

---

**Analytic Solutions**  
**of**  
**Some Partial Differential Equations**

---

A Synopsis submitted to  
GUJARAT TECHNOLOGICAL UNIVERSITY  
for the Award of  
Doctor of Philosophy  
in  
Science-Maths  
by

MAULIK S. JOSHI

Enrollment No.: 149997673009

under supervision of

**Dr. Rashmi R. Keshvani**

Retired Professor,

Department of Mathematics,

Sarvajanik College of Engineering and Technology, Surat.



GUJARAT TECHNOLOGICAL UNIVERSITY

AHMEDABAD

June, 2019

## **Contents**

<b>1</b>	<b>Abstract</b>	<b>3</b>
<b>2</b>	<b>Brief description on the state of the art of the research topic</b>	<b>5</b>
<b>3</b>	<b>Definition of the Problem</b>	<b>6</b>
<b>4</b>	<b>Objectives and Scope of work</b>	<b>7</b>
<b>5</b>	<b>Original contribution by the thesis</b>	<b>8</b>
<b>6</b>	<b>Methodology of Research and Results/Comparisons</b>	<b>9</b>
<b>7</b>	<b>Achievements with respect to objectives</b>	<b>10</b>
<b>8</b>	<b>Conclusions</b>	<b>11</b>
<b>9</b>	<b>List of Publications</b>	<b>14</b>
<b>10</b>	<b>References</b>	<b>15</b>

## 1 Abstract

Theory of differential equations, whether of ordinary or partial, is the heart of Calculus- the most omnipotent branch of Mathematics. The formulation of representative problems that arise in any physical sciences and engineering, are modelled by partial differential equations. So to study partial differential equations is of main interest and necessity to researchers.

Real world problems in many diverse areas, like aero-dynamics, elasticity fluid dynamics, heat transfer, optics, acoustic, meteorology etc. are modelled by partial differential equations. All the basic equations behind any physical phenomena are either linear or non-linear, partial differential equations. Hence to understand any physical problem, thorough study of theory of differential equations is definitely required. The class of differential equations, which can be solved analytically, is much more limited, compared to that of, which can be solved by numerical methods. But, it is the theory of analytic methods only, which provides a platform to develop various numerical methods, too. So, one is tempted to study analytic methods, with the aim to develop deep understanding of fundamental concepts.

According to *G.F.Roach*, “A newcomer to the subject is often required to read through a great deal of mathematical analysis spread throughout a number of different books, where perhaps rigour rather than potential application is the first consideration, before he is in any position to judge whether or not any particular technique is applicable to his own specific problem.” Keeping all these points in mind, here in this research work, a comparative study of different analytic methods is considered, with a view, to develop insight.

Partial differential equations, particularly second order partial differential equations, with their types, Sturm - Liouville Eigen Value problems, Problems with Dirichlet conditions, Neumann Conditions, Robin conditions, Problems with homogeneous boundary conditions, non-homogeneous boundary conditions, Non-homogeneous problems, Problems with finite domains and infinite domains, problems in polar form, Bessel's equations, all the varieties are covered, rather more weightage is given to infinite domain problems in this research work. Modifications required to be done to the solutions obtained for infinite domain problems, to solve semi-infinite domain problems, are also derived, introducing method of images. Dirac Delta function, Gaussian curves and Gauss-Weierstrass Kernel representations are powerful proven tools to understand behavior of solutions. Their relations with Fourier analysis are discussed in details. A few boundary value problems, collected from different branches of engineering, are solved to illustrate the methods, out of which four are published as research papers in distinct journals.

This research work is organized within six chapters.

**(1) Introduction:**

- (a) Partial Differential equations, Definitions, Their types, Types of problems, Types of conditions, General Sturm-Liouville Problems
- (b) Some useful functions:
  - (i) Dirac Delta Function (ii) Heaviside Unit Step Function (iii) Normal distribution function

**(2) Introduction to various Analytic Methods with applications:**

- (a) Eigen function Expansion Method using separation of variables
- (b) Fourier Transform Method
- (c) Green's function method
- (d) Laplace transform method

**(3) Heat Equation and Solution to Heat Equation:**

Homogeneous, nonhomogeneous equations on finite or infinite or semi-infinite domain using various methods. Explicit representation of a solution using Gauss- Weierstrass Kernel, Solution of some examples

**(4) Poisson's Equation and Solution to Poisson's Equation**

Solution of homogeneous, nonhomogeneous problems on finite or infinite or semi-infinite domain using Green's function method. Determination of Green's function, using various methods such as:

- (a) Method of Eigen function expansions
- (b) Direct method
- (c) Using infinite space Green's function by removing singularity
- (d) Method of images using infinite space Green's function

**(5) Wave Equation and Solution to wave Equation:**

For one-dimensional wave equation, method of characteristics is introduced firstly on infinite region and then required modifications to be done to solve semi-infinite and finite domain problems are discussed. How to solve wave equation using Green's function Method, is also discussed.

**(6) Conclusions- Summary - Scope of this research work:**

As this research work is regarding comparative study of analytic methods, those points, which need to be highlighted as summary, are listed. Not only in Mathematics, in any applied sciences, boundary value problems are inescapable, the scope of study of analytic methods, used to solve partial differential equations, remains widely open.

## 2 Brief description on the state of the art of the research topic:

Real world problems in any physical situations of different discipline of engineering and science are modelled by partial differential equations along with the conditions. In other words, boundary value and/or initial value problems are generated. To solve those modelled problems we need to develop methods, which can be applied to obtain solutions and justify the stated initial and boundary conditions corresponding to the problem.

Nowadays, to solve a boundary/ initial value problem, trend is towards numerical methods. There are many numerical methods which are helpful to solve partial differential equations and to give approximate solutions of the problem. Numerical methods are programmable also. But, it goes without saying, that the platform to solve the problem numerically is being provided by the theory of Analytic methods only. More ever analytic methods provide us exact solutions of initial and boundary value problems under some suitable assumptions. The theory of Analytic methods has its own beauty and importance.

For the given problem, there may be many different analytical methods available, one needs to understand that out of those methods which one would be more suitable and applicable to the problem under discussion. The conceptual understanding plays vital role in the selection of an appropriate analytic method. As *Sir David Hilbert, 1900 (In Paris Lecture)* said “ Each Progress in Mathematics is based on the discovery of stronger tools and easier methods, which at the same time makes it easier, to understand earlier methods. By making these stronger tools and easier methods his own, it is possible for the individual researcher to orientate himself in the different branches of Mathematics.”

Among all analytic methods, like Eigen function expansion method, Fourier series and Fourier transform method, Laplace Transform method, Green’s function method and many others, available to deal with boundary value problems, Green’s function method possesses a remarkable place. Historically differential equations were used to be solved by converting those into integral equations. Even today also, an integral equation representation of a boundary value problem together with corresponding boundary conditions, is much more preferred to apply numerical techniques. The Green’s function method was introduced by G. Green in 1828. However method was introduced a long back ago, it was somewhat less learnt. Comparatively this method has begun to emerge as a potential tool after the usage of the abstract mathematical analysis and advanced theory of differential and integral operators. In

this method, we have to determine a certain function, which will remain same for all boundary value problems, having same governing equation. That is this method gives explicit solution, means once, corresponding Green's function to a boundary value problem is determined, we simply have to substitute it, along with initial and/or boundary conditions in derived formula, which is most probably in integral form, to find out required solution. So, it is not the case that, for every distinct problem, but with same governing equation, we need to determine distinct Green's functions.

In this research work, more weightage is given to Green's function method compared to other analytic methods, but at the same time, as there prevails a large variety of boundary value problems and initial and/or boundary conditions, it is discussed that for which type of problems, it would be better to use other than this method, with thorough explanation. The Dirac Delta function, Heavy side step function, Gaussian and Normal distribution and their Fourier transforms, are introduced. Their relationship with Green's function method is also discussed.

It is my humble effort to provide a comparative study of analytic methods, by which one can acquire more comprehensive understanding.

### **3 Definition of the Problem:**

To introduce and study different analytic methods, the most common partial order equations, particularly second order partial differential equations are utilized, with their types and varieties. Sturm - Liouville Eigen Value problems, Laplace equation, heat equation, wave equation, Problems with Dirichlet conditions, Neumann Conditions, Robin conditions, Problems with homogeneous boundary conditions, non-homogeneous boundary conditions, Non-homogeneous problems, Problems on finite domains and infinite domains, all these varieties and the methods to solve them are gradually discussed, starting from elementary and well-known methods to advanced methods. Fourier Analysis, with its very powerful tool, namely "convolution" is introduced and utilized freely. Help of Dirac Delta function and Gaussian curves, together with Fourier analysis is taken. Solutions are obtained for finite domain problems as well as infinite domain problems. For semi-infinite domain problems, solutions obtained for infinite domain problems are modified using method of images.

## 4 Objectives and Scope of the work:

**The objectives of this research work are as follows:**

- (1) To provide conceptual understanding about prevailing analytic methods to solve partial differential equations.
- (2) To provide an approach based on differential and integral operators introduced by modern functional analysis.
- (3) To provide an introduction to Fourier Transform method - Fourier Analysis and to highlight, its strong relations with electrical engineering, civil engineering.....
- (4) To provide thorough understanding for the Green's function method and to select an appropriate expression of Green's function, subject to the conditions and the corresponding location of the point of the system.
- (5) To provide a self-contained and comprehensive literature to acquire insight to tackle with initial and boundary value problems.
- (6) To make realize a feel for the unity and integrity of Mathematics by highlighting interrelations between apparently seeming, quite different subjects.

### **Scope of this research work :**

Nowadays, Green's function method has emerged as a powerful analytic method.

We can summarize scope of this research work as follows:

- (1) In any branch of engineering or applied science, any researcher has to encounter partial differential equations and boundary value problems in his own field and study. This research work can serve as a guideline to solve those problems.
- (2) Theory of Green's function method is not only a tool to solve a boundary value problem, it is also a means of reducing a differential equation into an integral equation. This point seems to be very attractive to those, who are primarily interested in developing numerical techniques and algorithms to obtain solutions of boundary value problems.
- (3) Green's function method also helps in studying Schwartz Distribution Theory and the space of testing functions. (A testing function  $\varphi$  is a continuous function, which has continuous derivatives of all orders and which vanishes outside a certain finite interval)

## 5 Original contribution by the thesis

While solving the problems under discussion, appropriate expressions of solutions are selected with explanation. Expressions involving Dirac delta functions and Heaviside unit function are derived with full details. The Dirac Delta function, denoted by  $\delta$ , is defined

$$\text{as } \delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{with } \int_{-\infty}^{\infty} \delta(x) dx = 1$$

The Heaviside unit function, denoted by  $H$ , is defined as  $H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

- ❖ The representation of  $\delta$  function in terms of the derivative of  $H$ , is also shown.
- ❖ While studying Green's function method, all the steps of derivations of the following formulas are shown:

### (a) For Poisson Equation:

- **Green's function for infinite plane**

$$G(X, X_0) = \frac{1}{2\pi} \ln |X - X_0| = \frac{1}{4\pi} \ln [(x - x_0)^2 + (y - y_0)^2]$$

- **Green's function for a semi-infinite plane ( $y > 0$ )**

$$G(X, X_0) = \frac{1}{2\pi} \ln \frac{|X - X_0|}{|X - X_0^*|} = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}$$

- **Green's function for quarter plane  $x > 0, y > 0$ :**

$$G(X, X_0) = \frac{1}{2\pi} \ln \frac{|X - X_0| |X - X_0^{***}|}{|X - X_0^*| |X - X_0^{**}|} = \frac{1}{4\pi} \ln \frac{[(x - x_0)^2 + (y - y_0)^2][(x + x_0)^2 + (y + y_0)^2]}{[(x - x_0)^2 + (y + y_0)^2][(x + x_0)^2 + (y - y_0)^2]}$$

- **Green's function for a semi-infinite plane but with insulated boundaries.**

$$\begin{aligned} G(X, X_0) &= \frac{1}{2\pi} (\ln |X - X_0| + \ln |X - X_0^*|) \\ &= \frac{1}{4\pi} \ln [((x - x_0)^2 + (y - y_0)^2)((x - x_0)^2 + (y + y_0)^2)] \end{aligned}$$

Where,  $X_0 = (x_0, y_0)$ ,  $X_0^* = (x_0, -y_0)$ ,  $X_0^{**} = (-x_0, y_0)$ ,

$$X_0^{***} = (-x_0, -y_0)$$

- **Green's function for a circle**

$$G(X, X_0) = \frac{1}{4\pi} \ln \frac{(r^2 + r_0^2 - 2rr_0 \cos \varphi)a^2}{(r^2 r_0^2 + a^4 - 2rr_0 a^2 \cos \varphi)} \quad \text{where } r = |X| \text{ and } r_0 = |X_0|$$

- **Green's function for rectangle ( $0 < x < L, 0 < y < H$ ) - the method of multiple images**

$$G(X, X_0) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{1+n+m} \ln|X - X_{nm}|$$

Where sources

$$X_{nm} = \left[ \frac{L}{2} + \left( x_0 - \frac{L}{2} \right) (-1)^n + nL \right] + i \left[ \frac{H}{2} + \left( y_0 - \frac{H}{2} \right) (-1)^m + mH \right]$$

**(b) n -dimensional infinite space Green's function for heat equation**

$$G(X, t; X_0, t_0) = \frac{1}{(2\pi)^n} \left[ \frac{\pi}{k(t-t_0)} \right]^{\frac{n}{2}} e^{-\frac{(X-X_0)^2}{4k(t-t_0)}}$$

**(c) The Green's function for one dimensional wave equation for infinite domain**

$$G(x, t; x_0, t_0) = \frac{1}{2c} \int_{-\infty}^{\infty} \frac{e^{-i\omega(x-x_0)} \sin c\omega(t-t_0)}{\pi\omega} d\omega$$

$$= \begin{cases} 0 & |x - x_0| > c(t - t_0) \\ \frac{1}{2c} & |x - x_0| < c(t - t_0) \end{cases}$$

Expressing  $G(x, t; x_0, t_0)$  in terms of Heaviside unit step function,

$$G(x, t; x_0, t_0) = \frac{1}{2c} \{-H[(x - x_0) - c(t - t_0)] + H[(x - x_0) + c(t - t_0)]\}$$

**(d) the Green's function for three dimensional wave equation for infinite domain**

$$G(X, t; X_0, t_0) = \int_{-\infty}^{\infty} \bar{G}(\omega, t; X_0, t_0) e^{-i\omega X} d^3\omega$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{e^{-i\omega(X-X_0)} \sin c\omega(t-t_0)}{c\omega} d^3\omega$$

## 6 Methodology of Research and Results/Comparisons:

To introduce and illustrate the method, firstly very simple problems are selected and theory to solve them is developed, step-by-step, from easy to complex.

As for example, to introduce Green's function, firstly a simple boundary value problem is solved by an elementary, well-known method and then it is shown that how the Green's function for that problem, arises in quite natural way and what physical meaning, it bears. The concrete framework is then furnished by including necessary definitions and functions. Finally what properties are being enjoyed by these Green's functions, are derived with their physical meanings.

As for example,

(1) For the differential equation,  $\frac{d^2u}{dx^2} + k^2u = -f(x)$   $0 \leq x \leq L$ , the required two-point Green's function  $G(x, y)$  should satisfy  $G'' + k^2G = -\delta(x - y)$ , where  $\delta$  is Dirac Delta Function. Here  $G(x, y)$  can be interpreted as that displacement of the system which is due to an applied force of density  $\delta$  concentrated at the point  $x = y$  [2]

(2) For the non-homogeneous heat equation,  $\frac{\partial u}{\partial t} = k\nabla^2u + Q(X, t)$ , with initial condition  $u(X, 0) = f(X)$ , where  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2} + \dots$ ,

$X = (x, y, z, w, \dots)$  and  $X_0 = (x_0, y_0, z_0, w_0, \dots)$ , depending upon the value of dimension  $n$ , the required Green's function  $G(X, t; X_0, t_0)$  should satisfy,

$$\frac{\partial G}{\partial t} = k\nabla^2G + \delta(X - X_0)\delta(t - t_0)$$

Here  $G(X, t; X_0, t_0)$  can be interpreted as the temperature response at  $X$  at time  $t$  due to a source located at  $X_0$  at starting time  $t = t_0$  [1]

## 7 Achievements with respect to objectives

- (1) A comprehensive and self-contained literature introducing analytic methods to solve partial differential equations of various types has been prepared.
- (2) Achieved, a sense of profound learning of these techniques and to acquire capability to solve BVPs.
- (3) Achieved a sense of high appreciation towards excellent ideas innovated and cultivated by great mathematicians.
- (4) Achieved capability of expressing a situation involving an impulse, using Dirac Delta function and its derivative. This point helps to express and understand many engineering problems.
- (5) Realized that Fourier analysis and Electrical Engineering are very deeply related.

## 8 Conclusion

- (I) If problem under study has circular symmetry and boundary conditions are such that the required solution is finite at origin, then it is advisable to solve problem using Bessel's functions. Many problems of this type are solvable using Bessel's Functions.

As for example:

To solve the two dimensional heat equation  $\frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + Q(r)$ ,

with circularly symmetric time independent sources, and boundary

condition as  $u(a, t) = T$  and initial condition  $u(r, 0) = f(r)$ , then using

Bessel's function, solution will be

$$u(r, t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{z_n r}{a}\right)^2 kt} J_0\left(\frac{z_n r}{a}\right) + \int_r^a \frac{1}{r} \left( \int_0^r \frac{\tau Q(\tau)}{k} d\tau \right) dr + T$$

$$\text{where } a_n = \frac{\int_0^a (f(r) - \left( \int_r^a \frac{\tau Q(\tau)}{k} d\tau \right) dr + T) J_0\left(\frac{z_n r}{a}\right) dr}{\int_0^a J_0^2\left(\frac{z_n r}{a}\right) dr}$$

- (II) The  $n$ -dimensional infinite space Green's function for non-homogeneous heat equation  $\frac{\partial u}{\partial t} = k \nabla^2 u + Q(X, t)$ , on infinite domain, is

$$G(X, t; X_0, t_0) = \frac{1}{(2\pi)^n} \left[ \frac{\pi}{k(t-t_0)} \right]^{\frac{n}{2}} e^{-\frac{(X-X_0)^2}{4k(t-t_0)}}$$

(a)  $G$  is a Gaussian function centred at  $X = X_0$ .

(b)  $G$  is symmetric about response and source position if the elapsed time is same.

The solution  $u(X, t)$  to this equation (for  $n=3$ ) is:

$$\begin{aligned} u(X, t) = & \int_0^t \iiint G(X, t; X_0, t_0) Q(X_0, t_0) d^3 x_0 dt_0 \\ & + \iiint u(X, 0) G(X, t; X_0, 0) d^3 x \\ & + k \int_0^t \oint (G(X, t; X_0, t_0) \nabla_{X_0} u - u \nabla_{X_0} G(X, t; X_0, t_0)) \cdot \hat{n} dS_0 dt_0 \end{aligned}$$

Here,

- (a) The term  $\int_0^t \iiint G(X, t; X_0, t_0) Q(X_0, t_0) d^3 x_0 dt_0$  implies the contribution due to the source term  $Q(X_0, t_0)$  in which the Green's function  $G(X, t; X_0, t_0)$  acts as the influence function.
- (b) The term  $\iiint u(X, 0) G(X, t; X_0, 0) d^3 x$  implies the contribution due to initial

function  $u(X, 0)$  in which the Green's function  $G(X, t; X_0, 0)$  is the influence function.

(c) The term  $k \int_0^t \iint (G(X, t; X_0, t_0) \nabla_{x_0} u - u \nabla_{x_0} G(X, t; X_0, t_0)) \cdot \hat{n} dS_0] dt_0$

implies the contribution due to non-homogeneous boundary conditions.

Thus all the types of non-homogeneity are covered.

(III) The Green's function for one-dimensional heat equation on semi- infinite domain, is

$$G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}} \left( e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} - e^{\frac{-(x+x_0)^2}{4k(t-t_0)}} \right)$$

Here  $k$  is physically measurable quantity. If  $xx_0$  is very large compared to

$k(t-t_0)$ , that is the product of distances of response location  $x$  and source location  $x_0$  from origin is sufficient large compared to  $k(t-t_0)$ ,

(provided neither  $x$  nor  $x_0$  is near to boundary ) then the contribution from

the term  $e^{\frac{-2xx_0}{4k(t-t_0)}}$  can be neglected. Hence for such cases, Green's function for semi-infinite domain can be approximated by infinite domain Green's function.

That is, for semi-infinite problem also,

$$G(x, t; x_0, t_0) \approx \frac{1}{\sqrt{4\pi k(t-t_0)}} \left( e^{\frac{-(x-x_0)^2}{4k(t-t_0)}} \right), \text{ if } xx_0 \gg k(t-t_0).$$

(IV)  $G(X, t; X_0, t_0) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{e^{-i\omega(X-X_0)} \sin c\omega(t-t_0)}{c\omega} d^3\omega$  is 3 –dimensional,

infinite domain Green's function. So the solution  $u(X, t)$  to 3 –dimensional, infinite domain inhomogeneous wave equation,  $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + Q(X, t)$ , is

$$u(X, t) = \int_0^{t_0^+} (\iiint G(X, t; X_0, t_0) Q(X_0, t_0) d^3 x_0) dt_0 + \iiint \left[ \frac{\partial u}{\partial t_0}(X_0, 0) G(X, t; X_0, 0) - u(X_0, 0) \frac{\partial}{\partial t_0} G(X, t_0; X_0, 0) \right] d^3 x_0 - c^2 \int_0^{t_0^+} \left[ \iint (u(X_0, t_0) \nabla_{x_0} G(X, t; X_0, t_0) - G(X, t; X_0, t_0) \nabla_{x_0} u(X_0, t_0)) \cdot \hat{n} dS_0 \right] dt_0$$

This formula covers all the three types of non-homogeneous terms. The first integral on R.H.S. shows effect of nonhomogeneous source term  $Q(x, t)$  to the response, the second integral shows effect of initial conditions and the third integral shows effect of nonhomogeneous boundary condition ( if it is given) and as on the boundary,  $G$  will be zero, contribution of this third term will be only  $-c^2 \int_0^{t_0^+} \left[ \oint \left( u(X_0, t_0) \nabla_{x_0} G(X, t; X_0, t_0) \right) \cdot \hat{n} dS_0 \right] dt_0$ .

So any kind of non-homogeneity can be treated by this single formula.

- (V) However, using Green's function method, the wave equations, with all the types of non-homogeneity, can be solved, but, if problem has no any non-homogeneous boundary or initial conditions, it would be better to solve wave equation using method of characteristics. In this method, the general solution of one-dimensional wave equation,  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$  is,

$$u(x, t) = F(x - ct) + G(x + ct), \text{ where } F \text{ and } G \text{ are arbitrary functions.}$$

Thus the general solution is expressed as sum of two waves  $F(x - ct)$  and  $G(x + ct)$ , where  $F(x - ct)$  is a wave of fixed shape traveling to the right with velocity  $c$  and  $G(x + ct)$  is a wave of fixed shape traveling to the left with velocity  $-c$ . Here  $F$  and  $G$  can be determined using initial conditions. Say, initial conditions are given as:  $u(x, 0) = f(x) \quad -\infty < x < \infty,$

$$\text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad -\infty < x < \infty,$$

$$\text{Then } u(x, t) = \frac{1}{2} (f(x - ct) + f(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\tau) d\tau$$

This expression of solution to wave equation seems more natural.

It is known as D'Alembert's Solution.

- (VI) The solution to the one-dimensional wave equation on semi-infinite domain ( either on  $x > 0$  or on  $x < 0$  ) with  $u(0, t) = 0$ , is the same as an infinite domain problem with the initial positions  $u(x, 0) = f(x)$ , and initial velocities  $\frac{\partial u}{\partial t}(x, 0) = g(x)$  extended as odd functions.

(VII) To obtain the solution for a vibrating string of fixed length  $L$ , using method of characteristics, first the problem must be extended to infinite domain problem by extending the initial conditions as odd functions (around  $x = 0$ ) which are periodic with period  $2L$ . Then utilizing method of characteristics with D’Almberts solution, solution to the wave equation on finite length  $L$  can be obtained. If the string is initially at rest, that is,

$$g(x) = \frac{\partial u}{\partial t}(x, 0) = 0, \text{ then its periodic extension is also zero for all } x.$$

Hence  $u(x, t) = \frac{1}{2}(f(x - ct) + f(x + ct))$ , where  $f(x)$  is the odd periodic extension of the given initial position. This solution is much simpler than the expression of the solution obtained, by using Fourier sine series.

## 9 List of Research Papers Published

### (1) *Meteorite Shooting as a Diffusion Problem*

Journal for Research | Volume 03| Issue 01 | March 2017 ISSN: 2395-7549

### (2) *A STUDY ABOUT SOIL TEMPERATURE VARIATION AT DISTINCT SUB-SURFACES*

IMPACT: International Journal of Research in Engineering & Technology

ISSN(P): 2347-4599; ISSN(E): 2321-8843 Vol. 6, Issue 3, Mar 2018,

29-32 © Impact Journals



### (3) *Solution of Nonhomogeneous Wave Equation Using Green’s Function*

CIKITUSI JOURNAL FOR MULTIDISCIPLINARY RESEARCH

Volume 6, Issue 2, February 2019

ISSN NO: 0975-6876

### (4) *The Green’s Function Method For Non-homogeneous Heat Equation*

JASC: Journal of Applied Science and Computations

Volume VI, Issue III, March/2019

ISSN NO: 1076-5131

Page No:2213

## 10 References

- [1] Richard Habberman,  
“Elementary Applied Partial Differential Equations with Fourier series and  
Boundary value problems”,  
Prentice Hall Inc. 1983
- [2] G.F.Roach,  
“Green’s functions: Introductory Theory with Applications”,  
Van Nostrand Reinhold Company. 1967
- [3] Mark A, Pinsky, “ Partial Differential Equations and Boundary-Value Problems with  
Applications” McGraw-Hill, 1998
- [4] Ronald N. Bracewell,  
“The Fourier Transform and its applications”,  
International edition 2000, McGraw-Hill Education.
- [5] Richard A. Johnson,  
“ Miller & Freund’s Probability and Statistics For Engineers”, sixth edition
- [6] [www.meteorlab.com/METEORLAB2001dev/whatmeteorites.htm](http://www.meteorlab.com/METEORLAB2001dev/whatmeteorites.htm)
- [7] Y. Elmahroug, B. Tellili & C. Souga,  
Calculation of Total Mass Attenuation Coefficients, Effective Atomic Numbers and  
Effective Electron Densities for the Earth Atmosphere,  
International Journal of Physics and Research (IJPR),  
volume 3, Issue 3, July-August 2013, pp. 77-86
- [8] *Ground Temperatures as a Function of Location, Season, and Depth*  
(<https://www.builditsolar.com/Projects/Cooling/Earthtemperatures.html> )
- [9] Definitions of: Phasor, Propagation Constant, Attenuation constant : Wikipedia

[10] Nwagbara, Moses Okemini&Uzowulu, Onyinyechi, Climate Change, Soil Temperature and Cassava Yield in Abia State, Southeastern Nigeria, IASET: International Journal of Agricultural & Bio-Chemical Science (IASET: IJABCS), Volume 1, Issue 1, January-June 2017, pp. 39-48