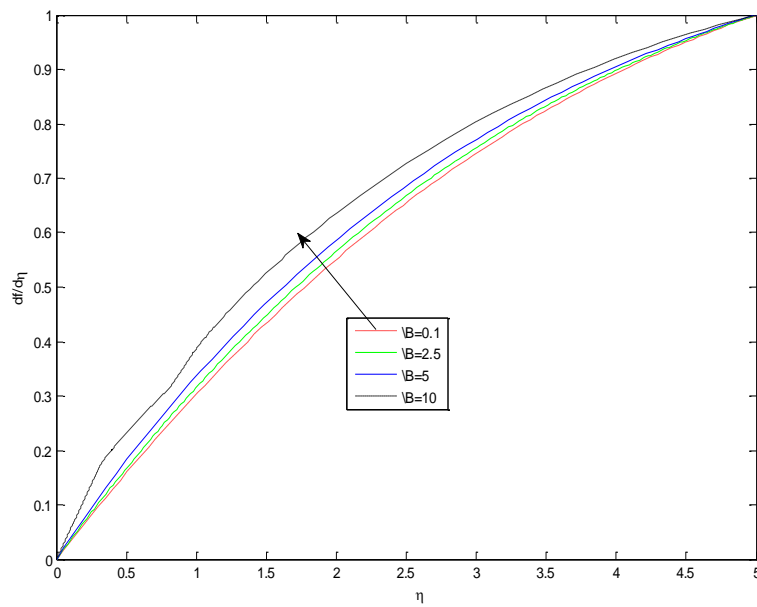


FIGURE 7. 7 Effect of fluid parameter B on velocity profile for $A = 5, B = 0.1, pr = 1, \xi = 0.1$

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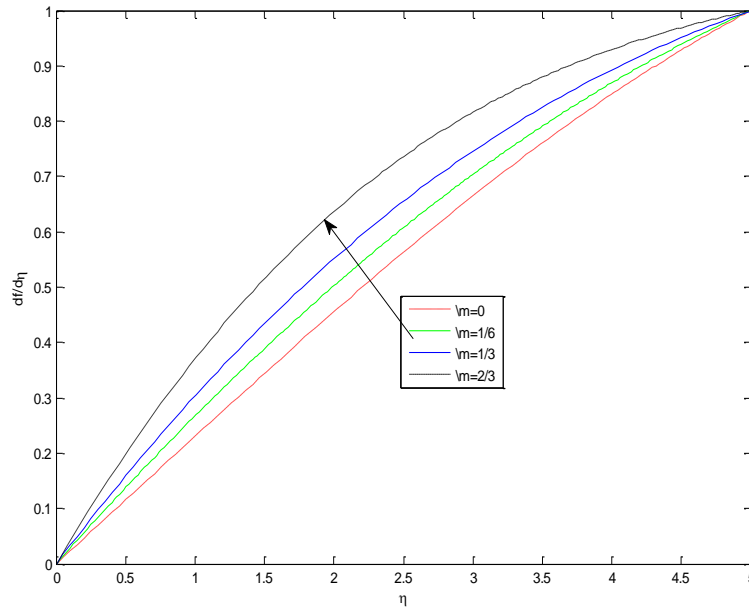


FIGURE 7. 8 Effect of parameter m on velocity profile for $A = 5, B = 0.1, pr = 1, \xi = 0.1$

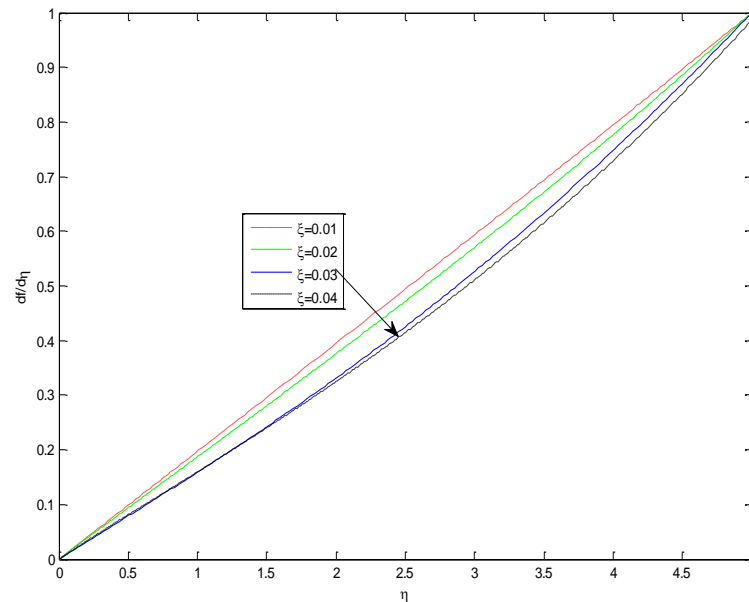


FIGURE 7. 9 Effect of parameter ξ on velocity profile for $A = 1, B = 0.1, pr = 0.7, m = 0$

7.10 Conclusion

In this chapter, the impact of various physical parameters on Powell-Eyring fluid flow is analyzed. The Powell-Eyring model is explored by considering free-stream velocity in power-law form. The non-similarity solution is found using the local non-similarity method at other than the 90-degree wedge.

The governing equations derived via the local non-similarity solution technique are in the same form as those confronted in the treatment of similarity boundary layers. This is nothing but the ordinary differential equations. These equations are solved utilising MATLAB bvp4c solver and displayed using graphs.

Temperature and velocity profile for different values of power-law index m are compared, that means at different body shape. The effect of stream-wise co-ordinate on velocity and temperature profile is also observed. This obviously indicates that the streamwise co-ordinate influences flow considerably and shows the presence of non-similarity solutions.

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List of publications

List of Publications

1. On the Transformation of Non-Newtonian Boundary value problem to Initial value problem. *International journal of Engineering Science Invention Research and development* Vol.III, (sept-2016), pp.180-189, e-ISSN: 2349-6185.
2. Similarity Solution of Forced Convection Flow of Powell-Eyring & Prandtl-Eyring Fluids by Group-Theoretic Method. *Mathematical Journal of Interdisciplinary Sciences* Vol-5, No-2, (March2017) pp.151–165, ISSN: 2278-9561.
3. Finite Difference Method on third order non-linear Differential Equation: Magneto hydro dynamic flow of Sisko fluid. *Indian Journal of Industrial and Applied Mathematics* Vol. 8, No. 2, (July–December 2017), pp.191-200, ISSN: 0973-4317.
4. Similarity analysis of three dimensional nanofluid flow by deductive group theoretic method. *Applications and Applied Mathematics: An International Journal (AAM)* Vol. 13, Issue 1 (June 2018), pp.1260-1272, ISSN: 1932-9466.
5. Local Non-Similar Solution of Powell-Eyring Fluid flow over a Vertical Flat Plate. *Applications and Applied Mathematics: An International Journal (AAM)* Vol. 14, Issue 2 (December 2019), pp.973-984, ISSN: 1932-9466.
6. Similarity Analysis of MHD Three Dimensional Nanofluid Flow for Non-Newtonian Power-law Model over Linearly Stretching Sheet with Convective Boundary Conditions. *International Journal of Heat and Technology* Vol. 38, No. 1, (March, 2020), pp. 203-212, ISSN: 0392-8764.

Details of the work presented in conference

1. On the solution of forced heat convection Powell-Eyring fluid flow by group-theoretic method. *PRIME-Conference*, March-2017, ISBN: 978-81-933591-5-0.