APPLICATIONS OF SPLINE COLLOCATION METHOD TO PARTIAL DIFFERENTIAL EQUATIONS

A Synopsis Submitted to

GUJARAT TECHNOLOGICAL UNIVERSITY

Ahmedabad

For the Degree

of

Doctor of Philosophy

in

Mathematics

By

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Content

1. Abstract:

The study of differential equations is an extensive field in pure and applied Mathematics, Physical Sciences, Biological Technology and Engineering. All of these disciplines are related with the properties of differential equations. Pure Mathematics focuses on existence and uniqueness of solutions, whereas applied mathematics mainly focuses on the rigorous justification of the methods approximating solutions. Differential Equations play an important role in every physical, technical or biological modeling process. Differential equations like those wont to solve real life issues might not be solvable analytically or terribly tough to possess closed form solutions are often approximated by numerical methods.

During the last few years, piecewise polynomial approximations have become very important in engineering applications. The most popular of such approximating functions are spline functions. The various features of the Spline collocation technique enhance the applicability in the field of numerical analysis to partial differential equations.

The present work deals with the use of Spline collocation method to various types of linear as well as non-linear Partial Differential Equations (PDEs) under the different set of boundary conditions. Linear PDEs are solved using Spline explicit and implicit schemes while non-linear PDEs are handled with Hofp-Cole transformation and Orlowski and Sobczyk transformation (OST) to apply Spline collocation method. The method is successfully applied to the problems which describes the flow of electricity in the transmission lines, heat conductions in a thin rod, heat flow in a rectangular plate, finite vibrating string, vibrating membrane problems, viscous Burgers' equations and Navier-Stokes equations.

Here attempts are made to solve the problems in the form of PDE only without converting into Ordinary Differential Equations (ODE) and it is observed that the method can be successfully applied.

This PhD thesis would be helpful for others to obtain effective approximate solutions to increasingly more challenging and complicated PDE problems arising in engineering sciences, applied sciences and other fields.

2. Brief description on the state of the art of the research topic:

In modern practice majority of unsolved problem in life science, physical science and engineering are described mathematically by nonlinear partial differential equations [1]. Partial differential equations arise in many branches of applied mathematics for example, in hydrodynamics, elasticity, quantum mechanics, electromagnetic theory, etc. The analytical treatment of these equations is a rather involved process and requires applications of advanced mathematical methods; on the other hand, it is generally easier to provide sufficiently approximate solutions by simple and efficient numerical methods. Many techniques are available for their solutions and in absence of exact solution; generally, we apply either approximate methods or numerical methods to solve such equations. The methods in which the solution is approximated by a finite number of terms of an infinite expansion of functions are known as approximate methods whereas the method in which the solution is approximated numerically at a number of discrete points are known as numerical methods [2].

The importance of the numerical methods has increased with the development of high speed digital and analog computers. In the early days of research in numerical analysis because of restricted capacity of computing machines, the applications of numerical methods were possible to a limited set of problems. Today the situation is different. The computing devices available are sufficiently advanced and developed to deal with almost an unlimited range of problems what is really needed, which is the main requirement of numerical methods. In a numerical simulation process of a physical problem, the methods mostly used are Finite difference method, Finite element method, Milne's methods, predictor-corrector method, perturbation method, Runge-Kutta method, Taylor's series expansion, etc. Several numerical methods have been proposed for the solution of partial differential equations (PDEs), but we will restrict ourselves to a treatment of the spline collocation method and will discuss the numerical procedures with simple PDEs.

In many numerical interpolation methods, an nth order polynomial passing through (n+1) given data points is fitted [3]. Because of round-off errors, these polynomial are found to give erroneous results, especially when the function undergoes sudden changes in the vicinity of a point in its range, further it was found that a low order polynomial approximation in each subinterval provides a better approximation to the tabulated function, than fitting a high-order polynomial to the entire range. These connecting piecewise polynomials are called spline functions, named after the draftsman's device using a thin flexible strip (called a spline) to draw a smooth curve through given points. The points at which two connecting splines meet are called knots. The connecting polynomial could be of any degree and so there are different types of spline functions, viz., linear, quadratic, cubic, quantic, etc. Spline functions possess all the basic mathematical properties. The spline functions along with their successive derivatives are continuous, differentiable and more generally they are analytic functions.

Literature survey

The notion of 'Spline' was first mathematical referred by I.J. Schoenberg in [4], which is probably the first place that the word "spline" is used in connection with smooth, piecewise polynomial approximation. Whitney [1949, 53], a student of Schoenberg working on spline functions, derived a criterion that only certain splines exist for the purpose of interpolation through given data. General spline functions with their minimum norm properties were discussed by Lynch et al [1964, 66]. The splines of even order interpolating the data at junctions appear in a very simple fashion and their existence criterion was developed by Ahlberg et al [5, 6, 7]. The spline functions are also useful in the methods developed in mathematical programming by Ahlberg J. H. and Nilson E.N. [8, 9]. The period of 1960 to 1972 was significant in the field of spline theory. A remarkable research on existence, uniqueness, minimum norm property and best approximation properties was done by Kalthia N.L and Bulsari A.B. [10]. The theory of the spline has been extended in a number of directions of considerable importance is the extension to several dimensions by Ahlberg E. N. Nilson and J. Walsh [11]. To find the numerical solution for the Parabolic Partial Differential Equations, Cubic Spline Collocation Method was used by O. A. Taiwo, O.S. Odetunde [12]. Numerical Solution of a class of non-linear Partial Differential Equations by using Barycentric Interpolation Collocation Method was done by Hongchun Wu, Yulan Wang, and Wei Zhang [13]. Quadratic spline solution was applied for boundary value problem of fractional order by Waheed K. Zahra and Samah M. Elkholy [14]. Maria Munguia and Dambaru Bhatta [15] used a Cubic B-Spline method in Approximating solutions of Boundary Value Problems. Spline Collocation approach to Partial Differential Equation exhibited by Pathak A. K. and Doctor H. D. [16].

Three fundamental second order PDE frequently show up in many applications like, one dimensional and two dimensional heat equation, one dimensional and two dimensional wave equation and Laplace equation depending on the classification of equation. Certain type of boundary conditions and initial conditions are required for their solutions [17]. Several numerical methods have been proposed for the solution of partial differential equations [18, 19].

Burgers' equation is a fundamental partial differential equation occurring in various area of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. The equation was introduced by harry Batman in 1915 and later studied by Johannes Martinus Burgers in 1948.The viscous Burgers equation can be converted to a linear equation by Cole-Hofp transformations by Sachdev P.L. [20,21]. The Variational iteration method and homotopy perturbed method for solving Burgers' equation in fluid dynamic was studied by Noorzad Reza, A. Tahmasebi Poor and Mahdi Omidvar [22]. The application of the Method of line (MOL) has been illustrated to solve burgers equation by Biazar et al.[23]. The exact solutions of the one dimensional Burgers equation have been surveyed by Berton and Platzman. Many authors have used a variety of numerical techniques based on finite difference method, finite element and boundary element methods in attempting to solve the equation particularly for small values of the kinematic viscosity which correspond to step fronts in the propagation of dynamic wave forms. Generating exact solution of two dimensional coupled viscous Burgers' equations was solved by Fletcher and Clive A. J. [24]. The numerical solution of coupled viscous Burgers' equation has been solved by many researchers [25, 26, 27, 28, 29, 30].

The Navier-Stokes equation is an important governing equation in fluid dynamics which describes the motion of the fluid. The Navier-Stokes equation is non-linear; there cannot be a general method to solve analytically. Azad M. A. K. and L.S. Andallah [31] presented exact solution to a one dimensional Navier-Stokes equation which exhibit erratic turbulent like behavior. Kim, John and Paviz Moin [32] have applied a fractional step method to incompressible Navier-Stokes equations. Azad M. A. K. and L.S. and Allah [33] presented an explicit finite difference scheme for one dimensional Navier-Stokes equations. Gorguis Alice [34] exhibited a reliable approach to the solution of Navier-Stokes equations. However, due to the great mathematical difficulty of these equations, very few approaches had been found to the mathematical treatment of viscous flows (except in a few special cases) using approximate solutions.

The present study emphasizes the use of spline collocation method (explicit and implicit scheme) to a variety of Partial differential equations (PDEs) under the different set of boundary conditions. The linear PDEs are solved using spline explicit and implicit schemes directly whereas non-linear PDEs are handled with Hofp-Cole transformation and Orlowski and Sobczyk transformation (OST) to apply Spline collocation method. The thesis demonstrates a numerical method to solve linear as well as nonlinear PDEs.

3. Definition of the Problem:

The applications of differential equations are in form of mathematical models of real world problems arising in medical science, engineering and physical science. Solution of the differential equation is an important task for any problem. The solutions are possible by the different methods namely the analytic, approximate and numerical method. Our physical world is too complex and the study of nature motivates Researchers, scientists, engineers and others with an inherent spirit of challenge. Today our resources are quite rich with sophisticated devices which include software and hardware for the study of more and more physical systems. Many of the physical problems can be viewed mathematically. Mass-balances, heat and momentum-balances for each phase and equilibrium-relations between the phases, laws of conservation are normally the main modeling tools for the study of physical systems. Most of the problems are in the form of differential equations, mainly Partial Differential Equations and thus solution of these equations ultimately turns out to be the solution of the problem.

4. Objectives and Scope of the work:

Objectives:

From the literature survey it is found that, there are many real world problems in which the unknown function involved, depends on several independent variables and gives rise to partial differential equation when they are modeled mathematically. Most of the problems from fluid and solid mechanics, heat transfer, vibrations, electromagnetic theory etc. result into this category. The primary objective of the present work is to determine the successful application of Spline collocation method to solve such equations.

Most of the researchers convert Partial Differential Equations (PDEs) into Ordinary Differential Equations (ODEs) using similarity method and then solve by any numerical method or using available tools like MAPLE, Mathematica, etc. The main objective is to obtain the solution of PDE directly without converting it into ODE to reduce computational complexity using spline collocation method.

Scope of the work:

- Successfully solve linear one dimensional as well as two dimensional parabolic and hyperbolic partial differential equations using spline explicit and implicit schemes.
- Solution of nonlinear boundary value problems in partial differential equation is a new challenge in flow analysis. Most of the researcher converts partial differential equation in ordinary differential equation using similarity method then solve by any methods, but the new scope is that, without converting into ordinary differential equation, one can directly solve partial differential equation using different transformations like Hofp-Cole transformation and OST.
- Our aim is to propose quite a new idea in the theory of splines and that is numerical technique. We hope that our proposed numerical technique will be useful to deal with the non-linear partial differential equations.
- Scopes of the new proposed technique are wide. It will be useful to solve linear as well as nonlinear boundary value problems.

5. Original Contribution by the thesis:

Most of the natural phenomena can be governed by Partial differential equations and thus solution of these partial differential equations ultimately turns out to be the solution of the problem to be studied. The class of partial differential equations is too wide and obtaining their solutions is an important task in the study of such differential equations. There are several methods for solving partial differential equation. The thesis emphasizes the use of spline implicit and explicit scheme to various types of initial and boundary value problems arising in engineering and sciences.

Chapter 1 represents a general discussion about Partial Differential Equations (PDE), Classification of PDE, Numerical Methods, approximate and numerical methods for solving PDE. In this chapter, we also introduce Spline functions, Hofp-Cole Transformation, OST transformations and Reynolds numbers. It shields the investigation of spline function (explicit scheme and spline implicit scheme). Development of spline function is expressed in the chapter. The chapter also includes fundamentals of fluid dynamics.

Chapter 2 demonstrates an application of spline function to solve parabolic PDEs with one as well as two space variables. The flow of electricity in a transmission lines, heat conductions in a rod of finite length and heat flow in a rectangular thin plate are discussed and solved by cubic spline explicit and implicit methods. The range of boundary value problem was divided into a number of subintervals. The solutions are obtained by explicit and implicit cubic spline methods and the results are compared with the available analytic solutions. They are quite satisfactory, which are presented in tabular as well as graphical forms.

Chapter 3 deals with the application of spline collocation technique to solve hyperbolic partial differential equation with one and two space variables. The vibrating string and the vibrating membranes of finite length as well as width are discussed as the case study. The range of boundary value problem is divided into a number of subintervals. The solution are obtained by explicit and implicit cubic spline methods and compared with the available analytic solutions. The solutions of the problems are presented in tabular as well as graphical forms, proving the reliability of spline collocation method.

Chapter 4 designates the proficiency of spline collocation method to obtain a numerical solution of one, two & three dimensional viscous coupled Burger's equation. An approximate solution of viscous Burger's equation is given by spline collocation method. The new method based on the Hofp-Cole transformation is used to transform the system of one, two & three dimensional coupled Burger's equations into linear heat equations. The linear heat equation is then solved by spline implicit scheme and spline explicit formula. It has been then compared with the available solution to validate the method.

Chapter 5 boons in finding the effective application of spline collocation method in deciphering the system of nonlinear PDEs. The study of one dimensional (1D) and two dimensional (2D) Navier-Stokes equation is done here, in which the governing equation in fluid dynamics describes the motion of fluid. Applying OST and Hofp-Cole transformation combined; we have transformed one dimensional and two dimensional Navier-stokes equations into linear partial differential equations. These equations are solved numerically using spline collocation method. The solutions are analyzed and reflected in graphs.

6. Methodology of Research and Results:

A low order polynomial approximation in each subinterval provides a better approximation to the tabulated function than fitting a high-order polynomial to the entire range. These connecting piecewise polynomials are called spline functions, named after the draftsman's device using a thin flexible strip (called a spline) to draw a smooth curve through given points. The points at which two connecting splines meet are called knots. The connecting polynomial could be of any degree and so there are different types of spline functions, viz., linear, quadratic, cubic, quantic, etc. Spline functions possess all the basic mathematical properties. The spline functions along with their successive derivatives are continuous, differentiable and more generally they are analytic functions.

The Spline collocation method is used to solve parabolic type and hyperbolic type PDEs with one as well as two space variables.

To investigate the behavior of one, two and three dimensional Burgers' equation the spline explicit and implicit scheme along with Hofp-Cole Transformation is applied and the results are compared with the available results in the literature, which are either in tabular form or graphical form.

The spline explicit and implicit scheme along with OST transformation is used to investigate the behavior of one and two dimensional Navier-Stokes equation and compared the results with available results in the literature. The results are shown in graphical form.

7. Achievements with respect to objectives:

During the research work, following work is completed to achieve the research objectives:

- Solving linear parabolic and hyperbolic one dimensional as well as two dimensional partial differential equations by spline explicit and implicit schemes, accurate results can be successfully obtained. Comparative analysis is made for the problems.
- The proposed algorithm efficiently handles different dimensional PDE problems.
- The numerical solution using spline collocation method can be successfully applied to solve Coupled nonlinear partial differential equations with the help of Hofp-Cole transformation and OST transformation.
- Well-known Burgers' equation and Navier-Stokes equations are solved using spline explicit and implicit scheme based on Hofp-Cole and OST transformations.
- The spline collocation method gives closed form solution to PDEs without converting them into linear one.

8. Conclusion:

This thesis demonstrates the study of spline collocation method to solve the linear and non-linear boundary value problems. The various features of the spline collocation technique are expanded the field of numerical solution to the Partial differential equations. A special emphasize is given to the applicability and reliability of the method of spline collocation. To check the applicability of spline functions, the solutions are obtained for the various types of Partial differential equations occurring in the study of several physical phenomena in engineering sciences. In all, it is worthy to mention that the spline collocation technique with all the above approaches are helpful to solve various types of fluid flows problems.

We observed that, Spline implicit scheme gives better approximation in comparison to the explicit scheme. It is also noticed that, as a particular case, if problems are defined with Neumann and Dirichlet boundary conditions, then spline explicit and implicit scheme gives better solution as compared to homotopy perturbation method.

According to the experience during the research work, it can be concluded that, Splines schemes show better results for the one, two and three dimensional partial differential problems. The benefit of the Spline collocation method is, nonlinear problems can be solved directly without converting PDE to ODE. More accurate results can be obtained by reducing the mesh size of the interval. The restriction of this method is without using Hofp-Cole and OST transformations we could not solve nonlinear PDEs.

The Present work justifies the applicability and reliability of the spline collocation method. We conclude that spline explicit and implicit schemes are effective and quite encouraging than the finite difference method. It will be possible that same technique can be applied for higher order partial differential equation. Thus the wide applicability of spline collocation is sought in this thesis.

9. List of Publications:

1. Nileshkumar A. Patel, Jigisha U. Pandya. (2016). A Numerical approach for solving nonlinear boundary value problems in finite domain using Spline collocation Method. International Journal for Innovative Research in Science & Technology, 3, 318-321.
2. Nileshkumar A. Patel, Jigisha U. Pandya. (2017). Spline Collocation method for solving Burgers' equation in fluid dynamics. International Journal of Emerging Technology and Advanced Engineering, 7, 185-189.
3. Nileshkumar A. Patel, Jigisha U. Pandya. (2017). Vibrating Membrane Problem solved using Spline collocation method with Dirichlet conditions. International Journal of Mechanical and Production Engineering Research and Development, 7, 147-154.
4. Nileshkumar A. Patel, Jigisha U. Pandya. (2017).One dimensional heat equation subject to both Newman and Dirichlet initial boundary conditions and used a Spline collocation method. Kalpa Publication in Computing, 2, 107-112.
5. Nileshkumar A. Patel, Jigisha U. Pandya. (2018).A Numerical Solution of Two-dimensional coupled Burgers' equation using Spline Explicit and Implicit Scheme. American International Journal of Research in Science, Technology, Engineering and Mathematics, 22, 122-134.

Papers Presented in Conferences:

1. Patel Nileshkumar A., & Pandya Jigisha U, "A Numerical Method for the Heat Equation with Dirichlet and Neumann Conditions," International conference on research and innovations in science, Engineering & technology, BVM, Vidyanagar, February – 2017.
2. Patel Nileshkumar A., & Pandya Jigisha U, "Vibrating String Problem Solved Using Spline Collocation Method with Dirichlet and Neumann Conditions," Incompressible Fluid Past Flat a Plate, National Conference on Progress, Research and Innovation in Mechanical Engineering, SCET, Surat, March – 2017.

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