MODELING THE PROBLEM OF MULTIFLUID FLOW IN POROUS MEDIA AND THEIR SOLUTION BY COMPUTATIONAL METHODS

A Thesis submitted to Gujarat Technological University

For the Award of

Doctor of Philosophy

In

Science - Maths

Researcher

Megha Kamlesh Tailor

Enrollment No. : 149997673010

Supervisor

Dr. Shailesh S. Patel

Professor and Head

ASH Department

GIDC Degree Engineering College, Navsari

GUJARAT TECHNOLOGICAL UNIVERSITY

AHMEDABAD

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I declare that the thesis entitled "**Modeling the Problem of Multifluid Flow in Porous Media and their Solution by Computational Methods"** submitted by me for the degree of Doctor of Philosophy is the record of research work carried out by me during the period from 2014 to 2020 under the supervision of **Dr. Shailesh S. Patel**, Professor and Head of the ASH Department, GIDC Degree Engineering College, Abrama, Navsari and this has not formed the basis for the award of any degree, diploma, associate ship, fellowship, titles in this or any other University or other institution of higher learning.

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ABSTRACT

The thesis entitled “MODELING THE PROBLEM OF MULTIFLUID FLOW IN POROUS MEDIA AND THEIR SOLUTION BY COMPUTATIONAL METHODS” is the description of my research work carried out on numerical simulation of the problem fluid flow through porous media.

The fluid flow through porous media plays very important role in the field of engineering and applied science such as petroleum engineering, soil mechanism, chemical engineering, oil reservoir engineering, civil engineering, geophysics etc.

Generally, three types of phenomena in the displacement problems like instabilities phenomenon, imbibition phenomenon and fingero-imbbibition phenomenon in porous media with some physical effects are considered. In all these investigated problems, the governing differential equations of the phenomenon raised from the developed mathematical model which has been solved numerically. The hydrodynamics of single phase flow and multiphase flow through porous media has gained three to six ugh porous media, the specific problems are almost unlimited and therefore it is reasonable to select such type of problems discussed here. The present study investigates the flow of incompressible and immiscible fluids.

The thesis consists basically divided into six chapters.

Following to these introductory words, Chapter 2 gives an overview of fluid flow through porous media which is encountered in many engineering fields. The mathematical problems of different physical phenomenon arising one dimensional partial differential equations. These nonlinear partial differential equations solved using Finite Differential Crank-Nicolson Method.

The next chapter 3 is targeted to the problem fingering phenomenon arising in fluid flow through porous media in homogeneous porous medium. We obtain the solution of governing equation with the help of initial and boundary conditions.

Chapter 4 discusses the fingero-imbibition phenomenon in double phase flow through homogeneous as well as heterogeneous porous media. The finite difference Crank-Nicolson scheme is used to solve the governing equation with initial and boundary conditions. The numerical and graphical interpretation of solution given by SCILAB coding.

Chapter 5 is focused on mathematical model of cocurrent imbibition phenomenon with effect of inclination. This problem has been solved Finite Difference Method. The graphical representation shows the saturation of water increase with fixed distance X at different time level with angle.

In chapter 6, mathematical model of counter current imbibition phenomenon in homogeneous porous media in secondary oil recovery process has been studied. The saturation of injected water is used by Finite difference Crank-Nicolson scheme with suitable initial and boundary conditions.

This work has been dedicated to study of one dimensional flow in porous media and the development of mathematical model of fluid flow through porous media. These nonlinear partial differential equations are solved using finite difference Crank-Nicolson scheme with the help of initial and boundary conditions. The solutions have been represented graphically as well as numerically using SCILAB. This thesis would be beneficial for solving the nonlinear problems arising in fluid flow through porous media.

ACKNOWLEDGEMENT

My name appears on the cover of this thesis, but a great many people have contributed to its production. I owe my gratitude to all those people who have made this thesis possible and because of whom my research experience has been one that I will cherish forever.

First and foremost I would like to thank God, who has given me the strength and patience during this research work. I am sure without his sheer blessing, it is impossible for me to complete this research work. He has always showered his choicest blessings on me, to enable me, for the work.

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Words cannot express my gratitude towards my mother Priti K. Tailor who helped me in every difficult situation. Thanks to my father Kamlesh I. Tailor for their unparallel support and blessing for every moment. Also, thanks to my brother, mother-in law & father-in-law, for all of the sacrifices that they have made only for my success.

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Megha K. Tailor

Contents

LIST OF SYMBOLS

P	Porosity of the porous medium
β	Constant parameter
k_{sat}	Permeability at full saturation
k_{eo}	Effective permeability of oil
k_{ew}	Effective permeability of water
k_w	Relative permeability of water
k_o	Relative permeability of oil
P_c	Capillary pressure
Q	Rate of water flow.
K	Permeability of the porous medium
$\mathrm{P_w}$	Pressure of water
$\mathrm{P_o}$	Pressure of oil
δ_w	Constant viscosity of injecting water
δ_o	Constant viscosity of native oil
V_w	Velocity of water
$\mathrm{V_o}$	Velocity of oil
k	Hydraulic conductivity
$\overline{\mathrm{P}}$	Mean pressure
K_a	Constant parameter
$\mathrm{S_w}$	Saturation of water
$\mathrm{S_o}$	Saturation of oil
θ	The angle of inclination with the porous matrix
g	Acceleration due to the gravity
$\mathrm{V_t}$	Total velocity

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Dedicated to my parents and husband…

CHAPTER 1

INTRODUCTION

CHAPTER 1

Introduction

1.1 Introduction:

The present study deals with the study of fluid flow through porous media. It focuses on approximate numerical solutions of various problem of current interest in two phase flow system through homogeneous as well as heterogeneous porous medium. The nonlinear partial differential equations results from the mathematical formulation of the physical phenomenon whose approximate numerical solutions are obtained using finite difference Crank-Nicolson scheme with appropriate initial and boundary conditions. To study and solve various problems, the researcher has refereed as well as studied various researcher works to understand the problems well.

During the past four decades theoretical research in flow in porous media has received noticeable attention. This research area plays significant role in social sciences and engineering field such as civil engineering, geoscience, petroleum engineering, hydrogeological engineering etc. we investigate a selection of mathematical models, by starting with basics details of porous medium, also the behavior of flowing fluids is the fundamental study of the research. In particular, an attempt is made to give the link between the growth of mathematics and recent displacement problems with their applications. We have tried to study four different mathematical models in details and solved by finite difference method.

1.2 Structure of the thesis:

The mathematical models which are formulated with suitable initial and boundary conditions to predict the injected water saturation distribution patterns in finite aquifers are solved using finite difference method. To study “Modeling the problem of multi fluid flow and their solution by computational method" the researcher has divided the study into six chapters as follows.

Chapter 1: "Introduction":

Chapter one contains introductory part of the research work. It's including objectives, scope, and significant contribution of this research project. It shows the scope of the present work in various fields such as civil engineering, petroleum engineering, and hydrological engineering and in different branches of the earth science. Also, it shows the important of mathematical methods through which problem were defined.

Chapter 2: "A Prerequisite for Fluid Flow through Porous Media and Brief Description of Finite Difference Method":

Second chapter build up a stronger structure in logical manner to provide knowledge of fundamental of fluid flow through porous media, which is essential part of the study for better understanding of porous media. This chapter deals with necessity of the study of flow through Porous medium with basics definitions, their properties, Darcy's law plays most important role in the study of the subject. Also, the numerical method called finite difference method has been discussed.

Chapter 3: "Mathematical Analysis of Fingering Phenomenon in Homogeneous Porous Media":

This chapter discusses the phenomenon of fingering which arises in the flow of two immiscible fluids through porous media by finite difference Crank-Nicolson scheme. The present investigation is that there is uniform water injected into an oil saturated porous medium. The injected water shoots through the oil formation and give rise to fingers. The mathematical formulation has been constructed for the problem each cases. In the present work the nonlinear partial differential equation has been solved numerically. The numerical solution has been obtained by finite difference method. The behavior of numerical calculations has been intercepted by plotting various graphs from saturation of water $(S) \rightarrow X$ and saturation of water $(S). \rightarrow T$ Keeping X and T fixed respectively.

Chapter 4: "Mathematical Modeling of Fingero-Imbibition Phenomenon and its Solution by Finite Difference Method."

Chapter four is the discussion of the phenomenon of fingero-imbibition in homogeneous as well as heterogeneous porous medium. This phenomenon is the simultaneous occurrence of two phenomenon fingering and imbibition phenomenon. The mathematical problem yields to one dimensional nonlinear partial differential equation and solve it using appropriate initial and boundary conditions by finite difference method. The graphical and numerical represents has been done by Scilab.

Chapter 5: "Numerical Analysis of Co-Current Imbibition Phenomenon with Effect of Inclination."

Chapter five focused on co-current imbibition phenomenon in inclined homogeneous porous media. Such a spontaneous imbibition may occur in the form of co-current and counter current imbibition. The main difference between these crucial mechanisms for imbibition is the direction of flow. In co-current imbibition phenomenon describe as both wetting and non-wetting phase flow in the same direction with the non-wetting phase being pushed out ahead of the wetting phase. The saturation rate of wetting phase has been discussed by the effect of inclination. The governing equations have been solved finite difference method. The solutions represent saturation of injected water is increase with specific distance and different time. The numerical solution and graphical representation also discussed.

Chapter 6: "Mathematical Investigation of Counter-Current Imbibition Phenomenon in Homogeneous Porous Media."

Chapter six is dedicated to counter current imbibition phenomenon arising fluid flow through homogeneous porous matrix with inclination effect during secondary oil recovery process. Counter-current imbibition phenomenon occurs due to the difference of viscosity of injected fluid and native fluid. The finite difference method is used to solve the governing equations with help of initial and boundary conditions. The graph shows that the saturation of water behavior with respect to distance at different time (T) with different angle ($\theta = 5^{\circ}, 10^{\circ}, 15^{\circ}, 20^{\circ}$)

The references are given alphabetically at the end.

1.3 Brief Description on the State of the Art of the Research Topic:

The historical development of the ground water is the basics route of fluid flow through porous media. It is well known that an interconnect pores of homogeneous as well as heterogeneous porous media constitute capillary with irregular walls and fluid flowing in the interconnected capillaries is called fluid flow in porous media. The term fluid flow through porous media has vast amount of literature. The purpose of this thesis is to illustrate for a variety of problems, the well-recognized importance in the oil industry and its growing importance in hydrology, geotechnical engineering and environment protection.

The physical problems of the production of oil and gas from underground sources are nothing more than that of fluid flow through porous media which is greatly imagined topic of current research. This research focuses on the flow problems that are valid to the single and double phase flow systems with great special features.

In the figure 1.1 shows that self-explanatory about the vast scope of the multiphase fluid flow through porous media in different branches of earth sciences. The concept of mathematics, and in numerical analysis, is of well interest to know the behavior of physical problems. The groundwater hydrology is observed as specialized science which includes geology, hydrology and fluid mechanics also concerned with this part

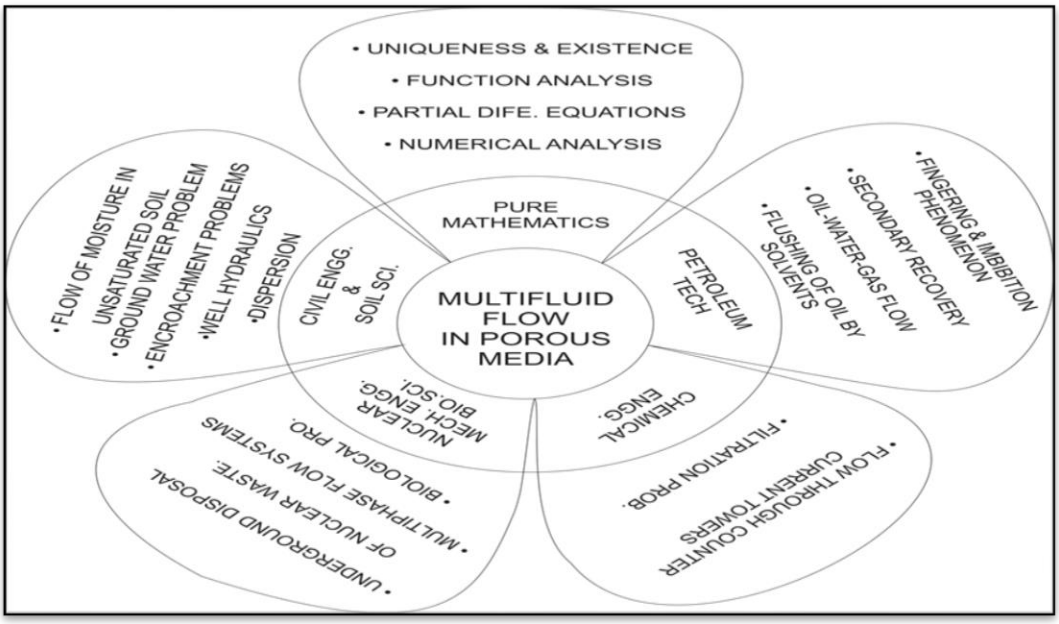


FIGURE 1.1: Author’s view of the scope of subject

The science of reservoir engineering, which developed in recent decades, no draws on many advanced techniques in data acquisition and reservoir simulation. A good understanding of the subject is compulsory for efficient oil production and conservation. The thesis provides the basic principles of geology, science and mathematics. The problems considered miscible-immiscible fluids and homogeneous as well as heterogeneous cracked porous mediums.

The single phase flow and multiphase flow with different viewpoints have been discussed by many researchers. Muskat [78] has been discussed different type of problems of fluid flow through porous media in various fields. The theory of dynamics of fluids in porous media, as applicable to many discipline of science and engineering is given book in the book “Dynamics of fluids in porous media” by Bear [24].

The phenomenon of fingering has great importance in various engineering fields like soil mechanics, groundwater hydrology, agricultural engineering and petroleum engineering [16]. The statistical behavior of instabilities in homogeneous porous media without capillary pressure has been observed by Scheidegger and Johnson (1961) [5]. Saffman and Taylor (1958) derived a classical result for the shape of finger in the absence of capillary. The average cross-sectional area occupied by the fingers while size and shape of the individual fingers neglected examined by Scheidegger (1960) [2]. In the fractured porous medium, consisting of an extensively developed system of randomly oriented fractures, the entire porous blocks due to the action of capillary force depends on time and in terms of relevant flow parameter is given by Mattax and Kyte [85] and is included in the continuity equation. Cardwell and Parsons [36] considered various aspects of gravity drainage theory. McEven [74] has solved the linear displacement equation numerically with capillary pressure included in the analysis. The problem of fluid flow through fractured porous media was discussed by Barenblatt and Zheltov [124]. The physics of oil-water flow through cracked porous media was described by Bokserman, Zheltov and Kocheshkov (BZK) [31]. The graphical relationship between capillary pressure and saturation has been defined by Richardson [99]. Hovanassian [60] has obtained a numerical solution of oil-water displacement problem with capillary pressure. Engelberta and Klinkenberg [52] observed the occurrence of fingers in their experiments on the displacement of oil and water from packs of granular material..

Current interest of another phenomenon is imbibition phenomenon in porous medium. Cocurrent and counter current are the classification of spontaneous imbibition. Many researchers have been discussed different point of views [21,95,49,91,56,68, 87,81,1,113,80].

Fingero-imbibition phenomenon happens in secondary oil recovery process, when fingering- imbibition phenomenon occurs simultaneously is known as fingero-imbibition phenomenon and it has been discussed viewpoints [8,11,12,69,70,81,91,92,106,112].

1.4 Definition of the Problem:

Many researchers have discussed mathematical studies of fluid flow through porous media with different physical phenomenon. To describe water saturation a general model for the two-phase immiscible fluid flow in porous media is presented. In the secondary oil recovery process, physical phenomenon like as fingering phenomenon, fingero-imbibition phenomenon, cocurrent imbibition phenomenon, counter current imbibition phenomenon occur in the two phase flow through porous media. The goal of the present work is to investigate the behavior of the saturation of injected water in different physical phenomenon which is arising in fluid flow through homogeneous as well as heterogeneous porous media.

1.5 Objective and Scope of Work:

The prime objective of the present work is to find the numerical solutions of the mathematical models arising during the secondary oil recovery process by finite difference method. The solutions provided the saturation of injected water which helps us to predict the amount of water required to inject for recovering oil. The solution of one dimensional nonlinear partial differential equation has been solved by finite difference method (FDM).

1.6 Original Contribution by the Thesis:

The thesis is aimed to achieve the solution of one dimensional mathematical models arising in fluid flow through porous media with appropriate initial and boundary conditions by finite difference Crank-Nicolson scheme. An approximate numerical solution has been obtained to find saturation of the injected water with respect to time (T) and distance (X).

The researcher has aimed to study following mathematical models;

1. Fingering phenomenon in homogeneous porous media.
2. Counter current imbibition phenomenon in inclined homogeneous porous media.
3. Cocurrent imbibition phenomenon in inclined homogeneous porous media.
4. Fingero- imbibition phenomenon in the homogeneous as well as heterogeneous porous media.

The study will surely contribute new idea as well as solutions to solve different problems arise in petroleum engineering, ground water porous media, hydrogeological engineering etc.

1.7 Methodology of Research and Results/ Comparisons:

We have studied different literature review related to fluid flow through porous media and find out the research gap and statement of the problems. The literature reviews advantage us to find define an objective of the research.

We have used scilab for solving nonlinear partial differential equation using finite difference method. The survey of the literatures helped us to define the area of the research. The mathematical model has been solving for using different physical phenomenon arising in homogeneous as well as heterogeneous porous medium.

During the secondary oil recovery process the nonlinear partial differential equation arising in the oil-water displacement process in homogeneous porous media. The finite difference Crank-Nicolson scheme has been used to find solution of this equation with initial and boundary conditions.

The mathematical model is developed for the problem of fingero-imbibition phenomenon arising in homogeneous and heterogeneous porous medium. To solve nonlinear partial differential equation using suitable initial and boundary conditions. The scilab coding has been used to find out numerical and graphical interpretations.

One dimensional nonlinear partial differential equation for cocurrent imbibition phenomenon in the inclined oil formatted homogeneous porous media. This phenomenon has been discussed through numerical solution and graphical representation.

The problem of counter current imbibition phenomenon in inclined homogeneous porous media. We have achieved numerical solution of nonlinear partial differential equation arising during oil recovery process with appropriate conditions using finite difference Crank-Nicolson scheme.

1.8 Achievement with Respect to Objectives:

Fluid flow through porous media has modified the problem of the mathematical models for oil and water displacement process in homogeneous porous media, fingero-imbibition in inclined homogeneous as well as heterogeneous porous media, co current imbibition phenomenon in inclined homogeneous porous media and the phenomenon of counter current imbibition in inclined homogeneous porous media.

The mathematical models of the problem in the fluid flow through porous media have been solved appropriate initial and boundary conditions by finite difference Crank-Nicolson scheme.

1.9 Conclusion:

There are different mathematical models in fluid flow through porous media. It is difficult task to choose right model depends on the geological properties, demand for accuracy and computational efficiency. From the various phenomenon described in literature review, but here we choose some common phenomenon like fingering phenomenon, fingero-imbibition phenomenon, concurrent imbibition phenomenon, counter current imbibition phenomenon.

The numerical solutions of mathematical models have been got using initial and boundary conditions. The solutions are represented graphically as well as numerically using SCILAB coding for finite difference method and hence get exact idea of saturation of water in secondary oil recovery process and observed that saturation of water increases then oil come out of with the water at each different space (X) and time (T). The solutions will be useful to determine the amount of water required for injection and for the predication of the oil recovered.

Through the study, the researcher came to the conclusion that the saturation of water level gets increases with time and space as we inject the water.

CHAPTER 2

A Prerequisite for Fluid Flow through Porous Media and Brief Description of Finite Difference Method

CHAPTER 2

A Prerequisite for Fluid Flow through Porous Media and Brief Description of Finite Difference Method

2.1 Introduction:

The chapter focuses on introduction of fluid flow through porous media and its properties as well as Darcy’s law which is very important property of fluid flow through porous media. This chapter also includes study of physical phenomenon and description of mathematical method which is used to solved different physical problems.

2.2 Fluid Flow through Porous Medium:

Solid materials involved the interconnected pores in it, generally known as porous medium. The skeletal portion of material is usually a solid and is called the matrix. Pore space can be filled with one or more fluids (oil, water or gas). Soil, sand, limestone, bread, lungs or kidney are examples of a porous medium. Pores are complex network of void spaces of various size and shapes circulated more or less often throughout the substance and that substance is known as porous.

A porous medium is a substance having pores. The holes (voids or pores) are usually pore with fluids like liquid or gas. The porous medium is considered as a solid body with pores. The pore space is defined as the non-solid space in a solid body.

A porous medium can restrictively be defined as follows satisfying three conditions:

1. The non-solid space within the solid matrix is interconnected.
2. The smallest dimension of the non-solid space must be large enough to contain fluids particles; that is, it must be large compared to the mean-free path of fluid molecules.
3. The dimensions of the non-solid space must be small enough so that when interfaces between two fluids occur within the non-solid space, the orientations of interfaces will be controlled largely by interfacial forces. Which is shows an orientation of interface in pipe as compared to that in capillary pressure.

The non-solid space is called pore space. The grains of the material are sometimes cemented at points of contact with a variety of connecting agents, in which case they are said to be consolidated e.g. sandstone. In other cases, the grains are not cemented at points of contact and such materials are said to be unconsolidated e.g. soil and sand.

Multiphase system defined as when the void space occupied by more than two fluids that are immiscible with each other (i.e. water and oil). They are maintaining distinct boundary between themself.

The single phase flow system can be expressed as the void space of a porous medium filled by only one fluid (water) or by several fluids completely miscible with each other. (e.g. salt water and fresh water)

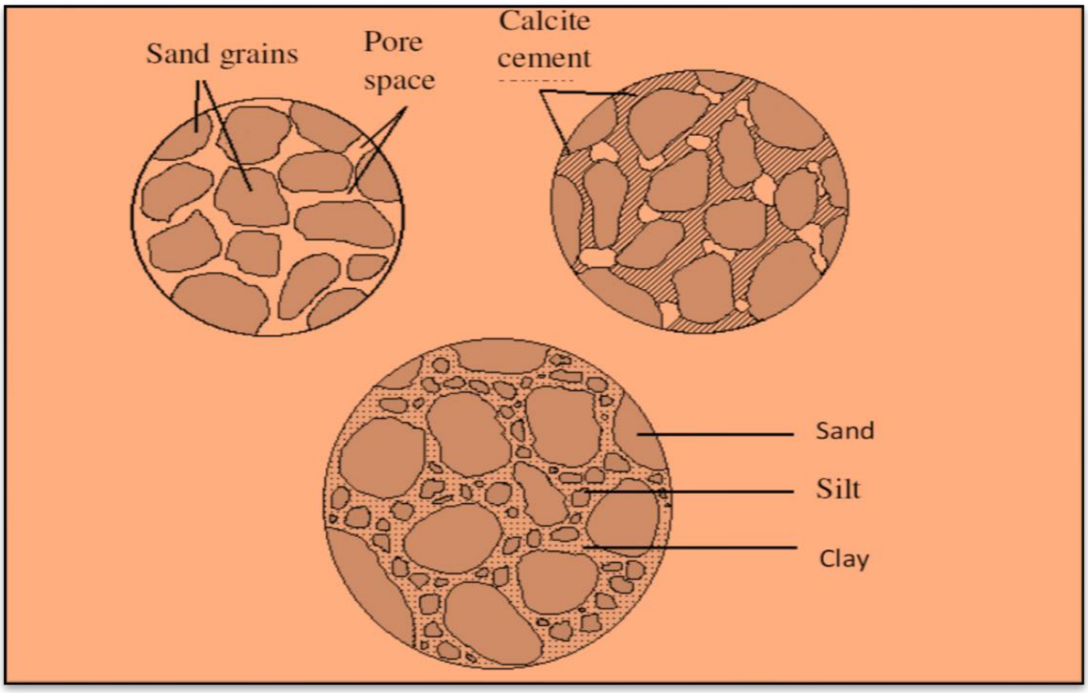


FIGURE 2.1: Representation of porous media.

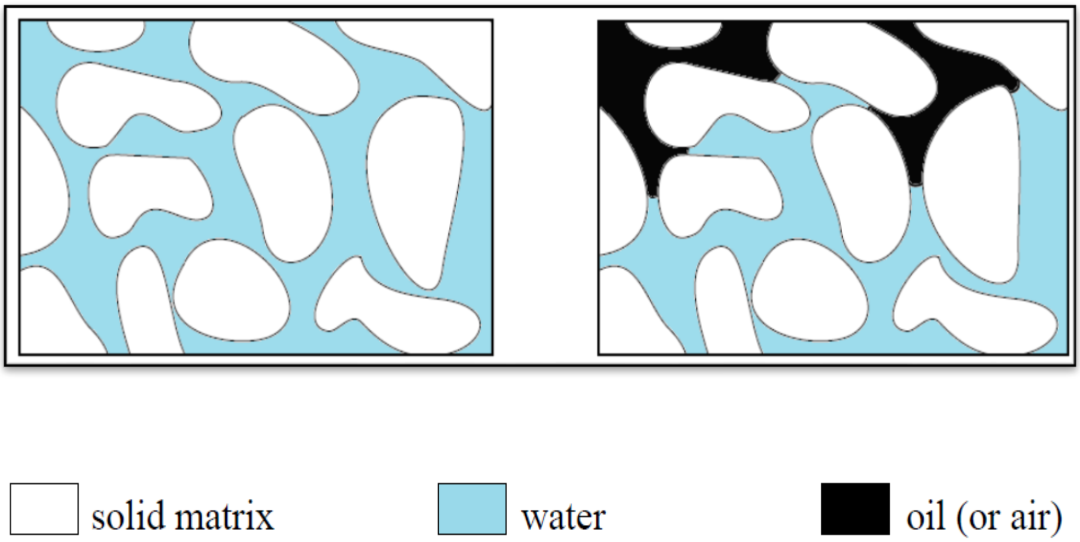


FIGURE 2.2: Schematic drawing of a porous medium filled with one or two fluids.

Our common research topic is fluid flow in porous media encountered in many branches like chemical engineering, petroleum engineering, medical and biological science, groundwater hydrology etc.

2.3 Fundamental Properties of Porous Media:

For any porous material to form a reservoir:

(a) It must have a certain storage capacity; this property is described by the porosity.

(b) The fluids must be able to flow in the medium; this property is characterized by the permeability.

(c) It must contain a sufficient quantity of fluids a sufficient concentration; the impregnated volume is a factor here, as well as saturations.

The porous material is characterized by many geometrical properties, but for purpose of fluid flow through it only two properties are important porosity and permeability are sufficient, which are described below.

2.3.1 Porosity:

Porosity is the most important property in fluid flow through porous media. Porosity is the index of how much fluids can be stored within the medium. It is defined as ratio of the volume of the voids the grains to the volume of the material. The porosity is dimensionless quantity. This important rock property is determined mathematically by the following generalized relationship,

$$\mathrm{P} = \frac{\text{pore volume}}{\text{bulk volume}} = 1 - \frac{\text{grain volume}}{\text{bulk volume}}$$

Bulk volume = grain volume + pore volume

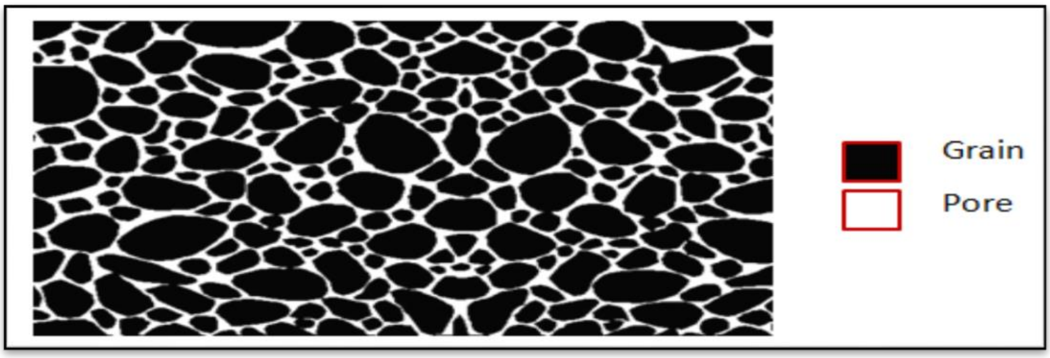


FIGURE 2.3: Microscopic cross section image of a porous medium.

P is the porosity of the porous medium.

2.3.2 Permeability:

Permeability is described as the measures the capacity and ability of the formation to transmit fluids. The permeability k is a very important rock property because it controls the directional movement and the flow rate of the reservoir fluids in the formation. This rock characterization was first defined mathematically by Henry Darcy's in 1856. The units of permeability are $\mathrm{cm}^2, \mathrm{ft}^2$, Darcy, $^{\mathrm{m}}/_{\mathrm{s}}$ and $^{\mathrm{f}}/_{\mathrm{s}}$.

2.3.3 Relationship between Porosity and Permeability:

A correlation has been established between porosity and permeability for a given material. An attempt is made to write an equation of the type

$$\log k = a\emptyset + b$$

2.3.4 Effective Permeability:

It is called absolute permeability when there is only single fluid moving through porous medium. However, when there is more than one fluid present in a rock, the permeability of each fluid to flow is reduced because another fluid will also be flowing in the rock. The effective permeability of that fluid can be defined the capacity to preferentially flow or transmit a particular fluid when other immiscible fluids (oil & water) are present in a rock.

2.3.5 Relative Permeability:

The ratio of effective permeability of a particular fluid at a particular saturation to absolute permeability of that fluid at total saturation is called relative permeability in multiphase flow through porous media. If relative permeability is 1 then the single fluid present in a rock. Oil and water relative permeability may be defined as

$$k_w = \frac{k_{ew}(S_w)}{k_{sat}} \text{ and } k_o = \frac{k_{eo}(S_o)}{k_{sat}}$$

k_{sat}= Permeability at full saturation.

k_{eo} = Effective permeability of oil.

k_{ew}= Effective permeability of water.

2.2 Types of Fluid Flow:

2.4.1 Single phase Flow:

The void space is filled with a single fluid in porous media, it is known as single phase flow. For e.g., water, air.

2.4.2 Multiphase Flow:

Multiphase flow refers to the interactive flow of two distinct phases with common interfaces in a channel, with each phase representing a mass or volume of matter. The two phases can exist as combinations of solid, gas and/or liquid phases. Although multiphase flow involving three phases can also exist, most multiphase engineering applications are two-phase flow.

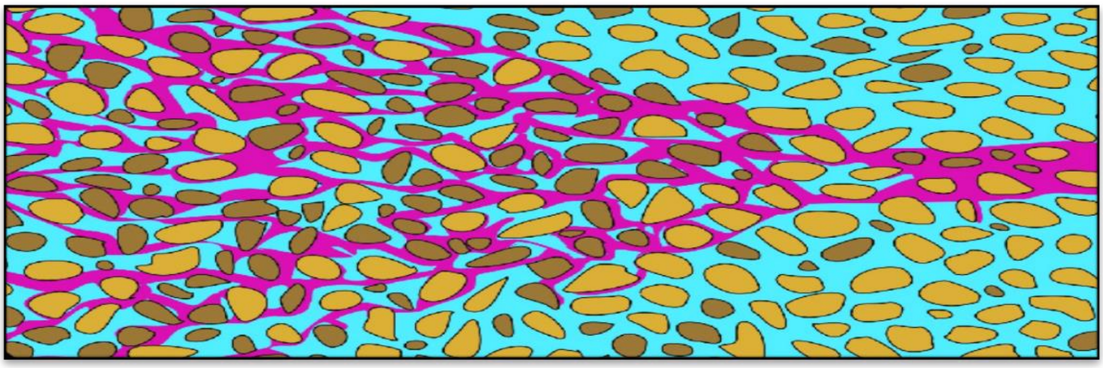


FIGURE 2.4: Multiphase phase flow of oil and water.

2.4.3 Steady and Unsteady Flow:

The terms steady and unsteady are used in engineering field. The flow regime is identified as a steady flow if the pressure at every location in the reservoir remains constant, i.e. it does not change with time.

Unsteady flow is defined as the fluid flowing condition at which the rate of change of pressure with respect to time at any position.

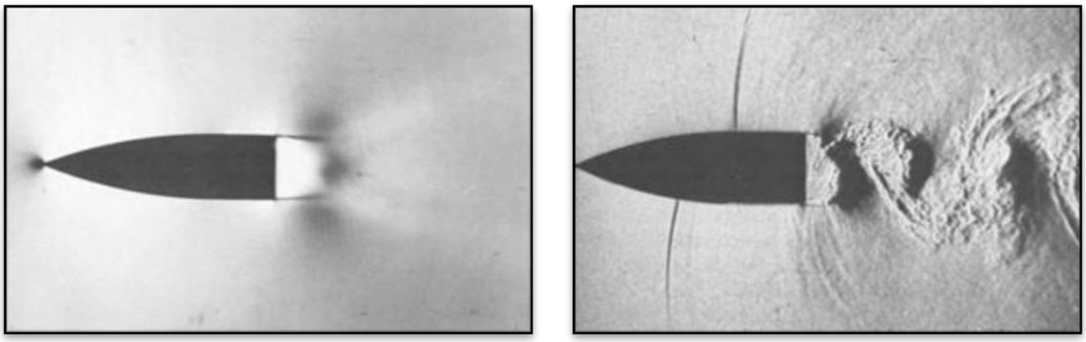


FIGURE 2.5: Steady flow & unsteady flow.

2.4.4 Uniform and Non-Uniform. Flow:

The flow is called uniform flow when the velocity of flow unchanged in both magnitude and direction at any point of fluid flowing, with respect to time.

The velocity of the flow changes with different points in a flowing fluid, for a given time is known as non-uniform flow.

2.4.5 Compressible and Incompressible Flow:

A fluid has divided into compressible or incompressible depending on the level of variation of density during the fluid flow. Compressible flow can be defined as the fluid flow in which the density remains constant throughout the fluid flow. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion. Gases are compressible fluid.

The density remains unchanged with each point then that type of flow is known as incompressible flow.

2.4.6 Miscible Flow:

Two fluids are defined as miscible if the molecules of the one fluid are free to mix with the molecules of the other fluid. There is no interface between two miscible fluids. A common example of two miscible fluids is water and ethanol. In any proportions it is possible to mix the water and ethanol together to form a single homogeneous phase. When two gases meet, they are always miscible; for example oxygen and nitrogen readily mix in air.

2.4.7 Immiscible Flow:

Two fluids are defined as immiscible if the two fluids scarcely mix at all at the molecular level, and not at all at the microscale. The two phases remain distinct and there is a well-defined interface between the two fluids. Common examples of two immiscible fluids are water and oil.

2.5 Capillary Mechanism:

2.5.1 Wettability:

When two immiscible fluids such as oil and water are together in contact with a rock face the situation is as shown in [Figure- 2.6].The angle θ measured through the water is called the contact angle.

If contact angle is less than 90° reservoir rock is called as being water wet. The contact angle is greater than 90° it is oil wet. The wet ability, as defined as the angle, is a measure of which fluid preferentially adheres to the rock.

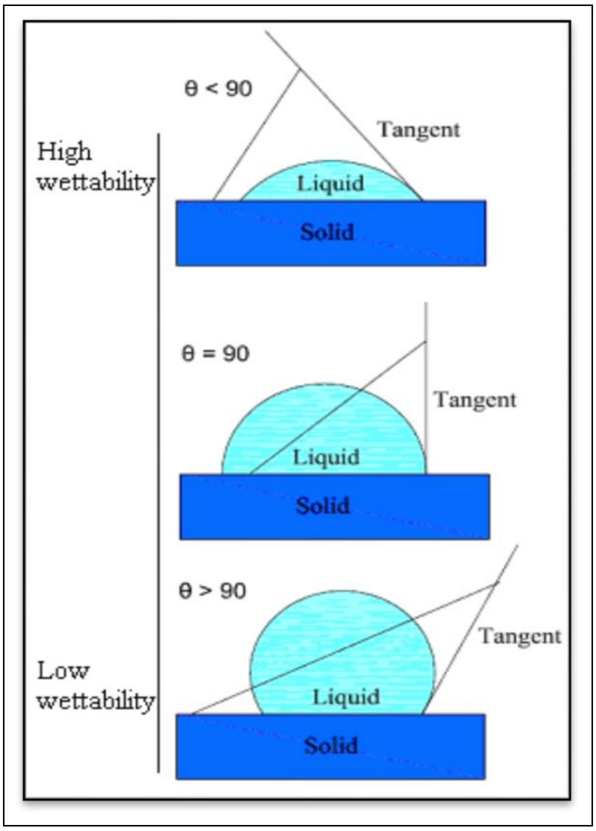


FIGURE 2.6: Diagram of contact angle and wettability measurements.

In other words, whenever two or more fluid phases occupy a porous medium, it is inevitable that one of the fluids will be adsorbed on the solid surface more strongly than the other fluid. The fluid which is most absorbed and which displace the other fluid is called wetting phase. The displaced phase is called non-wetting phase. In water/oil flow, water is often the wetting fluid. In some cases, the oil may be wetting preferentially to the water, especially for a number of limestone's.

2.5.2 Capillary Pressure:

When two immiscible fluids are in contact a clearly defined interface exists between them. An interface is made by existence of forces, called interfacial forces, that act at only boundaries between separate phases and are tangential to the boundaries, means the molecules near the interface are unevenly attracted by their neighbors and give rise to such forces. When the interface is curved the pressure on the concave side exceeds that on the convex (pressure discontinuity) and this difference is known as capillary pressure.

The difference in pressure between phases occupying the pore space of a porous medium is related to gravity, saturation, pore size, pore shape, interfacial forces, the angle at which fluid-fluid interface contact solid-solid surface, the density difference between phases and the radius of curvature of interface. These factors are not all independent variables in respect to their effect on $\mathrm{P_c}$. There is, however, an inter relation among them. Liquid-air interface across a capillary tube, in which is the contact angle related to the three interfacial forces.

(i) Liquid – solid (ii) gas – solid (iii) liquid – gas

The change in interfacial curvatures and P_c are accompanied by a change in fluid saturation. A functional relationship between saturation and capillary pressure can be visualized by considering a model of a cross sectional of an element of pore space.

Capillary pressure is the most important functional relationship in respect to the mechanics of mixed fluids in porous media close together and situated on either side of the interface

$$\mathrm{P_c} = \mathrm{P_{non-wetting\ phase}} - \mathrm{P_{wetting\ phase}}$$

The forgoing discussion shows that, for a medium saturated with a fluid and surrounded by another fluid:

a) If the saturating fluid is wetting, it is displaced by surrounding fluid only if the excess pressure applied to the capillary pressure for the largest pores.

b) If the saturating fluid is non-wetting, it is displaced spontaneously by the surrounding fluid.

2.5.3 Saturation:

Saturation can defined as that fraction, or percent, of the pore volume occupied by a particular fluid (oil, gas, or water). This property is expressed mathematically by the following relationship:

$$\text{fluid satuartion} = \frac{\text{total volume of the fluid}}{\text{pore volume}}$$

All saturations are based on pore volume not gross volume of the reservoir. The saturation of each individual phase ranges between 0 % to 100%.

2.6 Darcy’s law:

In this section, we shall discuss an important factor for the discussion of fluid flow through porous media, known as Darcy’s law, which forms the basis of our present investigation. Darcy’s law covers the principal that governs how fluid moves in the subsurface. The volumetric flow rate (Q) of the fluid through homogeneous sand column is (a) proportional to the cross sectional area A of the column and proportional to the difference in fluid level elevations at inflow and outflow of the column, (b) inversely proportional to the length of the column, i.e.

$$\mathrm{Q} = \mathrm{K\,A}\frac{\mathrm{h}^{(a)} - \mathrm{h}^{(b)}}{\mathrm{L}}$$

Q = rate of water flow.

K = constant of proportionality.

A = cross sectional area.

L= length of sample.

$h^{(a)}$= measure the pressure at point a.

$h^{(b)}$= measure the pressure at point b.

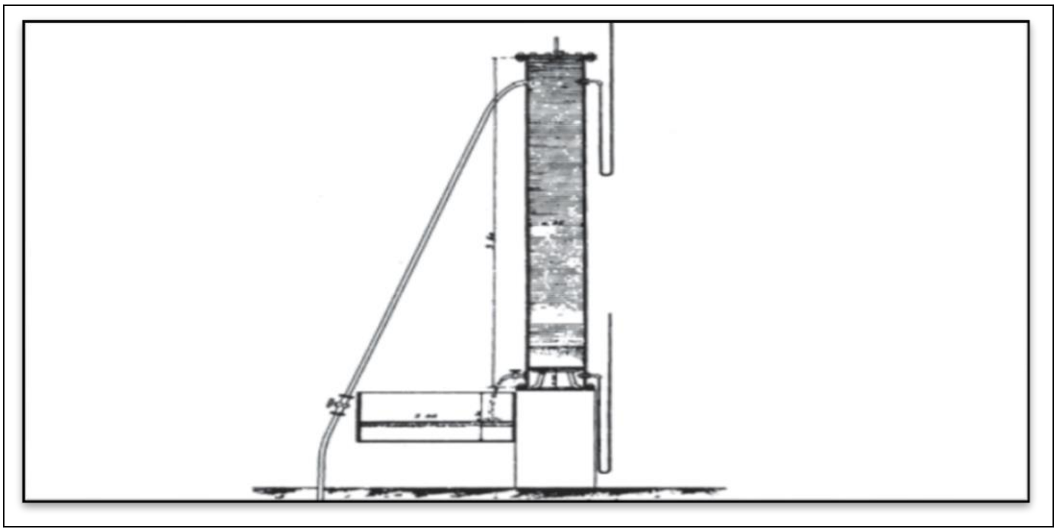


FIGURE 2.7: Darcy’s column experiment.

2.7 Physical phenomenon:

This chapter also discusses three special kinds of phenomenon arising in **the low of two immiscible fluid flow.**

2.7.1 Fingering Phenomenon:

The phenomenon of fingering, due to the difference of viscosities of the flowing fluids has a great importance in secondary oil recovery process of petroleum technology. It has been observed that, when a fluid contained in a porous medium is displaced by another fluid of less viscosity instead of regular displacement of the whole front, fingers may occur which shoot through the porous medium at relatively very high speed. This phenomenon is called fingering (instability) phenomenon. Figure (2.8), it is vital role in secondary oil recovery process of petroleum technology.

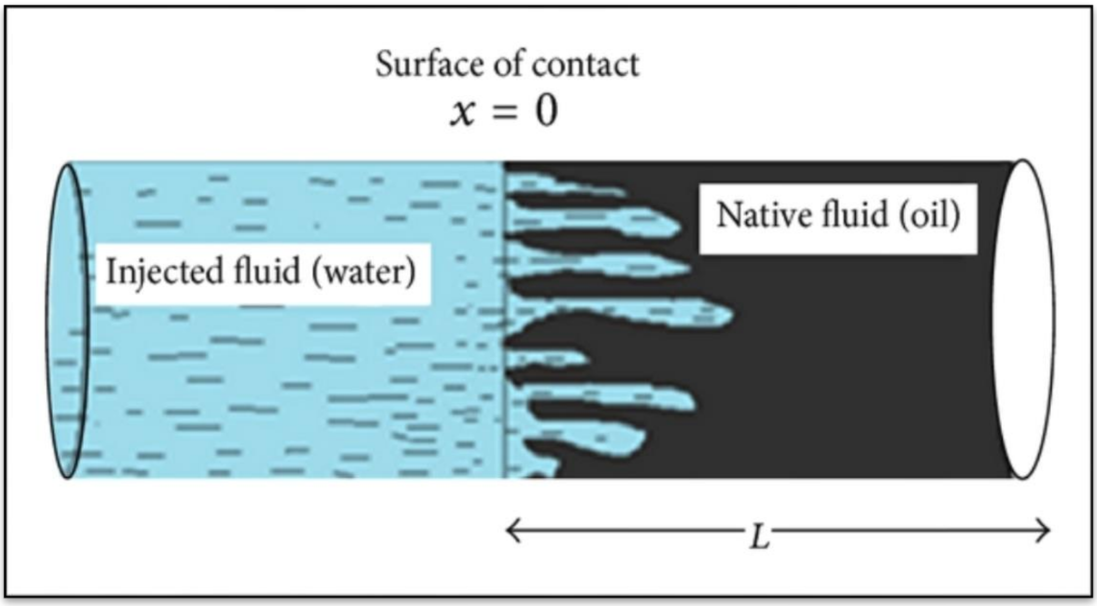


FIGURE 2.8: Occurrences of fingers.

2.7.2 Imbibition Phenomenon:

When non wetting phase of porous medium filled with some fluid is brought into contact with another fluid which preferentially wets the medium, and then there is a spontaneous flow of the wetting fluid into the medium and a counter flow of the resident fluid from the medium. Such phenomenon is known as imbibition phenomenon. Oil recovery process, printing process, food industry, biological sciences, surface chemistry, textiles and construction all these fields depends on the applications of this phenomenon.

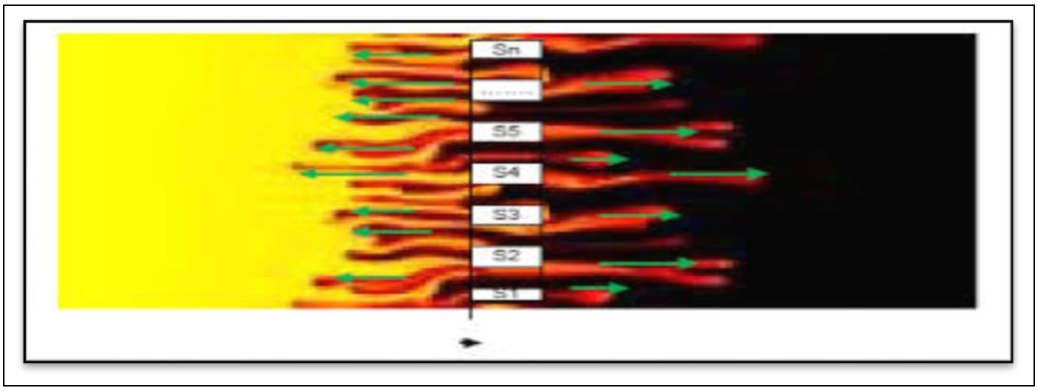


FIGURE 2.9: Representation of imbibition phenomenon.

2.7.3 Fingero- imbibition Phenomenon:

During the secondary oil recovery process fingero-imbibition phenomenon occurs. When fingering and imbibition phenomenon occurs simultaneously it is called fingero-imbibition phenomenon. For example, if a finite cylindrical porous matrix containing native liquid N is completely surrounded by an impermeable surface except for one end of the cylindrical and this end exposed to an adjacent formation of another fluid I which is preferentially wetting and less viscous than the injection of I is initiated by imbibition and consequent displacement of N produces fingering. Such a phenomenon is called fingero-imbibition phenomenon.

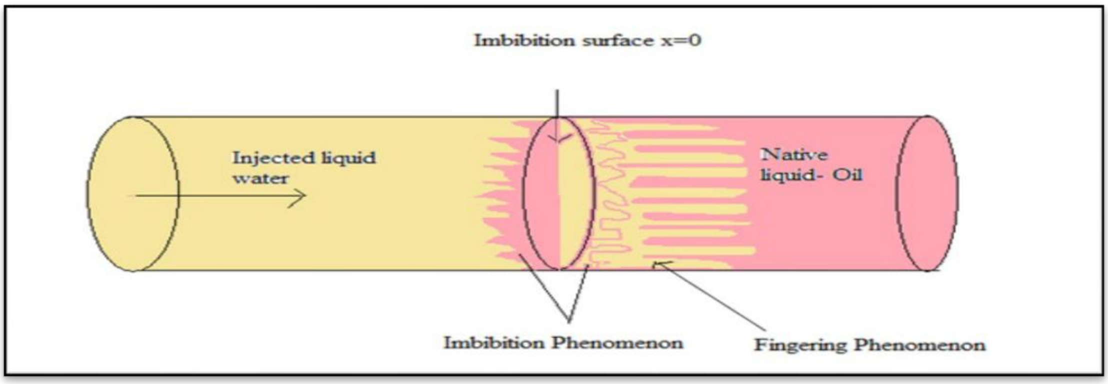


FIGURE 2.10: Representation of fingero-imbibition phenomenon.

2.8 Finite Difference Method:

The solution of Ordinary Differential Equations (ODEs) is found by two ways first is discrete method and second is numerical method. Discrete methods are one of the oldest and most successful areas of numerical calculation. One of the examples to need for such calculation is the problem of predicting the motions of planets or other bodies in the space.

The governing differential equations for the sun and the planet are known and Newton showed that how to solve it exactly. In the last three centuries no one has ever found its exact solution in the case of more than two bodies. So, the numerical calculation of such orbits is effortless by modern standards.

The most popular families of numerical methods for solution of ODEs are:

- Finite Difference Method
- Finite Element Method
- Runge-Kutta Method

For finding the solution of two ending point boundary value problem using finite difference method, we replace the derivatives occurring in the differential equation as well as in the boundary conditions by means of their finite difference approximations. By Substituting finite difference we obtain sets of linear equation and it is solve by any standard procedure.

Now procedure to find the appropriate finite difference (central difference) approximation of the derivatives as follows.

The Taylor’s series expansion of

$$f(X+\Delta X) = f(X) + \frac{\partial f}{\partial X}\Delta X + \frac{\partial^2 f}{\partial X^2}\frac{(\Delta X)^2}{2!} + \frac{\partial^3 f}{\partial X^3}\frac{(\Delta X)^3}{3!} + \cdots \quad (2.1)$$

Now above Taylor’s series convert to discretization form, we get

$$\emptyset_{i+1,j} = \emptyset_{i,j} + \left(\frac{\partial \emptyset}{\partial X}\right)_{i,j}\Delta X + \left(\frac{\partial^2 \emptyset}{\partial X^2}\right)_{i,j}\frac{(\Delta X)^2}{2!} + \left(\frac{\partial^3 \emptyset}{\partial X^3}\right)_{i,j}\frac{(\Delta X)^3}{3!} + \cdots \quad (2.2)$$

For which we obtain,

$$\left(\frac{\partial \emptyset}{\partial \mathrm{X}}\right)_{\mathrm{i,j}} = \frac{\emptyset_{\mathrm{i+1,j}} - \emptyset_{\mathrm{i,j}}}{\Delta \mathrm{X}} - \left(\frac{\partial^2 \emptyset}{\partial \mathrm{X}^2}\right)_{\mathrm{i,j}} \frac{(\Delta \mathrm{X})^2}{2!} + \ldots \tag{2.3}$$

Neglecting second and higher orders terms in above series we get,

$$\left(\frac{\partial \emptyset}{\partial \mathrm{X}}\right)_{\mathrm{i,j}} = \frac{\emptyset_{\mathrm{i+1,j}} - \emptyset_{\mathrm{i,j}}}{\Delta \mathrm{X}} \tag{2.4}$$

Equation (2.4) is called first order forward difference with respect to X.

If we replace ΔX by (- ΔX) in equation (2.2), we get

$$\emptyset_{\mathrm{i+1,j}} = \emptyset_{\mathrm{i,j}} - \left(\frac{\partial \emptyset}{\partial \mathrm{X}}\right)_{\mathrm{i,j}} \Delta \mathrm{X} + \left(\frac{\partial^2 \emptyset}{\partial \mathrm{X}^2}\right)_{\mathrm{i,j}} \frac{(\Delta \mathrm{X})^2}{2!} - \left(\frac{\partial^3 \emptyset}{\partial \mathrm{X}^3}\right)_{\mathrm{i,j}} \frac{(\Delta \mathrm{X})^3}{3!} + \cdots \tag{2.5}$$

Neglecting second and higher order terms, we obtain

$$\left(\frac{\partial \emptyset}{\partial \mathrm{X}}\right)_{\mathrm{i,j}} = \frac{\emptyset_{\mathrm{i,j}} - \emptyset_{\mathrm{i-1,j}}}{\Delta \mathrm{X}} \tag{2.6}$$

The above equation is said to be first order backward difference w.r.t to X.

Subtracting equation (2.2) & equation (2.5), we get

$$\left(\frac{\partial \emptyset}{\partial \mathrm{X}}\right)_{\mathrm{i,j}} = \frac{\emptyset_{\mathrm{i+1,j}} - \emptyset_{\mathrm{i-1,j}}}{2\Delta \mathrm{X}} \tag{2.7}$$

This is said to be central differences with respect to X.

In similar manner, it is possible to derive second order mixed derivative formula for finite difference method.

- Second order central difference with respect to X.

$$\left(\frac{\partial^2 \emptyset}{\partial \mathrm{X}^2}\right)_{\mathrm{i,j}} = \frac{\emptyset_{\mathrm{i+1,j}} - 2\emptyset_{\mathrm{i,j}} + \emptyset_{\mathrm{i-1,j}}}{(\Delta \mathrm{X})^2} \tag{2.8}$$

- Second order central difference with respect to Y.

$$\left(\frac{\partial^2 \emptyset}{\partial \mathrm{Y}^2}\right)_{\mathrm{i,j}} = \frac{\emptyset_{\mathrm{i,j+1}} - 2\emptyset_{\mathrm{i,j}} + \emptyset_{\mathrm{i,j-1}}}{(\Delta \mathrm{Y})^2} \tag{2.9}$$

- Second order central mixed difference with respect to X & Y.

$$\left(\frac{\partial^2 \emptyset}{\partial X\,\partial Y}\right)_{i,j} = \frac{\emptyset_{i+1,j+1}+\emptyset_{i-1,j-1}-\emptyset_{i-1,j+1}-\emptyset_{i+1,j-1}}{4\Delta X\Delta Y} \qquad (2.10)$$

Many of differential equations which result from using central difference analog. Accordingly, knowledge of the methods of obtaining numerical solutions of differential equation is important to the modern researchers. The more efficient methods for numerical solution of partial differential equations have been developed more recently with the start of high speed software.

The analytical solutions of the equations considered here, are specified with continuous variables, $(\text{e.g. } X, T, S_w \text{ etc.})$. To obtain a numerical solution, one replaces these continuous variables with discrete variables. The relations between those discrete variables are finite difference equations which are solved numerically using different software. The discrete variables are defined at a finite number of points spaced equally in the given interval $\left(e.g.\ X_i = i\,(\Delta X)\right)$ as shown in the figure (2.11).

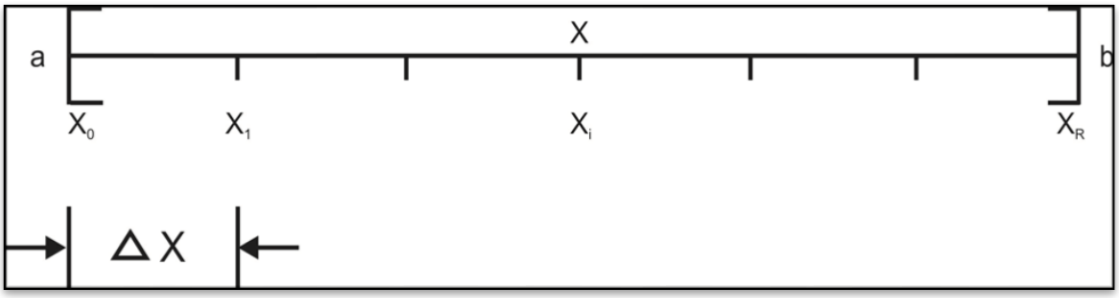
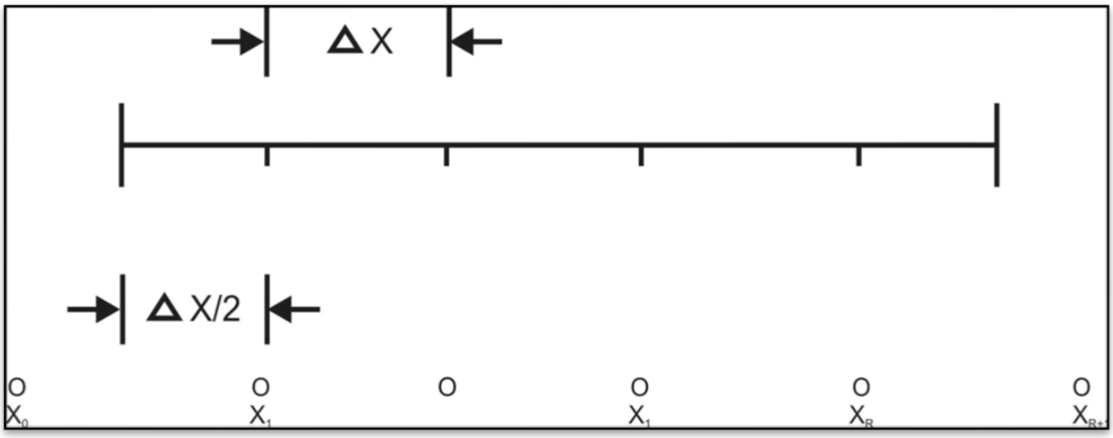


FIGURE 2.11: Diagram of equal space for finite difference method.

Where R is the total number of increments in the complete interval between a and b.

It is sometime desirable to develop a single set of equations which will handle a variety of boundary conditions. This objective is accomplished by spacing the first point one-half an increment from the boundary. The points (also called grid points) are arranged as shown in the figure below, and the value of the independent variable at each point is given by

$$X_i = \left(i - {}^{1}\!/_{2}\right)\Delta X$$



In this arrangement, there are no points on the boundaries, so the number of unknown values of S_w is equal to the number of increments for all types of boundary conditions.

2.8.1 Analog of the Derivatives:

We consider S_w as a function of two variables X and T. At the grid point $(X_i\,,T_n)$ the finite difference second order correct analog for the derivatives are given by

$$\left[\frac{\partial S_w}{\partial X}\right]_{i,n} = \frac{S_{w_{i,n}} - S_{w_{i-1,n}}}{2(\Delta X)}$$

$$\left[\frac{\partial^2 S_w}{\partial X^2}\right]_{i,n} = \frac{S_{w_{i+1,n}} - 2S_{w_{i,n}} + S_{w_{i-1,n}}}{(\Delta X)^2}$$

$$\left[\frac{\partial S_w}{\partial T}\right]_{i,n} = \frac{S_{w_{i,n+1}} - S_{w_{i,n}}}{\Delta T}$$

2.8.2 Crank-Nicolson Procedure:

Crank and Nicolson provided a method to obtain second- order correct finite differences which are written about the point$\left(X_i\,,T_{n+1/_2}\right)$ that is halfway between the known and the unknown time levels. In the figure below this point is known as a cross.

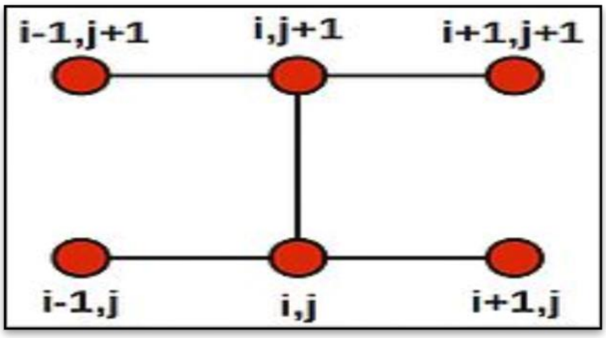


FIGURE 2.12: Crank-Nicolson scheme.

Values of the dependent variable, S_w, are computed only at the points designated by circles.

The second-order correct analogs for all the derivatives at a point $\left(\mathrm{X_i}\,,\mathrm{T}_{\mathrm{n}+1/_2}\right)$ are:

$$\left[\frac{\partial \mathrm{S_w}}{\partial \mathrm{T}}\right]_{\mathrm{i,n}+1/_2} = \frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i,n+1}}} - \mathrm{S}_{\mathrm{w}_{\mathrm{i,n}}}}{\Delta \mathrm{T}}$$

$$\left[\frac{\partial^2 \mathrm{S_w}}{\partial \mathrm{X}^2}\right]_{\mathrm{i,n}+1/_2} = \frac{1}{2(\Delta \mathrm{X})^2}\left[\mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n}}} - 2\mathrm{S}_{\mathrm{w}_{\mathrm{i,n}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n+1}}} - 2\mathrm{S}_{\mathrm{w}_{\mathrm{i,n+1}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n+1}}}\right]$$

$$\left[\frac{\partial \mathrm{S_w}}{\partial \mathrm{X}}\right]^2_{\mathrm{i,n}+1/_2} = \left[\frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n+1/2}}} - \mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n+1/2}}}}{2(\Delta \mathrm{X})}\right]^2$$

$$= \left[\frac{\left(\frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n+1}}}}{2}\right) - \left(\frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n+1}}}}{2}\right)}{2(\Delta \mathrm{X})}\right]^2$$

$$\left[\frac{\partial \mathrm{S_w}}{\partial \mathrm{X}}\right]_{\mathrm{i,n}+1/_2} = \left[\frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n+1/2}}} - \mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n+1/2}}}}{2(\Delta \mathrm{X})}\right]$$

$$= \left[\frac{\left(\frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i+1,n+1}}}}{2}\right) - \left(\frac{\mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n}}} + \mathrm{S}_{\mathrm{w}_{\mathrm{i-1,n+1}}}}{2}\right)}{2(\Delta \mathrm{X})}\right]$$

The Crank-Nicolson equation is stable for all the ratios of ΔX and ΔT. As a consequence, it can be shown that a stable solution can be obtained even when an unstable equation is the result of successive applications of the forward and backward equations.

CHAPTER 3

Mathematical Analysis of Fingering Phenomenon in Homogeneous Porous Media

Chapter 3

Mathematical Analysis of Fingering Phenomenon in Homogeneous Porous Media

3.1 Introduction:

The fingering phenomenon arises during the displacement process of two immiscible fluids like oil and water through homogeneous porous medium. This is frequently useful in many engineering fields and science. This phenomenon arises in the secondary oil recovery process.

This chapter discusses the importance of fingering phenomenon in double phase flow in homogeneous porous medium. If the porous medium filled with two immiscible fluids (oil and water) which is displaced by another less viscosity then regular displacement process may arise of whole front, protuberances take place which shoots through the porous medium at relatively very high speed. This type of phenomenon is called fingering phenomenon.

Many researchers have been discussed the fingering phenomenon of different point of view. The injected water, deliberated to push the native fluid forward, tends to penetrate the more viscous native oil through spontaneously formed multi-branched fingers by Marle (1981) [72]. A classical result for the shape of finger in the absence of capillary has been obtained by Saffman and Taylor (1958). The average cross-sectional area occupied by the fingers while size and shape of the individual fingers were neglected by Scheidegger (1960) [2]. The statistical behavior of instabilities in homogeneous porous media without capillary pressure has been studied by Scheidegger and Johnson (1961) [5].

We solved nonlinear partial differential equation which arises during fingering phenomenon in homogeneous porous medium. Finite difference Crank-Nicolson scheme has been used to solve nonlinear partial differential equation with the help of initial and boundary conditions.

3.2 Mathematical Formulation:

We choose a cylindrical piece of porous matrix whose length L whose three sides are impermeable except one end, from where water is injected in homogeneous porous medium.

Let the water is injected at x=0 then due to the injecting force and viscosity, different instability may arise which is due to the displacement of oil by water injection through interconnected capillaries. The length x of the fingers is being studied in the direction of displacement. Scheidegger and Johnson indicated replacing irregular fingers by schematic fingers of rectangular size.

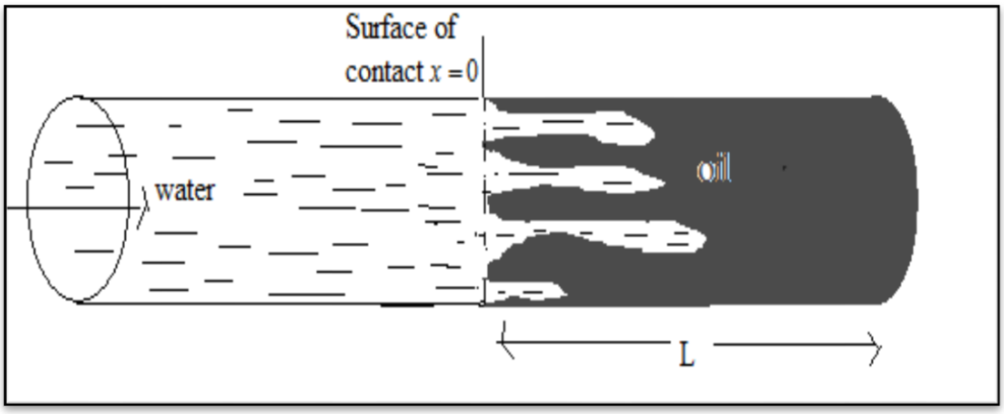


Figure 3.1 fingers formation in porous medium.

The average cross sectional area occupied by schematic fingers as a saturation of injected water for the average length of schematic fingers x=0, for given time $t > 0$.

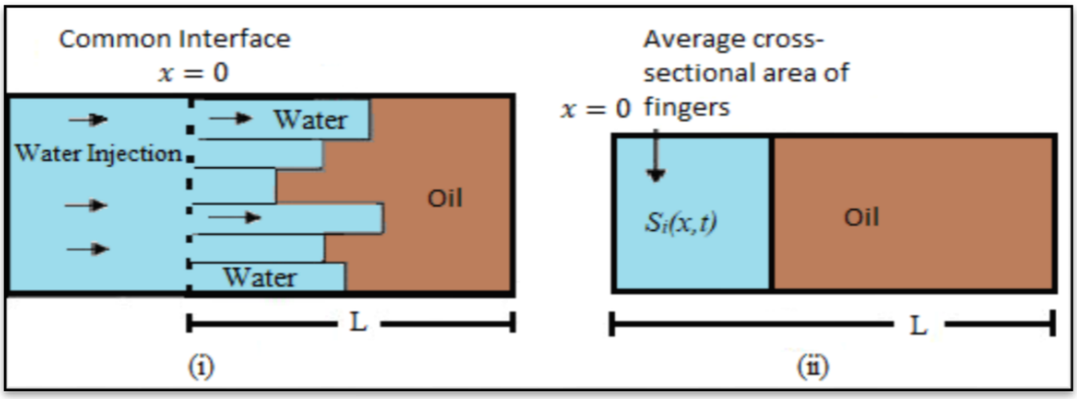


Figure 3.2: Schematic representation of the fingering phenomena

Darcy's law for these two immiscible fluids can be expressed as seepage velocities of injected water V_w and native oil V_o as Bear (1972) [24].

$$V_w = -\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} \tag{3.1}$$

$$V_o = -\frac{k_o}{\delta_o} K \frac{\partial P_o}{\partial x} \tag{3.2}$$

K = Permeability of the porous medium.

k_w= Relative permeability of water.

k_o = Relative permeability of oil.

P_w= Pressure of water.

P_o = Pressure of oil.

δ_w= Viscosity of injected water.

δ_o= Viscosity of native oil.

The continuity equation can be expressed as,

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{3.3}$$

$$P \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \tag{3.4}$$

P is the porosity of the porous matrix.

Putting the values of the seepage velocities V_w and V_o.

Now, the equations (3.1) and (3.2) into the equations (3.3) and (3.4) respectively, we get

$$\mathrm{P}\,\frac{\partial \mathrm{S_w}}{\partial \mathrm{t}} = \frac{\partial}{\partial \mathrm{x}}\left(\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\mathrm{K}\,\frac{\partial \mathrm{P_w}}{\partial \mathrm{x}}\right) \tag{3.5}$$

$$\mathrm{P}\,\frac{\partial \mathrm{S_o}}{\partial \mathrm{t}} = \frac{\partial}{\partial \mathrm{x}}\left(\frac{\mathrm{k_o}}{\delta_\mathrm{o}}\,\mathrm{K}\,\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}}\right) \tag{3.6}$$

When water is injected at common interface flow of injected water take place only due to the capillary pressure P_c.

The pressure difference between native oil and injected water is called capillary pressure (Scheidegger, 1960) [1, 2, 5];

$$\mathrm{P_c} = \mathrm{P_o} - \mathrm{P_w} \tag{3.7}$$

Eliminating $\frac{\partial P_w}{\partial x}$ from equations (3.5) & (3.7), we obtain

$$\mathrm{P}\,\frac{\partial \mathrm{S_w}}{\partial \mathrm{t}} = \frac{\partial}{\partial \mathrm{x}}\left(\mathrm{K}\,\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\left(\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}} - \frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}\right)\right) \tag{3.8}$$

For more expansion, adding (3.6) and (3.8) and using the relation saturation of water and oil ($S_w + S_o = 1$); we get

$$\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{K}\left(\frac{\mathrm{K_w}}{\delta_\mathrm{w}} + \frac{\mathrm{K_o}}{\delta_\mathrm{o}}\right)\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}} - \mathrm{K}\,\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}\right) = 0 \tag{3.9}$$

Now integrating the equation (3.9) w.r.t x, we got

$$\left(\mathrm{K}\left(\frac{\mathrm{k_w}}{\delta_\mathrm{w}} + \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\right)\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}} - \mathrm{K}\,\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\frac{\partial \mathrm{P_c}}{\partial x}\right) = -\mathrm{C(t)} \tag{3.10}$$

Where, $\mathrm{C(t)}$ is constant of integration.

For more Simplify the equation (3.10), we get

$$\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}} = -\frac{\mathrm{C(t)}}{\mathrm{K}\left(\frac{\mathrm{k_w}}{\delta_\mathrm{w}} + \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\right)} + \frac{\frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}}{1 + \frac{\mathrm{k_o}}{\mathrm{k_w}}\frac{\delta_\mathrm{w}}{\delta_\mathrm{o}}} \tag{3.11}$$

Putting the value (3.11) in (3.8) we have,

$$\mathrm{P}\,\frac{\partial \mathrm{S_w}}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\left[\frac{\mathrm{K}\,\frac{\mathrm{k_o}}{\delta_\mathrm{o}}\,\frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}}{1 + \frac{\mathrm{k_o}}{\mathrm{k_w}}\frac{\delta_\mathrm{w}}{\delta_\mathrm{o}}} + \frac{\mathrm{C(t)}}{1 + \frac{\mathrm{k_o}}{\mathrm{k_w}}\frac{\delta_\mathrm{w}}{\delta_\mathrm{o}}}\right] = 0 \tag{3.12}$$

The pressure of oil P_o can be defined as,

$$P_o = \frac{P_o + P_w}{2} + \frac{P_o - P_w}{2} = \overline{P} + \frac{1}{2} P_c \tag{3.13}$$

$\overline{P}$ is the constant mean pressure.

Now, taking differentiation the above equation w.r.t x, we get

$$\frac{\partial P_o}{\partial x} = \frac{1}{2} \frac{\partial P_c}{\partial x} \tag{3.14}$$

Putting the value (3.14) in (3.10), we get

$$P \frac{\partial S_w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left(K \frac{k_w}{\delta_w} \frac{\partial P_c}{\partial S_w} \frac{\partial S_w}{\partial x} \right) = 0 \tag{3.15}$$

For more simplify, we use the standard relations,

$K_w = S_w$ and $P_c = -\beta S_w$, Mehta [77]

The equation (3.15) become,

$$P \frac{\partial S_w}{\partial t} - \frac{\beta}{2} \frac{K}{\delta_w} \frac{\partial}{\partial x} \left(S_w \frac{\partial S_w}{\partial x} \right) = 0 \tag{3.16}$$

Choose dimensionless variables,

$$X = \frac{x}{L} \text{ and } T = \frac{K\beta t}{2PL^2\delta_w} \tag{3.17}$$

Equation (3.16) reduce to

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X} \left(S_w \frac{\partial S_w}{\partial X} \right) = S_w \frac{\partial^2 S_w}{\partial X^2} + \left(\frac{\partial S_w}{\partial X} \right)^2 \tag{3.18}$$

This is Boussinesq equation of fingering phenomenon in homogeneous porous media, which has been solved with appropriate initial and boundary conditions with help of finite difference Crank-Nicolson scheme.

Rewriting the equation (3.18) using $S_w (X, T) = S (X, T)$, we got

$$\frac{\partial S}{\partial T} = S \frac{\partial^2 S}{\partial X^2} + \left(\frac{\partial S}{\partial X} \right)^2 \tag{3.19}$$

The initial and boundary conditions are assumed as follows

$$S(X,0) = X\,,\ 0 \le X \le 1 \tag{3.20}$$

$$S(0,T) = T,\ T > 0 \tag{3.21}$$

$$S(1,T) = 1 - T,\ T > 0 \tag{3.22}$$

3.3 Solution by Finite Difference Method:

Using Finite difference Crank- Nicolson scheme is applied in equation (3.19), we get [32,101,102]

$$\frac{\partial S}{\partial T} = \frac{S_{i,n+1} - S_{i,n}}{\Delta T} \tag{3.23}$$

$$\frac{\partial^2 S}{\partial X^2} = \frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right] \tag{3.24}$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2\,(\Delta X)}\right]^2$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2 \tag{3.25}$$

Using the equations (3.23), (3.24) and (3.25) in (3.22), we get

$$\frac{S_{i,n+1} - S_{i,n}}{\Delta T} = S_{i,n+1/2}\,\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right] + \left[\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2\,(\Delta X)}\right]^2$$

$$= S_{i,n+1/2}\,\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right] + \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2$$

$$= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right]$$

$$+ \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2$$

$$= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right]$$

$$+ \left[\frac{1}{4\,(\Delta X)}\left(S_{i+1,n} + S_{i+1,n+1}\right) - \left(S_{i-1,n} + S_{i-1,n+1}\right)\right]^2$$

$$= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right]$$

$$+ \frac{1}{16\,(\Delta X)^2}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]^2$$

$$= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right]$$

$$+ \frac{1}{16\,(\Delta X)^2}\Big[\left(S_{i+1,n} - S_{i-1,n}\right)^2 + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)^2$$

$$+ 2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]\Big]$$

$$= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right]$$

$$+ \frac{1}{16\,(\Delta X)^2}\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ + 2\begin{bmatrix} \left(S_{i+1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n}\right)\left(-S_{i-1,n+1}\right) \\ + \left(-S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(-S_{i-1,n}\right)\left(-S_{i-1,n+1}\right) \end{bmatrix} \end{bmatrix}$$

$$= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right]$$

$$+ \frac{1}{16\,(\Delta X)^2}\left[\begin{array}{c} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{array}\right]$$

$$= \frac{1}{4\,(\Delta X)^2}\left[\begin{array}{c} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \\ + S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{array}\right]$$

$$+ \frac{1}{16\,(\Delta X)^2}\left[\begin{array}{c} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{array}\right]$$

$$\frac{S_{i,n+1} - S_{i,n}}{\Delta T}$$

$$= \frac{1}{4\,(\Delta X)^2}\left[\begin{array}{c} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \\ + S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{array}\right]$$

$$+ \frac{1}{16\,(\Delta X)^2}\left[\begin{array}{c} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{array}\right]$$

We assume that $C_0 = \frac{1}{16}\frac{\Delta T}{(\Delta X)^2}$

$$S_{i,n+1} - S_{i,n}$$

$$= 4C_0\left[\begin{array}{c} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \\ + S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{array}\right]$$

$$+ C_0\left[\begin{array}{c} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{array}\right]$$

$$= 4C_0 \left[\begin{array}{c} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \\ + S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{array}\right]$$

$$+C_0 \left[\begin{array}{c} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{array}\right]$$

$$= C_0 \Bigg[4\Big\{(-2)\left(S_{i,n+1}\right)^2 + \left(S_{i+1,n+1}\right)\left(S_{i,n+1}\right) + \left(S_{i-1,n+1}\right)\left(S_{i,n+1}\right)$$

$$+\left(S_{i,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i,n}\right)\left(S_{i-1,n+1}\right) + S_{i,n+1}\left(-2S_{i,n} + S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)$$

$$+ S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\Big\}$$

$$+ \left\{\begin{array}{c} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{array}\right\}\Bigg]$$

But the equation seems to be nonlinear which may have $\left(S_{i+1,n+1}\right)^2$ and $\left(S_{i+1,n}\right)\left(S_{i-1,n}\right)$ such terms also

For $i = 1, 2, 3, 4 \ldots N - 1$ terms,

We arrange one side $n + 1$ terms and other side n terms, we get

1) $+\left(S_{i+1,n+1}\right)^2 \rightarrow \{C_0\}$
2) $+\left(S_{i-1,n+1}\right)^2 \rightarrow \{C_0\}$
3) $+\left(S_{i,n+1}\right)^2 \rightarrow \{-8C_0\}$
4) $+\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \rightarrow \{-2C_0\}$
5) $+\left(S_{i+1,n+1}\right)\left(S_{i,n+1}\right) \rightarrow \{4C_0\}$
6) $+\left(S_{i-1,n+1}\right)\left(S_{i,n+1}\right) \rightarrow \{4C_0\}$
7) $+\left(S_{i+1,n+1}\right) \rightarrow \left\{4C_0S_{i,n} + 2C_0S_{i+1,n} - 2C_0S_{i-1,n}\right\}$
8) $+\left(S_{i,n+1}\right) \rightarrow \left\{-16C_0S_{i,n} + 4C_0S_{i+1,n} + 4C_0S_{i-1,n} - 1\right\}$
9) $+\left(S_{i-1,n+1}\right) \rightarrow \left\{4C_0S_{i,n} - 2C_0S_{i+1,n} + 2C_0S_{i-1,n}\right\}$

Other side n terms, overall negative sign,

1) $+\left(S_{i+1,n}\right)^2 \rightarrow \{C_0\}$
2) $+\left(S_{i-1,n}\right)^2 \rightarrow \{C_0\}$
3) $+\left(S_{i,n}\right)^2 \rightarrow \{-8C_0\}$
4) $+\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \rightarrow \{-2C_0\}$
5) $+\left(S_{i+1,n}\right)\left(S_{i,n}\right) \rightarrow \{4C_0\}$
6) $+\left(S_{i-1,n}\right)\left(S_{i,n}\right) \rightarrow \{4C_0\}$
7) $+\left(S_{i+1,n}\right) \rightarrow \{0\}$
8) $+\left(S_{i,n}\right) \rightarrow \{1\}$
9) $+\left(S_{i-1,n}\right) \rightarrow \{0\}$

This is the numerical expression of solution which gives the saturation of injected water for fingering phenomenon.

3.4 Numerical Solution and Graphical Interpretation:

The numerical representation of saturation of injected water has been solved by scilab coding. The numerical values of solution at different distance X and time T as shown in table (3.1). The graph of solution $S(X,T)$ versus distance X for a fixed time T= 0.001, 0.002, 0.003, 0.004, 0.005 in figure (3.3). We can see in Figure (3.4) represents the graph of solution $S(X,T)$ versus time T for a fixed distance.

Table 3.1: Numerical values of saturation of injected water $S(X,T)$ for different distance (X) and time (T)

X/T	T=0.001	T=0.002	T=0.003	T=0.004	T=0.005
0	0.0010	0.0020	0.0030	0.0040	0.0050
0.025	0.0260	0.0270	0.0280	0.0290	0.0300
0.05	0.0510	0.0520	0.0530	0.0540	0.0550
0.075	0.0760	0.0770	0.0780	0.0790	0.0800
0.1	0.1010	0.1020	0.1030	0.1040	0.1050
0.125	0.1260	0.1270	0.1280	0.1290	0.1300
0.15	0.1510	0.1520	0.1530	0.1540	0.1550
0.175	0.1760	0.1770	0.1780	0.1790	0.1800
0.2	0.2010	0.2020	0.2030	0.2040	0.2050
0.225	0.2260	0.2270	0.2280	0.2290	0.2300
0.25	0.2510	0.2520	0.2530	0.2540	0.2550
0.275	0.2760	0.2770	0.2780	0.2790	0.2800
0.3	0.3010	0.3020	0.3030	0.3040	0.3050
0.325	0.3260	0.3270	0.3280	0.3290	0.3300
0.35	0.3510	0.3520	0.3530	0.3540	0.3550
0.375	0.3760	0.3770	0.3780	0.3790	0.3800
0.4	0.4010	0.4020	0.4030	0.4040	0.4050
0.425	0.4260	0.4270	0.4280	0.4290	0.4300
0.45	0.4510	0.4520	0.4530	0.4540	0.4550
0.475	0.4760	0.4770	0.4780	0.4790	0.4800
0.5	0.5010	0.5020	0.5030	0.5040	0.5050
0.525	0.5260	0.5270	0.5280	0.5290	0.5300
0.55	0.5510	0.5520	0.5530	0.5540	0.5550
0.575	0.5760	0.5770	0.5780	0.5790	0.5800
0.6	0.6010	0.6020	0.6030	0.6040	0.6050
0.625	0.6260	0.6270	0.6280	0.6290	0.6300
0.65	0.6510	0.6520	0.6530	0.6540	0.6550
0.675	0.6760	0.6770	0.6780	0.6790	0.6800
0.7	0.7010	0.7020	0.7030	0.7040	0.7050
0.725	0.7260	0.7270	0.7280	0.7290	0.7300
0.75	0.7510	0.7520	0.7530	0.7540	0.7550
0.775	0.7760	0.7770	0.7780	0.7790	0.7800
0.8	0.8010	0.8020	0.8030	0.8040	0.8049
0.825	0.8260	0.8270	0.8280	0.8289	0.8298
0.85	0.8510	0.8520	0.8529	0.8538	0.8545
0.875	0.8760	0.8769	0.8778	0.8785	0.8791
0.9	0.9010	0.9018	0.9025	0.9030	0.9034
0.925	0.9259	0.9266	0.9270	0.9272	0.9273
0.95	0.9508	0.9510	0.9510	0.9509	0.9507
0.975	0.9752	0.9749	0.9745	0.9739	0.9733
1	0.9990	0.9980	0.9970	0.9960	0.9950

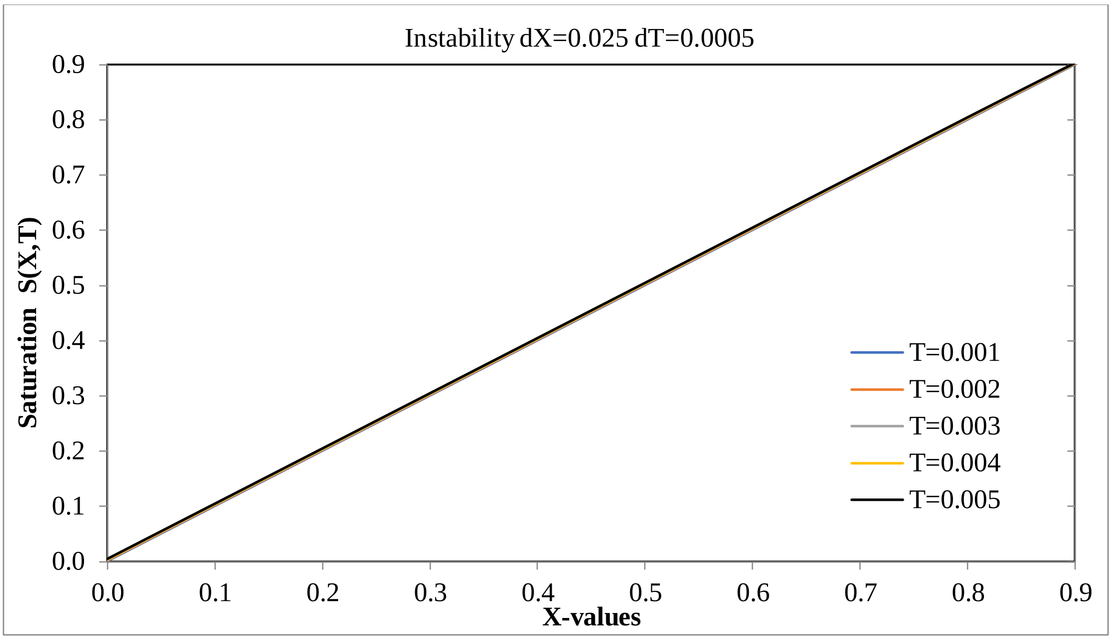


FIGURE 3.3: Saturation $S(X,T)$ → distance X at fixed time level.

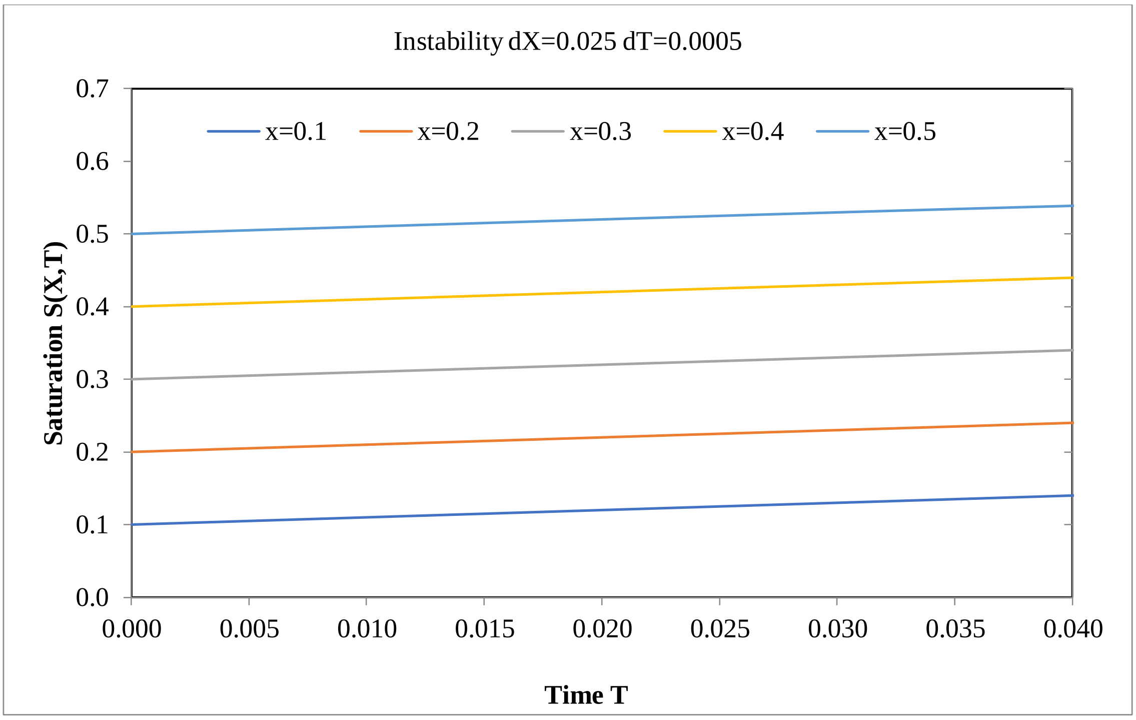


FIGURE 3.4: Saturation $S(X,T)$ → time T at fixed distance X.

3.5 Conclusion:

The numerical expression of solution of one dimensional nonlinear partial differential equation has been solved by finite difference Crank-Nicolson scheme. This equation is arising in fingering phenomenon in homogeneous porous medium.

The tabular values and graphical representation obtained by using scilab for the finite difference Crank-Nicolson scheme. The graph shows that the saturation of injected water is linearly increase with the different distance X for the time $T > 0$, and also it is linearly increase for time T for different length of fingers.

From the above discussion we have concluded the saturation of injected water is linearly increasing in instability phenomenon when water is injected at common interface which serves the purpose of secondary oil recovery process.

CHAPTER 4

Mathematical Modeling of Fingero-Imbibition Phenomenon and its Solution by Finite Difference Method

CHAPTER 4

Mathematical Modeling of Fingero-Imbibition Phenomenon and its Solution by Finite Difference Method

The fingero imbibition phenomenon divided in two parts,

(i) Fingero imbibition phenomenon in homogeneous porous media.

(ii) Fingero imbibition phenomenon in heterogeneous porous media.

In both the case, the approximate numerical solutions have been discussed using suitable initial and boundary conditions.

4.1 Part (A): Fingero-Imbibition Phenomenon in Homogeneous Porous Media:

4.1.1 Introduction:

This chapter represents an important phenomenon of fingero-imbibition in two phase flow through homogenous porous medium which is the combination of fingering phenomenon and imbibition phenomenon in porous medium.

When a porous medium filled with nonwetting phase (oil) is brought into the contact of another wetting phase (water) which is preferentially wetting, then the spontaneous flow of the wetting phase into the medium and counter flow of the resident phase from the medium without any external force. It is known as imbibition phenomenon. If a phase (oil) contained in a porous medium is moved by another phase lesser viscosity then, instead of regular displacement of the whole front, protuberances may occur which discharge through the porous medium at relatively very great speed giving rise to the fingering phenomenon. This immediate happening of both phenomena was called as fingero-imbibition phenomenon by Verma.

Many researchers have been studied this problem with different point of view and using different numerical methods. Mehta [76], Yadav and Mehta [78], Patel and Desai [82], Verma [9] etc. Fingero-imbibition phenomenon has received considerable current interest in hydrological systems as well as petroleum reservoir systems. The governing equation of fingero-imbibition is the nonlinear partial differential equation and its numerical solution is obtained by Crank-Nicolson scheme for finite difference method with suitable initial and boundary conditions.

4.1.2 Mathematical Formulation:

Using Darcy's law, the velocities of oil and water can be expressed as [24]

$$V_w = -\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} \tag{4.1}$$

$$V_o = -\frac{k_o}{\delta_o} K \frac{\partial P_o}{\partial x} \tag{4.2}$$

K = Permeability of the porous medium.

k_w= Relative permeability of water.

k_o= Relative permeability of oil.

δ_w= viscosity of injecting water

δ_o = Viscosity of native oil.

P_w= Pressure of water.

P_o= Pressure of oil.

The equation of continuity can be defined as,

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{4.3}$$

$$P \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \tag{4.4}$$

It can be obvious that fingero-imbibition phenomenon [1]

$$V_w + V_o = 0 \tag{4.5}$$

The capillary pressure may be written as,

$$P_c = P_o - P_w \tag{4.6}$$

$$\text{and } P_c = -\beta S_w \tag{4.7}$$

β is constant.

The relative permeability can be considered as a function of phase saturation .we can use following relationship [5]:

$$k_w = S_w \text{ and } k_o = S_o \tag{4.8}$$

The definition of phase saturation can be defined as,

$$S_w + S_o = 1 \tag{4.9}$$

Using equations (4.1) and (4.2) in (4.5),we get

$$\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} + \frac{k_o}{\delta_o} K \frac{\partial P_o}{\partial x} = 0 \tag{4.10}$$

Using the equation (4.6) we get,

$$\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} + \frac{k_o}{\delta_o} K \frac{\partial (P_c + P_w)}{\partial x} = 0$$

$$\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} + \frac{k_o}{\delta_o} K \frac{\partial P_c}{\partial x} + \frac{k_o}{\delta_o} K \frac{\partial P_w}{\partial x} = 0$$

$$\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right) K \frac{\partial P_w}{\partial x}+\frac{k_o}{\delta_o} K \frac{\partial P_c}{\partial x}=0 \tag{4.11}$$

Simplifying equation (4.11) we obtain

$$\frac{\partial P_w}{\partial x}= -\frac{\frac{k_o}{\delta_o}\frac{\partial P_c}{\partial x}}{\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)}$$

$$\frac{\partial P_w}{\partial x}= -\frac{k_o}{\delta_o}\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)^{-1}\frac{\partial P_c}{\partial x} \tag{4.12}$$

Using the value of $\frac{\partial P_w}{\partial x}$ in (4.1), we get

$$V_w = K\frac{k_w}{\delta_w}\frac{k_o}{\delta_o}\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)^{-1}\frac{\partial P_c}{\partial x} \tag{4.13}$$

Expending (4.13) in (4.3), we get

$$P\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left[K\frac{k_w}{\delta_w}\frac{k_o}{\delta_o}\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)^{-1}\frac{\partial P_c}{\partial x}\right]=0 \tag{4.14}$$

According to Scheidegger, we get

$$\frac{k_w}{\delta_w}\frac{k_o}{\delta_o}\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)^{-1} \approx \frac{k_o}{\delta_0}$$

Equation (4.14) reduce to,

$$P\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial x}\left[K\frac{k_o}{\delta_o}\frac{dP_c}{dS_i}\frac{\partial S_i}{\partial x}\right]=0 \tag{4.15}$$

Using the equations (4.7), (4.8) and (4.9) into (4.15), we get

$$P\frac{\partial S_w}{\partial t}=\frac{\beta K}{\delta_o}\frac{\partial}{\partial x}\left[(1- S_w)\frac{\partial S_w}{\partial x}\right] \tag{4.16}$$

Using dimensionless variables,

$$\mathrm{X} = \frac{\mathrm{x}}{\mathrm{L}}, \mathrm{T} = \frac{\beta \mathrm{Kt}}{\mathrm{P}\delta_{\mathrm{o}}\mathrm{L}^2}$$

Equation (4.16) reduce to,

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[(1 - S_w)\frac{\partial S_w}{\partial X}\right] \tag{4.17}$$

Suppose $\mathrm{S_w(X,T) = S(X,T)}$

Rewriting the equation (4.17), we got

$$\frac{\partial \mathrm{S}}{\partial \mathrm{T}} = \frac{\partial}{\partial \mathrm{X}}\left[(1 - \mathrm{S})\frac{\partial \mathrm{S}}{\partial \mathrm{X}}\right] \tag{4.18}$$

$$\frac{\partial \mathrm{S}}{\partial \mathrm{T}} = \frac{\partial^2 \mathrm{S}}{\partial \mathrm{X}^2} - \mathrm{S}\frac{\partial^2 \mathrm{S}}{\partial \mathrm{X}^2} - \left(\frac{\partial \mathrm{S}}{\partial \mathrm{X}}\right)^2 \tag{4.19}$$

Equation (4.18) represents the equation of fingero- imbibition phenomenon.

Let at the common interface, the saturation of water can be defined as

Let at the common interface saturation of injected water at X=0

$$\mathrm{S}(0,\ \mathrm{T}) = \mathrm{S}_1(\mathrm{T}), \tag{4.20}$$

Also, let the saturation of injected water at end of the matrix x=L,

$$\mathrm{S}(\ 1\ ,\ \mathrm{T}) = \mathrm{S}_2(\mathrm{T}), \tag{4.21}$$

Also, we assume that initial saturation is function of X,

$$\mathrm{S}(\mathrm{X},\ 0) = \mathrm{e}^{-\mathrm{X}}, \tag{4.22}$$

4.1.3 Solution by Finite Difference Method:

Using finite difference Crank- Nicolson scheme, [32,101,102];

$$\frac{\partial \mathrm{S}}{\partial \mathrm{T}} = \frac{\mathrm{S}_{\mathrm{i,n+1}} - \mathrm{S}_{\mathrm{i,n}}}{\Delta \mathrm{T}} \tag{4.23}$$

$$\frac{\partial^2 S}{\partial X^2} = \frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right] \tag{4.24}$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left(\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}\right)^2$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2 \tag{4.25}$$

Using equation (4.23), (4.24) and (4.25) in equation (4.18) we get,

$$\begin{aligned}
\frac{S_{i,n+1} - S_{i,n}}{\Delta T} &= \frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right] \\
&\quad - \left(S_{i,n+{}^{1}/_{2}}\right)\left[\frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right.\right. \\
&\quad \left.\left. + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right]\right] \\
&\quad - \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2\,(\Delta X)^2} &\left[\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right] \\
&- \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2\,(\Delta X)^2}\ \left[\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right)\right.\right. \\
&\left.\left. + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right]\right] \\
&- \frac{1}{16\,(\Delta X)^2}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]^2
\end{aligned}$$

$$
\begin{aligned}
= \frac{1}{2\,(\Delta X)^2}\ &\left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] \\
&- \left(\frac{\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i,n+1}}}{2}\right) \Big[\frac{1}{2\,(\Delta X)^2}\ \left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right)\right. \\
&\left.+ \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right]\Big] \\
&- \frac{1}{16\,(\Delta X)^2}\Big[\left(\mathrm{S}_{\mathrm{i+1,n}}\right)^2 + \left(\mathrm{S}_{\mathrm{i-1,n}}\right)^2 - 2\left(\mathrm{S}_{\mathrm{i+1,n}}\right)\left(\mathrm{S}_{\mathrm{i-1,n}}\right) \\
&+ \left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)^2 + \left(\mathrm{S}_{\mathrm{i-1,n+1}}\right)^2 - 2\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)\left(\mathrm{S}_{\mathrm{i-1,n+1}}\right) \\
&+ 2\left[\left(\mathrm{S}_{\mathrm{i+1,n}}\right)\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}}\right)\left(-\mathrm{S}_{\mathrm{i-1,n+1}}\right)\right. \\
&\left.+ \left(-\mathrm{S}_{\mathrm{i-1,n}}\right)\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right) + \left(-\mathrm{S}_{\mathrm{i-1,n}}\right)\left(-\mathrm{S}_{\mathrm{i-1,n+1}}\right)\right]\Big]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2\,(\Delta X)^2}\ &\left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] \\
&- \frac{1}{4(\Delta X)^2}\ \Big[\mathrm{S}_{\mathrm{i,n}}\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) \\
&+ \mathrm{S}_{\mathrm{i,n}}\left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right) + \mathrm{S}_{\mathrm{i,n+1}}\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) \\
&+ \mathrm{S}_{\mathrm{i,n+1}}\left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\Big] \\
&- \frac{1}{16\,(\Delta X)^2}\begin{bmatrix} \left(\mathrm{S}_{\mathrm{i+1,n}}\right)^2 + \left(\mathrm{S}_{\mathrm{i-1,n}}\right)^2 - 2\left(\mathrm{S}_{\mathrm{i+1,n}}\right)\left(\mathrm{S}_{\mathrm{i-1,n}}\right) \\ +\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)^2 + \left(\mathrm{S}_{\mathrm{i-1,n+1}}\right)^2 - 2\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)\left(\mathrm{S}_{\mathrm{i-1,n+1}}\right) \\ 2\begin{bmatrix} \left(\mathrm{S}_{\mathrm{i+1,n}}\right)\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}}\right)\left(-\mathrm{S}_{\mathrm{i-1,n+1}}\right) + \\ \left(-\mathrm{S}_{\mathrm{i-1,n}}\right)\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right) + \left(-\mathrm{S}_{\mathrm{i-1,n}}\right)\left(-\mathrm{S}_{\mathrm{i-1,n+1}}\right) \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

Suppose $\mathrm{C}_0 = \frac{1}{16}\,\frac{\Delta \mathrm{T}}{(\Delta X)^2}$

$$
\begin{aligned}
S_{i,n+1} - S_{i,n} &= 8C_0\left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] + \\
&4C_0\Big[S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) + \\
&S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\Big] - \\
&C_0\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) + \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\begin{bmatrix} \left(S_{i+1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n}\right)\left(S_{i-1,n+1}\right) + \\ \left(-S_{i-1,n}\right)\left(S_{i+1,n}\right) + \left(-S_{i-1,n}\right)\left(-S_{i-1,n+1}\right) \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}
S_{i,n+1} - S_{i,n} &\\
&= 8C_0 \left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right]\\
&+ 4C_0 \Big[(-2)\left(S_{i,n+1}\right)^2 + \left(S_{i+1,n+1}\right)\left(S_{i,n+1}\right) + \left(S_{i-1,n+1}\right)\left(S_{i,n+1}\right)\\
&+ \left(S_{i,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i,n}\right)\left(S_{i-1,n+1}\right)\\
&+ \left(-2\,S_{i,n} + S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) S_{i,n+1}\\
&+ \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) S_{i,n}\Big]\\
-C_0 &\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i-1,n}\right)\left(S_{i+1,n}\right) + \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i-1,n+1}\right)\left(S_{i+1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix}
\end{aligned} \tag{4.22}$$

Rewriting the Equation (4.22) with $(n+1)$ and n terms,

Only $(n+1)$ terms,

1) $+\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)^2 \rightarrow \{-\mathrm{C}_0\}$

2) $+\left(\mathrm{S}_{\mathrm{i-1,n+1}}\right)^2 \rightarrow \{-\,\mathrm{C}_0\}$

3) $+\left(\mathrm{S}_{\mathrm{i,n+1}}\right)^2 \rightarrow \{-\,8\mathrm{C}_0\}$

4) $+\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)\left(\mathrm{S}_{\mathrm{i-1,n+1}}\right) \rightarrow \{2\mathrm{C}_0\}$

5) $+\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right)\left(\mathrm{S}_{\mathrm{i,n+1}}\right) \rightarrow \{4\mathrm{C}_0\}$

6) $+\left(\mathrm{S}_{\mathrm{i,n+1}}\right)\left(\mathrm{S}_{\mathrm{i-1,n+1}}\right) \rightarrow \{4\mathrm{C}_0\}$

7) $+\left(\mathrm{S}_{\mathrm{i+1,n+1}}\right) \rightarrow \left\{8\mathrm{C}_0 + 4\mathrm{C}_0\,\mathrm{S}_{\mathrm{i,n}} - 2\mathrm{C}_0\,\mathrm{S}_{\mathrm{i+1,n}} + 2\mathrm{C}_0\,\mathrm{S}_{\mathrm{i-1,n}}\right\}$

8) $+\left(\mathrm{S}_{\mathrm{i,n+1}}\right) \rightarrow \left\{-8\mathrm{C}_0\mathrm{S}_{\mathrm{i,n}} + 4\mathrm{C}_0\,\mathrm{S}_{\mathrm{i+1,n}} - 8\mathrm{C}_0\,\mathrm{S}_{\mathrm{i,n}} + 4\mathrm{C}_0\,\mathrm{S}_{\mathrm{i-1,n}} - 16\,\mathrm{C}_0 - 1\right\}$

9) $+\left(\mathrm{S}_{\mathrm{i-1,n+1}}\right) \rightarrow \left\{8\mathrm{C}_0 + 2\mathrm{C}_0\,\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{C}_0\,\mathrm{S}_{\mathrm{i-1,n+1}} + 4\mathrm{C}_0\,\mathrm{S}_{\mathrm{i,n}}\right\}$

Only n terms (negative sign),

1) $+\left(S_{i+1,n}\right)^2 \rightarrow \{-C_0\}$
2) $+\left(S_{i-1,n}\right)^2 \rightarrow \{-\ C_0\}$
3) $+\left(S_{i,n}\right)^2 \rightarrow \{-\ 8C_0\}$
4) $+\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \rightarrow \{2C_0\}$
5) $+\left(S_{i+1,n}\right)\left(S_{i,n}\right) \rightarrow \{4C_0\}$
6) $+\left(S_{i,n}\right)\left(S_{i-1,n}\right) \rightarrow \{4C_0\}$
7) $+\left(S_{i+1,n}\right) \rightarrow \{8C_0\}$
8) $+\left(S_{i,n}\right) \rightarrow \{-16\ C_0 + 1\}$
9) $+\left(S_{i-1,n}\right) \rightarrow \{8C_0\}$

4.1.4 Results and Discussion:

The tabular values and graphical interpretation are discussed with the help of Crank-Nicolson scheme of finite difference method. Figure (4.1) shows the graph of saturation of injected water $S(X,T)$ versus X for fixed time 0, 0.1, 0.2, 0.3,…,,1.

X/T	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1	1	1	1	1	1	1	1	1	1	1
0.1	0.90484	0.9103	0.9127	0.91478	0.91637	0.91777	0.9189	0.9199	0.9208	0.9215	0.9221
0.2	0.81873	0.8257	0.8304	0.83399	0.83702	0.83953	0.8417	0.8435	0.8451	0.8464	0.8475
0.3	0.74082	0.7478	0.7537	0.7584	0.76235	0.76571	0.7686	0.771	0.7731	0.7748	0.7763
0.4	0.67032	0.6769	0.683	0.68824	0.69271	0.69653	0.6998	0.7026	0.7049	0.707	0.7087
0.5	0.60653	0.6126	0.6184	0.62363	0.62816	0.63205	0.6354	0.6382	0.6406	0.6427	0.6445
0.6	0.54881	0.5543	0.5596	0.56439	0.56855	0.57214	0.5752	0.5779	0.5801	0.582	0.5837
0.7	0.49659	0.5015	0.5061	0.51007	0.51349	0.51646	0.519	0.5212	0.5231	0.5247	0.526
0.8	0.44933	0.4537	0.4572	0.45997	0.46241	0.4645	0.4663	0.4678	0.4692	0.4703	0.4713
0.9	0.40657	0.4102	0.4118	0.4133	0.41452	0.4156	0.4165	0.4173	0.418	0.4186	0.419
1.0	0.369	0.369	0.369	0.369	0.369	0.369	0.369	0.369	0.369	0.369	0.369

TABLE 4.1: Numerical values of saturation of injected water $S(X,T)$ in homogeneous porous medium for fingero-imbibition phenomenon.

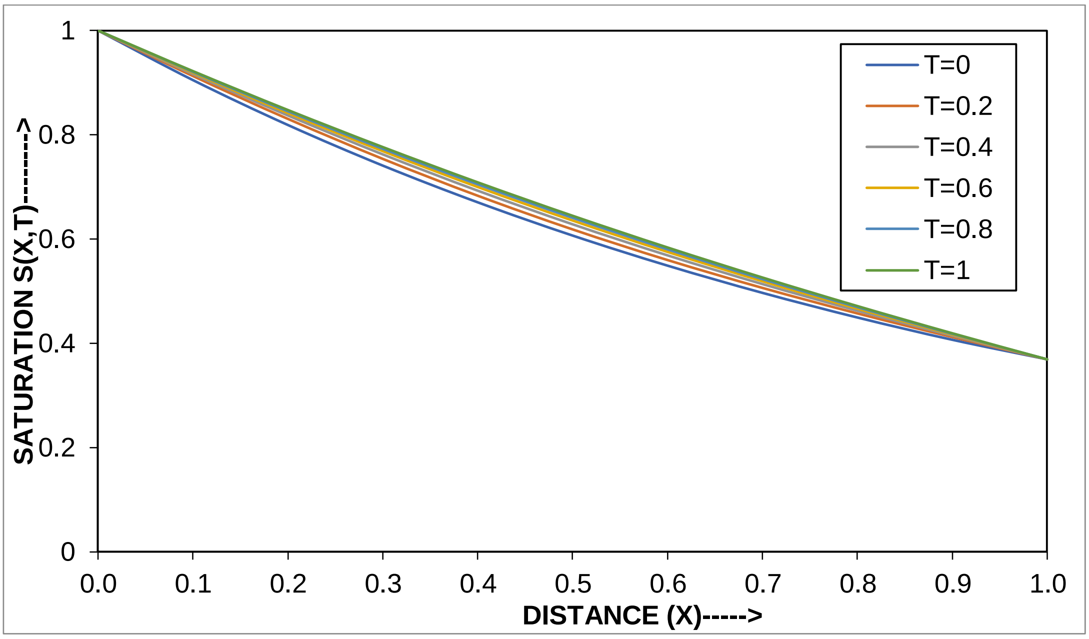


FIGURE 4.1: Graph of $S(X,T)$ versus X for a fixed time T

4.1.5 Conclusion:

Here, we have studied the Fingero-Imbibition equation of spontaneous imbibition of injected water and an oil saturated rock in a double phase flow system thorough porous media. The method finite difference Crank-Nicolson scheme was used to develop the numerical solution of the problem results show that water saturation is decreased with space and time which is shown in graph (4.1)

4.2 Part (B): Fingero-Imbibition Phenomenon in Heterogeneous Porous Media:

4.2.1 Introduction:

For the duration of the secondary oil recovery process Fingero- imbibition phenomenon rises. If a porous medium filled with nonwetting phase (oil) is brought contact with another wetting phase (water) then there is a spontaneous flow of wetting phase into the porous medium and a counter flow of nonwetting phase from the porous medium. This phenomenon is known as imbibition phenomenon. Further this if a porous medium is filled with some phase which is displaced by another less viscosity, instead of regular displacement of the whole front, protuberances may occur which shoot through porous medium at relatively great speed giving rise to the fingering. Thus the combination of two phenomena i.e. fingering and imbibition happen simultaneously which are known as fingero-imbibition phenomenon.

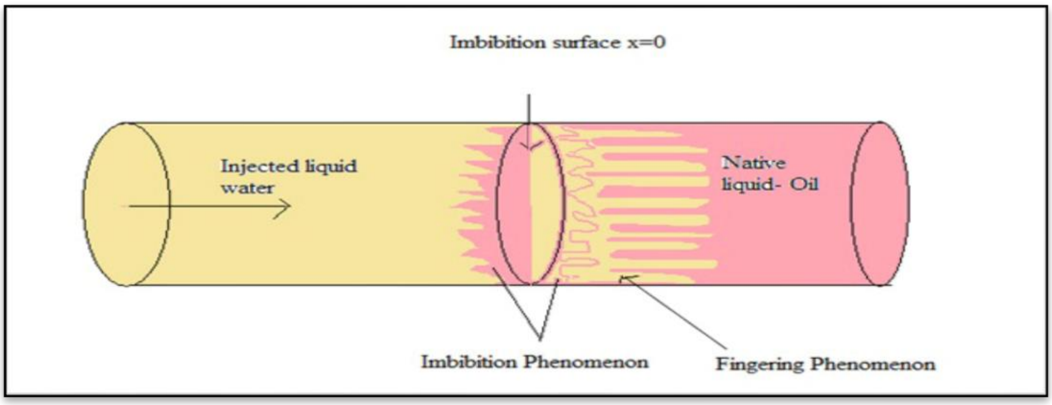


FIGURE 4.2: Representation of Fingero-imbibition Phenomenon.

This phenomenon is the important phenomenon which is frequently used in many engineering fields like petroleum engineering, soil science, agricultural engineering, and hydrogeology.

Our particular goal in present work is to measure the saturation of injected water for fingero-imbibition phenomenon in a heterogeneous porous medium. The problem gives us a one dimensional nonlinear partial differential equation using finite difference method with suitable conditions.

4.2.2 Modeling and its Mathematical Formulation:

Assume that there is a uniform water injection in oil formatted heterogeneous porous medium. The permeability and porosity are assumed as the functions of x only in heterogeneous porous medium.

Darcy's Law, as considered, is valid for studied problems. The findings of any problem depend on the shape and size of figure. When the figures are rectangle in shape only the average cross- sectional areas occupied by fingers are observed. The saturation of injected water for fingero-imbibition phenomenon can be found by the cross-sectional area occupied by injected water.

According to the Darcy's law, the velocity of injected water (V_w) and native oil (V_o) can be described as, [6, 50, 84]

$$\mathrm{V_w} = -\frac{\mathrm{k_w}}{\delta_\mathrm{w}} \mathrm{K} \frac{\partial \mathrm{P_w}}{\partial \mathrm{x}} \tag{4.1}$$

$$\mathrm{V_o} = -\frac{\mathrm{k_o}}{\delta_\mathrm{o}} \mathrm{K} \frac{\partial \mathrm{P_o}}{\partial \mathrm{x}} \tag{4.2}$$

K= Permeability of the porous medium.

$\mathrm{k_w}$ = Relative permeability of water.

$\mathrm{k_o}$= Relative permeability oil.

$\mathrm{P_w}$= Pressure of water.

$\mathrm{P_o}$ = Pressure of oil.

δ_w= Viscosity of injected water.

δ_o= Viscosity of native oil.

The equation of continuity defined as,

$$P\,\frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{4.3}$$

It is assumed that the permeability and porosity are the functions of x only for heterogeneous porous medium.

$$K = K_0\,(1 + ax) = K(x) \tag{4.4}$$

$$P = P(x) = \frac{1}{b_1 - b_2 x} \tag{4.5}$$

Where b_1, b_2, a, K_0 are positive constants.

For more generalization, we consider

$$K \propto P$$

$$K = K_a P \tag{4.6}$$

K_a is a constant.

Oil and water pressure difference can be defined as the capillary pressure P_c,

$$P_c = P_o - P_w \tag{4.7}$$

The capillary pressure considered as [41];

$$P_c = -\beta S_w \tag{4.8}$$

β is constant.

The relative permeability and phase saturation relationship can be defined as; [84, 86]

$$k_w = S_w \text{ and } k_0 = 1 - \alpha S_w \tag{4.9}$$

For counter current imbibition phenomenon condition can be expressed as, [1]

$$V_o + V_w = 0 \tag{4.10}$$

Using the equation (4.1) and (4.2) in (4.10), we get

$$\frac{k_o}{\delta_o} K \frac{\partial P_o}{\partial x} + \frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} = 0 \tag{4.11}$$

From the equation (4.7) and (4.11), we got

$$\frac{k_o}{\delta_o} K \frac{\partial}{\partial x} (P_c + P_w) + \frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} = 0$$

$$\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right) \frac{\partial P_w}{\partial x} + \frac{k_o}{\delta_o} \frac{\partial P_c}{\partial x} = 0 \tag{4.12}$$

Equation (4.12) reduce to $\frac{\partial P_w}{\partial x}$

$$\frac{\partial P_w}{\partial x} = -\frac{\frac{k_o}{\delta_o} \frac{\partial P_c}{\partial x}}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} \tag{4.13}$$

Putting the value of $\frac{\partial P_w}{\partial x}$ in equation (4.1), we get

$$V_w = K \frac{\frac{k_w}{\delta_w} \frac{k_o}{\delta_o} \frac{\partial P_c}{\partial x}}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} \tag{4.14}$$

Substituting the value (4.14) into (4.3), we get

$$P \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[K \frac{k_w}{\delta_w} \frac{k_o}{\delta_o} \left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)^{-1} \frac{\partial P_c}{\partial x}\right] = 0 \tag{4.15}$$

As per Scheidegger view [1], it is analyzed that

$$\frac{k_w}{\delta_w} \frac{k_o}{\delta_o} \left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)^{-1} \approx \frac{k_o}{\delta_o} \tag{4.16}$$

Equation (4.16) reduce to,

$$P \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[K \frac{k_o}{\delta_o} \frac{\partial P_c}{\partial S_w} \frac{\partial S_w}{\partial x}\right] = 0 \tag{4.17}$$

Using (4.6), (4.8), (4.9) into (4.17), we get

$$\frac{\partial S_w}{\partial t} = \frac{\beta K_a}{\delta_o P}\frac{\partial}{\partial x}\left[P\,(1-\alpha S_w)\,\frac{\partial S_w}{\partial x}\right]$$

$$= \frac{\beta K_a}{\delta_o}\frac{\partial}{\partial x}\left[(1-\alpha S_w)\,\frac{\partial S_w}{\partial x}\right] + \frac{\beta K_a}{\delta_o P}\,(1-\alpha S_w)\,\frac{\partial S_w}{\partial x}\,\frac{\partial P}{\partial x} \tag{4.18}$$

Since $\frac{1}{P}\frac{\partial P}{\partial x} = \frac{\partial\,(\log P)}{\partial x} = \frac{\partial}{\partial x}\left(-\log a_1 + \frac{b_2 x}{b_1}\right) = \frac{b_2}{b_1}$

From equation (4.18), we get

$$\frac{\partial S_w}{\partial t} = \frac{\beta K_a}{\delta_o}\frac{\partial}{\partial x}\left[(1-\alpha S_w)\frac{\partial S_w}{\partial x}\right] + \frac{b_2 \beta K_a}{b_1 \delta_0}\,(1-\alpha S_w)\,\frac{\partial S_w}{\partial x} \tag{4.19}$$

Choose dimensionless variables,

$X = \frac{x}{L}, T = \frac{\beta K_a t}{L^2 \delta_0}$

Equation (4.19) reduce to,

$$(1-\alpha S_w)\,\frac{\partial^2 S_w}{\partial X^2} - \alpha\left(\frac{\partial S_w}{\partial X}\right)^2 + A\left(\frac{\partial S_w}{\partial X}\right) - A\alpha S_w\,\frac{\partial S_w}{\partial X} - \frac{\partial S_w}{\partial T} = 0 \tag{4.20}$$

Where $A = \frac{L b_2}{b_1}$ and$S_w\,(X,T) = S(X,T)$.

Rewriting the equation (4.20), we got

$$(1-\alpha S)\,\frac{\partial^2 S}{\partial X^2} - \alpha\left(\frac{\partial S}{\partial X}\right)^2 + A\left(\frac{\partial S}{\partial X}\right) - A\alpha S\,\frac{\partial S}{\partial X} - \frac{\partial S}{\partial T} = 0 \tag{4.21}$$

The equation (4.21) defines the governing equation of fingero-imbibition phenomenon in heterogeneous porous medium. The equation (4.21) is the solution of saturation of injected water at distance X and time T. The following initial and boundary conditions are considered to solve (6.20) by finite difference crank-icolson scheme.

We consider initial and boundary conditions;

$$S\,(X,0) = 0\,, 0 \le X \le 1 \tag{4.22}$$

$$S\,(0,T) = 1, T > 0 \tag{4.23}$$

$$S\,(1,T) = 0, T > 0 \tag{4.24}$$

4.2.3 Solution by Finite Difference Method:

We apply finite difference Crank- Nicolson scheme in equation (4.21), we get

$$\frac{\partial S}{\partial T} = \frac{S_{i,n+1} - S_{i,n}}{\Delta T} \tag{4.25}$$

$$\frac{\partial^2 S}{\partial X^2} = \frac{1}{2}\left[\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right] \tag{4.26}$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}\right]^2$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2 \tag{4.27}$$

Put all these values in equation (4.21), we get

$$\begin{aligned}\frac{S_{i,n+1} - S_{i,n}}{\Delta T}\\ &= \frac{1}{2}\left[\left(\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right) + \left(\frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right)\right]\\ &- \alpha\left(S_{i,n+\frac{1}{2}}\right)\frac{1}{2}\left[\left(\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right)\right.\\ &\left.+\left(\frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right)\right] - \alpha\left[\frac{S_{i+1,n+\frac{1}{2}} - S_{i-1,n+\frac{1}{2}}}{2(\Delta X)}\right]^2\\ &+ A\left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]\\ &- A\alpha\left(S_{i,n+\frac{1}{2}}\right)\left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]\end{aligned}$$

$$
\begin{aligned}
= \frac{1}{2\,(\Delta X)^2}\,&\big[\big(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\big) + \big(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\big)\big] \\
&- \alpha\left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\Bigg[\frac{1}{2\,(\Delta X)^2}\,\big[\big(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\big) \\
&+ \big(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\big)\big]\Bigg] \\
&- \alpha\left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2 \\
&+ \frac{A}{4\,(\Delta X)}\,\big[\big(S_{i+1,n} + S_{i+1,n+1}\big) - \big(S_{i-1,n} + S_{i-1,n+1}\big)\big] \\
&- A\alpha\left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\Bigg[\frac{1}{4\,(\Delta X)}\,\big(S_{i+1,n} + S_{i+1,n+1}\big) \\
&- \big(S_{i-1,n} + S_{i-1,n+1}\big)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2\,(\Delta X)^2}\,&\big[\big(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\big) + \big(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\big)\big] \\
&- \frac{\alpha}{4(\Delta X)^2}\,\big(S_{i,n} + S_{i,n+1}\big)\big[\big(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\big) \\
&+ \big(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\big)\big] \\
&- \frac{\alpha}{16(\Delta X)^2}\,\big[\big(S_{i+1,n} - S_{i-1,n}\big) + \big(S_{i+1,n+1} - S_{i-1,n+1}\big)\big]^2 \\
&+ \frac{A}{4(\Delta X)}\,\big[\big(S_{i+1,n} - S_{i-1,n}\big) + \big(S_{i+1,n+1} - S_{i-1,n+1}\big)\big] \\
&- \frac{A\alpha}{8(\Delta X)}\,\big(S_{i,n} + S_{i,n+1}\big)\big[\big(S_{i+1,n} - S_{i-1,n}\big) + \big(S_{i+1,n+1} - S_{i-1,n+1}\big)\big]
\end{aligned}
$$

$$\begin{aligned}
= \frac{1}{2\,(\Delta \mathrm{X})^2}\,&\left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] \\
&- \frac{\alpha}{4(\Delta \mathrm{X})^2}\left(\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i,n+1}}\right)\left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right)\right. \\
&\left. + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] \\
&- \frac{\alpha}{16(\Delta \mathrm{X})^2}\left[\left(\mathrm{S}_{\mathrm{i+1,n}} - \mathrm{S}_{\mathrm{i-1,n}}\right)^2 + \left(\mathrm{S}_{\mathrm{i+1,n+1}} - \mathrm{S}_{\mathrm{i-1,n+1}}\right)^2\right. \\
&\left. + 2\left(\mathrm{S}_{\mathrm{i+1,n}} - \mathrm{S}_{\mathrm{i-1,n}}\right)\left(\mathrm{S}_{\mathrm{i+1,n+1}} - \mathrm{S}_{\mathrm{i-1,n+1}}\right)\right] \\
&+ \frac{\mathrm{A}}{4(\Delta \mathrm{X})}\left[\left(\mathrm{S}_{\mathrm{i+1,n}} - \mathrm{S}_{\mathrm{i-1,n}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n+1}} - \mathrm{S}_{\mathrm{i-1,n+1}}\right)\right] \\
&- \frac{\mathrm{A}\alpha}{8(\Delta \mathrm{X})}\left(\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i,n+1}}\right)\left[\left(\mathrm{S}_{\mathrm{i+1,n}} - \mathrm{S}_{\mathrm{i-1,n}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n+1}} - \mathrm{S}_{\mathrm{i-1,n+1}}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\,(\Delta \mathrm{X})^2}\left[\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + \left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] \\
&- \frac{\alpha}{4(\Delta \mathrm{X})^2}\left[S_{i,n}\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + S_{i,n}\left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right. \\
&\left. + S_{i,n+1}\left(\mathrm{S}_{\mathrm{i+1,n+1}} - 2\mathrm{S}_{\mathrm{i,n+1}} + \mathrm{S}_{\mathrm{i-1,n+1}}\right) + S_{i,n+1}\left(\mathrm{S}_{\mathrm{i+1,n}} - 2\mathrm{S}_{\mathrm{i,n}} + \mathrm{S}_{\mathrm{i-1,n}}\right)\right] \\
&- \frac{\alpha}{16(\Delta \mathrm{X})^2}\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix} \\
&+ \frac{A}{4(\Delta X)}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right] \\
&- \frac{A\alpha}{8(\Delta X)}\begin{bmatrix} S_{i,n}\left(S_{i+1,n} - S_{i-1,n}\right) + S_{i,n}\left(S_{i+1,n+1} - S_{i-1,n+1}\right) \\ S_{i,n+1}\left(S_{i+1,n} - S_{i-1,n}\right) + S_{i,n+1}\left(S_{i+1,n+1} - S_{i-1,n+1}\right) \end{bmatrix}
\end{aligned}$$

Assume that $C_0 = \frac{1}{16}\,\frac{\Delta T}{(\Delta X)^2}$

$$
\begin{aligned}
&S_{i,n+1} - S_{i,n} \\
&= 8C_0\left[\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right] \\
&- 4\alpha C_0\begin{bmatrix} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) + \\ S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{bmatrix} \\
&- \alpha C_0\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix} \\
&+ 4AC_0\Delta X\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right] \\
&- 2AC_0\alpha\Delta X\begin{bmatrix} S_{i,n}\left(S_{i+1,n} - S_{i-1,n}\right) + S_{i,n}\left(S_{i+1,n+1} - S_{i-1,n+1}\right) \\ S_{i,n+1}\left(S_{i+1,n} - S_{i-1,n}\right) + S_{i,n+1}\left(S_{i+1,n+1} - S_{i-1,n+1}\right) \end{bmatrix}
\end{aligned}
$$

The equation seems to be nonlinear which may have $\left(S_{i+1,n}\right)^2$ and $\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right)$ such terms also

Now, we arrange n+1 and n terms,

Only $(n+1)$ terms,

1) $+\left(S_{i+1,n+1}\right)^2 \rightarrow \{-\alpha C_0\}$

2) $+\left(S_{i-1,n+1}\right)^2 \rightarrow \{-\alpha C_0\}$

3) $+\left(S_{i,n+1}\right)^2 \rightarrow \{8\alpha C_0\}$

4) $+\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \rightarrow \{2\alpha C_0\}$

5) $+\left(S_{i+1,n+1}\right)\left(S_{i,n+1}\right) \rightarrow \{-4\alpha C_0 - 2A\alpha C_0\Delta X\}$

6) $+\left(S_{i-1,n+1}\right)\left(S_{i,n+1}\right) \rightarrow \{-4\alpha C_0 + 2A\alpha C_0\Delta X\}$

7) $+\left(S_{i+1,n+1}\right) \rightarrow \left\{8C_0 - 4\alpha C_0 S_{i,n} - 2\alpha C_0 S_{i+1,n} + 2\alpha C_0 S_{i-1,n} + 4AC_0\Delta X - 2A\alpha C_0\Delta X S_{i,n}\right\}$

8) $+\left(S_{i,n+1}\right) \rightarrow \left\{-16C_0 + 8\alpha C_0 S_{i,n} - 4\alpha C_0 S_{i+1,n} \mp 8\alpha C_0 S_{i,n} - 4\alpha C_0 S_{i-1,n} + 4A\alpha C_0\Delta X + 2A\alpha C_0\Delta X S_{i-1,n} - 1 - 2A\alpha C_0\Delta X S_{i+1,n}\right\}$

9) $+\left(S_{i-1,n+1}\right) \rightarrow \left\{8C_0 - 4\alpha C_0 S_{i,n} + 2\alpha C_0 S_{i+1,n} - 2\alpha C_0 S_{i-1,n} - 4AC_0\Delta X + 2A\alpha C_0\Delta X S_{i,n}\right\}$

Other side n terms, overall negative sign,

1) $+\left(S_{i+1,n}\right)^2 \rightarrow \{-\alpha C_0\}$
2) $+\left(S_{i-1,n}\right)^2 \rightarrow \{-\alpha C_0\}$
3) $+\left(S_{i,n}\right)^2 \rightarrow \{8\alpha C_0\}$
4) $+\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \rightarrow \{2\alpha C_0\}$
5) $+\left(S_{i+1,n}\right)\left(S_{i,n}\right) \rightarrow \{-4\alpha C_0 - 2A\alpha C_0\Delta X\}$
6) $+\left(S_{i-1,n}\right)\left(S_{i,n}\right) \rightarrow \{-4\alpha C_0 + 2A\alpha C_0\Delta X\}$
7) $+\left(S_{i+1,n}\right) \rightarrow \{8C_0 + 4AC_0\Delta X\}$
8) $+\left(S_{i,n}\right) \rightarrow \{1 - 16C_0\}$
9) $+\left(S_{i-1,n}\right) \rightarrow \{8C_0 - 4AC_0\Delta X\}$

4.2.4 Numerical Solution and Graphical Interpretation:

The numerical values of saturation of injected water for different distance X and for time T in table (4.2). Figure (4.3) represent the graph of $S(X,T)$ versus distance X for time T=0, 0.01, 0.02… 0.1.

X/T	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.0000	0.3201	0.3681	0.4133	0.4429	0.4658	0.4843	0.4997	0.5129	0.5245	0.5347
0.2	0.0000	0.0788	0.1693	0.2140	0.2499	0.2777	0.3008	0.3203	0.3372	0.3522	0.3656
0.3	0.0000	0.0204	0.0653	0.1065	0.1377	0.1642	0.1866	0.2062	0.2236	0.2391	0.2532
0.4	0.0000	0.0054	0.0230	0.0480	0.0719	0.0932	0.1125	0.1299	0.1457	0.1602	0.1735
0.5	0.0000	0.0014	0.0077	0.0198	0.0349	0.0502	0.0650	0.0791	0.0923	0.1047	0.1163
0.6	0.0000	0.0004	0.0024	0.0076	0.0157	0.0254	0.0358	0.0462	0.0564	0.0662	0.0756
0.7	0.0000	0.0001	0.0008	0.0028	0.0066	0.0121	0.0185	0.0255	0.0327	0.0399	0.0469
0.8	0.0000	0.0000	0.0002	0.0010	0.0026	0.0053	0.0089	0.0130	0.0175	0.0221	0.0266
0.9	0.0000	0.0000	0.0001	0.0003	0.0009	0.0019	0.0035	0.0054	0.0075	0.0097	0.0119
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 4.2: Numerical values of saturation of injected water $S(X,T)$ in heterogeneous porous medium for fingero-imbibition phenomenon

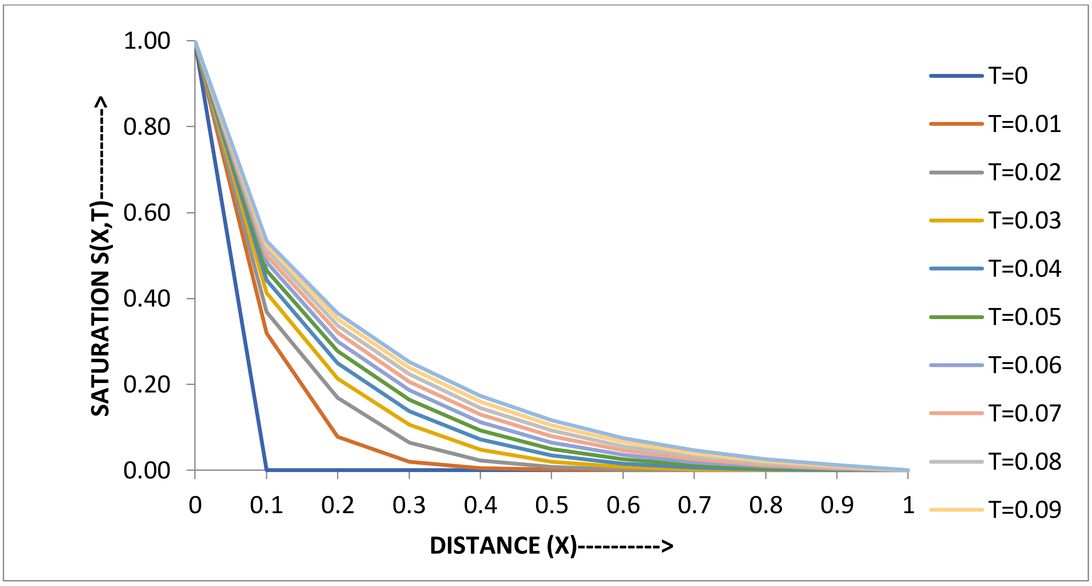


FIGURE 4.3: Graph of $S(X,T)$ versus X for a fixed time.

4.2.5 Conclusion:

We have discussed the phenomenon of fingero-imbibition in a heterogeneous porous medium. The Crank-Nicolson scheme of finite difference is obtained which represent the saturation of injected water. The numerical and graphical representations clearly indicate the saturation of water decrease when distance and time increase.

CHAPTER 5

Numerical Analysis of CoCurrent Imbibition Phenomenon with Effect of Inclination

Chapter 5

Numerical Analysis of CoCurrent Imbibition Phenomenon with Effect of Inclination

5.1 Introduction:

Spontaneous Imbibition is the one of the most important process in which the wetting phase (water) is drained into the porous medium by capillary pressure. Spontaneous imbibition may divide in two parts: (i) cocurrent imbibition phenomenon (ii) counter-current imbibition phenomenon.

In cocurrent imbibition phenomenon both the wetting (water) and non-wetting phase (oil) pass in the same direction and water forces oil out of matrix. To the opposite side in counter current imbibition phenomenon wetting (water) and non-wetting (oil) move in opposite direction and oil escape by flowing back in the same direction from which water has imbibed. Cocurrent imbibition phenomenon is more effective compare to counter current imbibition phenomenon.

Many researchers have studied this phenomenon with different point of view. The flow experiment which involves cocurrent and counter-current phenomenon has been performed by Bourblaux and Kalaydjian [21]. The physical and mathematical differences between cocurrent and countercurrent imbibition have been studied by Pooladi-Darvish and Firoozabadi [80]. The numerical solution of a specific imbibition phenomenon using Crank Nicolson scheme for finite differences have been obtained by Pradhan and Verma.

To inject water into fractured oil formatted porous medium is commonly used in the oil recovery process, and then cocurrent imbibition phenomenon occurs. The water and oil velocities are considered under inclination effects. For the flow problem, oil and water densities are constants. The porosity and permeability are also constants.

Here we have discussed cocurrent imbibition phenomenon in inclined homogeneous porous medium with the help of finite difference Crank Nicolson scheme. We have also discussed the numerical solution and graphical representation with appropriate initial and boundary conditions.

5.2 Formulation of the Problem:

By Darcy's law, the velocity of injected liquid (water V_w) and native oil (V_o) can be expressed as, [1, 24, 75]

$$V_w = -\frac{k_w}{\delta_w} K \left(\frac{\partial P_w}{\partial x} + \rho_w g \sin\theta\right) \tag{5.1}$$

$$V_o = -\frac{k_o}{\delta_o} K \left(\frac{\partial P_o}{\partial x} + \rho_o g \sin\theta\right) \tag{5.2}$$

k_w= Relative permeability of water.

k_o= Relative permeability of oil.

K= Permeability of the porous medium.

δ_w= Viscosity of injecting water.

δ_o= Viscosity of native oil.

P_w= Pressure of water.

P_o= Pressure of oil.

θ = The angle of inclination with the porous matrix.

g = Acceleration due to the gravity.

The equation of continuity,

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{5.3}$$

The relation between velocity of water and velocity of oil in cocurrent imbibition phenomenon is expressed as, [73, 87]

$$\mathrm{V_w} + \mathrm{V_o} = \mathrm{V_t} \tag{5.4}$$

The total velocity is $\mathrm{V_t}$.

The capillary pressure (P_c) is expressed as [1];

$$\mathrm{P_c} = \mathrm{P_o} - \mathrm{P_w} \tag{5.5}$$

The relation between capillary pressure and saturation of water is descried as; [76]

$$\mathrm{P_c} = -\beta \mathrm{S_w} \tag{5.6}$$

Where β is constant.

The relative permeability and phase saturation relationship can be defined as; [1,5]

$$\mathrm{k_w} = \mathrm{S_w} \ \text{ and } \mathrm{k_o} = 1 - \alpha \mathrm{S_w} \tag{5.7}$$

Where α is a constant.

Using (5.1), (5.2) and (5.4), we get

$$-\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\, \mathrm{K} \left(\frac{\partial \mathrm{P_w}}{\partial \mathrm{x}} + \rho_\mathrm{w} \mathrm{g} \sin\theta\right) - \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\, \mathrm{K} \left(\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}} + \rho_\mathrm{o} \mathrm{g} \sin\theta\right) = \mathrm{V_t} \tag{5.8}$$

Putting the equation (5.5) in (5.8), we get

$$\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\, \mathrm{K} \left(\frac{\partial \mathrm{P_w}}{\partial \mathrm{x}} + \rho_\mathrm{w} \mathrm{g} \sin\theta\right) + \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\, \mathrm{K} \left(\frac{\partial (P_c + P_w)}{\partial \mathrm{x}} + \rho_\mathrm{o} \mathrm{g} \sin\theta\right) = -\mathrm{V_t}$$

$$\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\, \mathrm{K} \left(\frac{\partial \mathrm{P_w}}{\partial \mathrm{x}} + \rho_\mathrm{w} \mathrm{g} \sin\theta\right) + \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\, \mathrm{K} \left(\frac{\partial P_c}{\partial \mathrm{x}} + \frac{\partial P_w}{\partial x} + \rho_\mathrm{o} \mathrm{g} \sin\theta\right) = -\mathrm{V_t} \tag{5.9}$$

Solving the equation (5.9) for $\frac{\partial \mathrm{P_w}}{\partial \mathrm{x}}$

$$\frac{\partial \mathrm{P_w}}{\partial \mathrm{x}} = -\frac{\left(\mathrm{K}\left(\frac{\mathrm{k_o}}{\delta_\mathrm{o}}\, \rho_\mathrm{o} + \frac{\mathrm{k_w}}{\delta_\mathrm{w}}\, \rho_\mathrm{w}\right) \mathrm{g} \sin\theta + \mathrm{K}\, \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\, \frac{\partial \mathrm{P_c}}{\partial \mathrm{x}} + \mathrm{V_t}\right)}{\left(\mathrm{K}\, \frac{\mathrm{k_w}}{\delta_\mathrm{w}} + \mathrm{K}\, \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\right)} \tag{5.10}$$

Equation (5.10) in (5.1), we have

$$V_w = -\frac{\frac{k_w}{\delta_w}\left(K\,\frac{k_o}{\delta_o}\,(\rho_w - \rho_o)\;g\sin\theta - K\,\frac{k_o}{\delta_o}\,\frac{\partial P_c}{\partial x} - V_t\right)}{\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)} \tag{5.11}$$

The pressure of water can be expressed as the relation between capillary pressure and mean pressure [52];

$$P_w = \frac{P_w + P_o}{2} + \frac{P_w - P_o}{2}$$

$$= \overline{P} - \frac{1}{2}\,P_c \tag{5.12}$$

Where $\overline{P}$ is a constant.

Equation (5.9) reduce to

$$K\left(\frac{k_w}{\delta_w}\,\rho_w + \frac{k_o}{\delta_o}\,\rho_o\right)\,g\sin\theta + \frac{K}{2}\left(\frac{k_o}{\delta_o} - \frac{k_w}{\delta_w}\right)\frac{\partial P_c}{\partial x} = -V(t) \tag{5.13}$$

Equation (5.11) reduce to

$$V_w = \frac{K}{2}\,\frac{k_w}{\delta_w}\,\frac{\partial P_c}{\partial x} - K\,\frac{k_w}{\delta_w}\,\rho_w g\sin\theta \tag{5.14}$$

Substituting the equation (5.14) into (5.3), we get

$$P\,\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{K}{2}\,\frac{k_w}{\delta_w}\,\frac{\partial P_c}{\partial x} - K\,\frac{k_w}{\delta_w}\,\rho_w g\sin\theta\right] = 0 \tag{5.15}$$

Using the relation between relative permeability and phase saturation, we get

$$P\,\frac{\partial S_w}{\partial t} - \frac{K\beta}{2\delta_w}\,\frac{\partial}{\partial x}\left[S_w\,\frac{\partial S_w}{\partial x}\right] - \frac{K\rho_w g\sin\theta}{\delta_w}\,\frac{\partial S_w}{\partial x} = 0 \tag{5.16}$$

Using dimensionless variables,

$$X = \frac{x}{L},\; T = \frac{Kt\beta}{2PL^2\delta_w}$$

Equation (5.16) reduce to

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[S_w \frac{\partial S_w}{\partial X}\right] + A\frac{\partial S_w}{\partial X} \tag{5.17}$$

$$\frac{\partial S_w}{\partial T} = S_w \frac{\partial^2 S_w}{\partial X^2} + \left(\frac{\partial S_w}{\partial X}\right)^2 + A\,\frac{\partial S_w}{\partial X}$$

Where $A = \frac{2Lg\sin\theta\,\rho_w}{\beta}$ and $S_w\,(x,t) = S_w\,(X,T)$

Suppose that$S_w\,(X,T) = S(X,T)$.

Rewriting the equation (5.17) we get,

$$\frac{\partial S}{\partial T} = S\left(\frac{\partial^2 S}{\partial X^2}\right) + \left(\frac{\partial S}{\partial X}\right)^2 + A\,\frac{\partial S}{\partial X} \tag{5.18}$$

The equation (5.18) is the governing equation for the cocurrent imbibition phenomenon and $S(X,T)$ gives the saturation of injected water.

Now, we solve equation (5.18) using appropriate initial and boundary with finite difference Crank-Nicolson scheme.

Using initial and boundary conditions,

$$S(X,0) = 1 - e^{-X} \qquad 0 \le X \le 1 \tag{5.19}$$

$$S(0,T) = S_0\,, \qquad X = 0\ and\ T > 0 \tag{5.20}$$

$$S(1,T) = f(T)\,, \qquad X = 1\ and\ T > 0 \tag{5.21}$$

5.3 Solution by Finite Difference Crank-Nicolson Scheme:

We apply finite difference Crank-Nicolson scheme in equation (5.17),

$$\frac{\partial S}{\partial T} = \frac{S_{i,n+1} - S_{i,n}}{\Delta T} \tag{5.22}$$

$$\frac{\partial^2 S}{\partial X^2} = \frac{1}{2}\left[\left(\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right) + \left(\frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right)\right] \tag{5.23}$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}\right]^2$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{\left(\frac{S_{i+1,n}+S_{i+1,n+1}}{2}\right)-\left(\frac{S_{i-1,n}+S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2 \qquad (5.24)$$

$$\frac{\partial S}{\partial X} = \frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}$$

$$= \left[\frac{\left(\frac{S_{i+1,n}+S_{i+1,n+1}}{2}\right)-\left(\frac{S_{i-1,n}+S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right] \qquad (5.25)$$

Using the equations (5.22), (5.23), (5.24) and (5.25) in equation (5.18), we get

$$\frac{S_{i,n+1} - S_{i,n}}{\Delta T}$$

$$= \left(S_{i,n+1/2}\right)\left[\frac{1}{2}\left(\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right.\right.$$

$$\left.\left. + \left(\frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right)\right)\right] + \left[\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}\right]^2$$

$$+ A\left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]$$

$$= \left(S_{i,n+1/2}\right)\left[\frac{1}{2\,(\Delta X)^2}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right]$$

$$+ \left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2$$

$$+ A\left[\frac{1}{4\,(\Delta X)}\left(S_{i+1,n} + S_{i+1,n+1}\right) - \left(S_{i-1,n} + S_{i-1,n+1}\right)\right]$$

$$
\begin{aligned}
= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right) & \left[\frac{1}{2\,(\Delta X)^2}\,\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right] \\
& + \frac{1}{16\,(\Delta X)^2}\,\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]^2 \\
& + \frac{A}{4(\Delta X)}\,\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right) & \left[\frac{1}{2\,(\Delta X)^2}\,\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right] \\
& + \frac{1}{16\,(\Delta X)^2}\,\Big[\left(S_{i+1,n} - S_{i-1,n}\right)^2 + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)^2 \\
& + 2\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1} - S_{i-1,n+1}\right)\Big] \\
& + \frac{A}{4(\Delta X)}\,\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \left(\frac{S_{i,n} + S_{i,n+1}}{2}\right) & \left[\frac{1}{2\,(\Delta X)^2}\,\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right] \\
+ \frac{1}{16\,(\Delta X)^2} & \begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix} \\
+ \frac{A}{4(\Delta X)} & \left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]
\end{aligned}
$$

$$= \frac{1}{4(\Delta X)^2}\left(S_{i,n}+S_{i,n+1}\right)\left[\left(S_{i+1,n+1}-2S_{i,n+1}+S_{i-1,n+1}\right)+\left(S_{i+1,n}-2S_{i,n}+S_{i-1,n}\right)\right]$$

$$+\frac{1}{16\,(\Delta X)^2}\begin{bmatrix} \left(S_{i+1,n}\right)^2+\left(S_{i-1,n}\right)^2-2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2+\left(S_{i-1,n+1}\right)^2-2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n}-S_{i-1,n}\right)\left(S_{i+1,n+1}\right)+\left(S_{i+1,n}-S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix}$$

$$+\frac{A}{4(\Delta X)}\left[\left(S_{i+1,n}-S_{i-1,n}\right)+\left(S_{i+1,n+1}-S_{i-1,n+1}\right)\right]$$

$$= \frac{1}{4(\Delta X)^2}\begin{bmatrix} S_{i,n}\left(S_{i+1,n+1}-2S_{i,n+1}+S_{i-1,n+1}\right)+S_{i,n}\left(S_{i+1,n}-2S_{i,n}+S_{i-1,n}\right) \\ S_{i,n+1}\left(S_{i+1,n+1}-2S_{i,n+1}+S_{i-1,n+1}\right)+S_{i,n+1}\left(S_{i+1,n}-2S_{i,n}+S_{i-1,n}\right) \end{bmatrix}$$

$$+\frac{1}{16\,(\Delta X)^2}\begin{bmatrix} \left(S_{i+1,n}\right)^2+\left(S_{i-1,n}\right)^2-2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2+\left(S_{i-1,n+1}\right)^2-2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n}-S_{i-1,n}\right)\left(S_{i+1,n+1}\right)+\left(S_{i+1,n}-S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix}$$

$$+\frac{A}{4(\Delta X)}\left[\left(S_{i+1,n}-S_{i-1,n}\right)+\left(S_{i+1,n+1}-S_{i-1,n+1}\right)\right]$$

Where $C_0 = \frac{1}{16}\frac{\Delta T}{(\Delta X)^2}$

$$S_{i,n+1}-S_{i,n}$$

$$= 4C_0\begin{bmatrix} S_{i,n}\left(S_{i+1,n+1}-2S_{i,n+1}+S_{i-1,n+1}\right)+S_{i,n}\left(S_{i+1,n}-2S_{i,n}+S_{i-1,n}\right) \\ S_{i,n+1}\left(S_{i+1,n+1}-2S_{i,n+1}+S_{i-1,n+1}\right)+S_{i,n+1}\left(S_{i+1,n}-2S_{i,n}+S_{i-1,n}\right) \end{bmatrix}$$

$$+C_0\begin{bmatrix} \left(S_{i+1,n}\right)^2+\left(S_{i-1,n}\right)^2-2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ +\left(S_{i+1,n+1}\right)^2+\left(S_{i-1,n+1}\right)^2-2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n}-S_{i-1,n}\right)\left(S_{i+1,n+1}\right)+\left(S_{i+1,n}-S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix}$$

$$+4A\Delta XC_0\left[\left(S_{i+1,n}-S_{i-1,n}\right)+\left(S_{i+1,n+1}-S_{i-1,n+1}\right)\right]$$

Only $n + 1$ terms,

1) $+ \left(S_{i+1,n+1}\right)^2 \rightarrow \{C_0\}$
2) $+ \left(S_{i-1,n+1}\right)^2 \rightarrow \{C_0\}$
3) $+ \left(S_{i,n+1}\right)^2 \rightarrow \{-8C_0\}$
4) $+ \left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \rightarrow \{-2C_0\}$
5) $+ \left(S_{i+1,n+1}\right)\left(S_{i,n+1}\right) \rightarrow \{4C_0\}$
6) $+ \left(S_{i,n+1}\right)\left(S_{i-1,n+1}\right) \rightarrow \{4C_0\}$
7) $+ \left(S_{i+1,n+1}\right) \rightarrow \left\{4C_0S_{i,n} + 4A\Delta XC_0 + 2C_0S_{i+1,n} - 2C_0S_{i-1,n}\right\}$
8) $+ \left(S_{i,n+1}\right) \rightarrow \left\{4C_0S_{i+1,n} + 4C_0S_{i-1,n} - 1 - 16C_0S_{i,n}\right\}$
9) $+ \left(S_{i-1,n+1}\right) \rightarrow \left\{4C_0S_{i,n} - 4A\Delta XC_0 + 2C_0S_{i-1,n} - 2C_0S_{i+1,n}\right\}$

(5.26)

Only n terms, (overall negative sign)

1) $+ \left(S_{i+1,n}\right)^2 \rightarrow \{C_0\}$
2) $+ \left(S_{i-1,n}\right)^2 \rightarrow \{C_0\}$
3) $+ \left(S_{i,n}\right)^2 \rightarrow \{-8C_0\}$
4) $+ \left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \rightarrow \{-2C_0\}$
5) $+ \left(S_{i+1,n}\right)\left(S_{i,n}\right) \rightarrow \{4C_0\}$
6) $+ \left(S_{i-1,n}\right)\left(S_{i,n}\right) \rightarrow \{4C_0\}$
7) $+ \left(S_{i+1,n}\right) \rightarrow \{4A\Delta XC_0\}$
8) $+ \left(S_{i,n}\right) \rightarrow \{1\}$
9) $+ \left(S_{i-1,n}\right) \rightarrow \{-4A\Delta XC_0\}$

(5.27)

Equation (5.26) and (5.27) gives the saturation of injected water for cocurrent imbibition phenomenon.

5.4 Numerical and Graphical Interpretation:

The numerical and graphical representation of solution of constant parameters have been considered $g = 9.8\ {}^{m}/_{s^2}, \beta = 2\ {}^{N}/_{m^2}, \rho_w = 0.1\ {}^{kg}/_{m^3}, L = 1\ m.$

5.4.1 Inclination with porous medium($\boldsymbol{\theta = 5^{\circ}}$)

The numerical values of saturation of injected water shows in table (5.1). In figure (5.1) is represented the saturation of injected water $S(X,T)$ versus distance X for fixed time T. The numerical values and graphical explanation of solution shows that the saturation of injected water increase when distance increase for given time T.

X/T	0	0.001	0.002	0.003	0.004	0.005
0	0	0	0	0	0	0
0.1	0.095163	0.095979	0.096802	0.097633	0.098472	0.099318
0.2	0.181269	0.182492	0.18374	0.185012	0.186311	0.187636
0.3	0.259182	0.261116	0.263092	0.26511	0.267171	0.269278
0.4	0.32968	0.332544	0.335472	0.338464	0.341525	0.344656
0.5	0.393469	0.39741	0.401444	0.405573	0.409802	0.414135
0.6	0.451188	0.456298	0.461536	0.466908	0.472418	0.478071
0.7	0.503415	0.509741	0.516237	0.522903	0.529743	0.536754
0.8	0.550671	0.558218	0.565932	0.573777	0.581725	0.589751
0.9	0.59343	0.601755	0.609643	0.617144	0.624305	0.631167
1	0.63	0.63	0.63	0.63	0.63	0.63

TABLE 5.1: Numerical value of $S(X,T)$ in homogeneous porous media for cocurrent imbibition phenomenon with $\theta = 5^{\circ}$

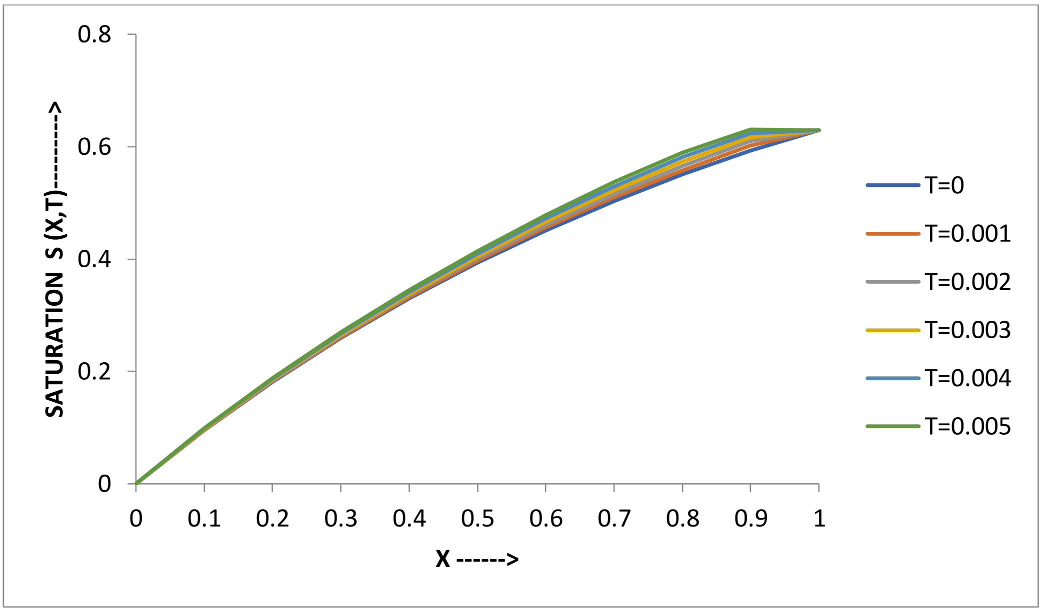


FIGURE 5.1: Saturation of injected water $S(X,T)$ versus distance X for fixed time T for $\theta = 5^{\circ}$.

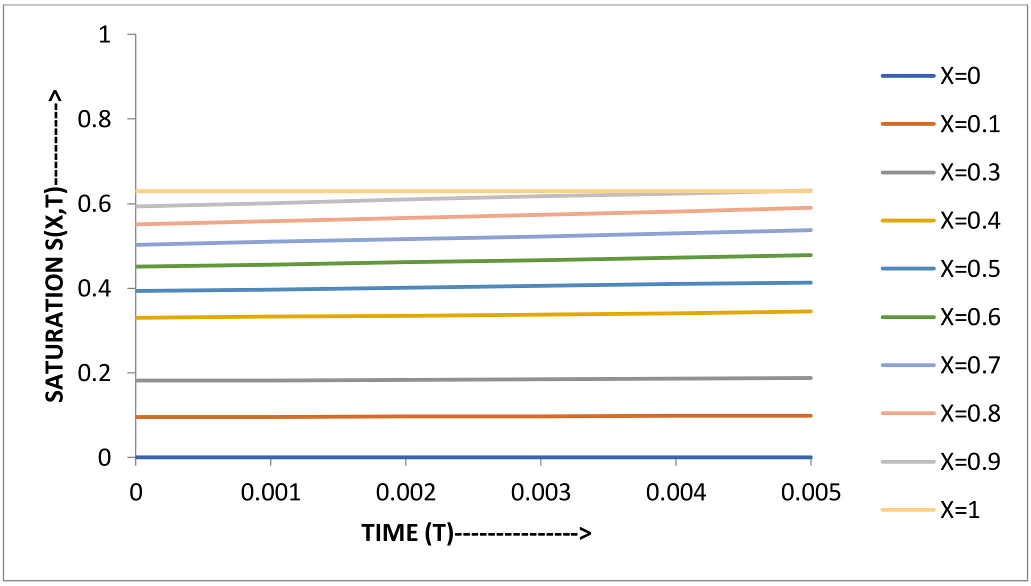


FIGURE 5.2: Saturation of injected water $S(X,T)$ versus time T for distance X= 0, 0.1, 0.2…, 1 for $\theta = 5^{\circ}$.

5.4.2 Inclination with porous medium($\theta = 10^{\circ}$).

The numerical values of saturation of injected water for $\theta = 10^{\circ}$ shows in table (5.2). The graph of solution $S(X,T)$ versus distance X for fixed time T= 0, 0.001, 0.002, 0.003, 0.004, 0.005.

X/T	0	0.001	0.002	0.003	0.004	0.005
0	0	0	0	0	0	0
0.1	0.095163	0.096056	0.096956	0.097865	0.098781	0.099707
0.2	0.181269	0.182562	0.183881	0.185226	0.186597	0.187997
0.3	0.259182	0.26118	0.263221	0.265305	0.267435	0.269611
0.4	0.32968	0.332602	0.335589	0.338643	0.341768	0.344964
0.5	0.393469	0.397463	0.401551	0.405737	0.410025	0.414419
0.6	0.451188	0.456346	0.461634	0.467057	0.472622	0.478333
0.7	0.503415	0.509785	0.516326	0.523042	0.529935	0.537003
0.8	0.550671	0.558261	0.56603	0.57394	0.58196	0.590063
0.9	0.59343	0.601945	0.610001	0.617653	0.624949	0.631932
1	0.63213	0.63213	0.63213	0.63213	0.63213	0.63213

TABLE 5.2: Numerical value of $S(X,T)$ in homogeneous porous media for cocurrent imbibition phenomenon with$\theta = 10^{\circ}$

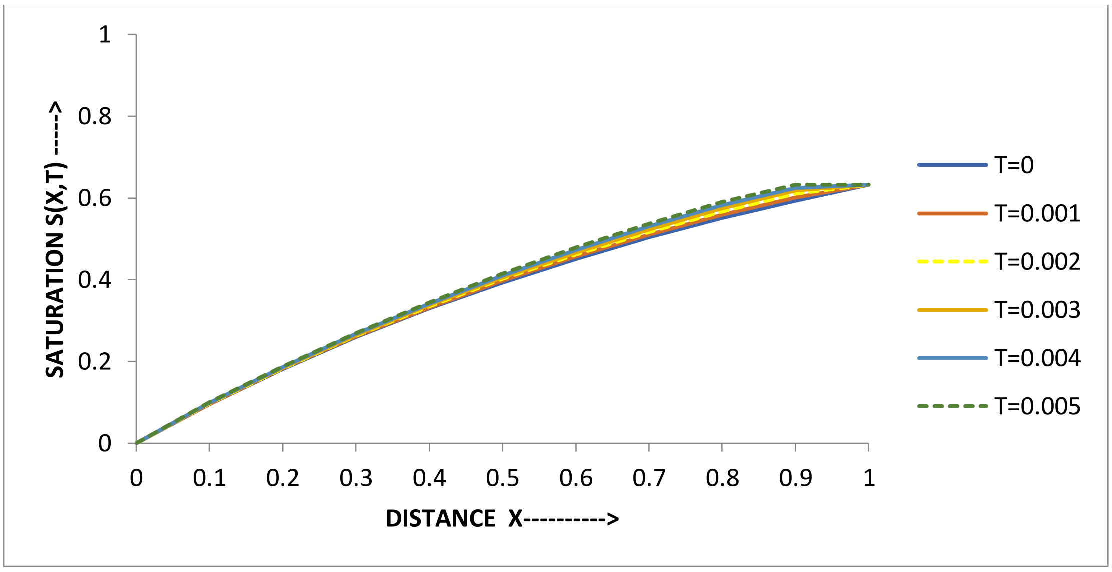


FIGURE 5.3: Graph $S(X,T)$ versus X for a fixed time T for $\theta = 10^{\circ}$.

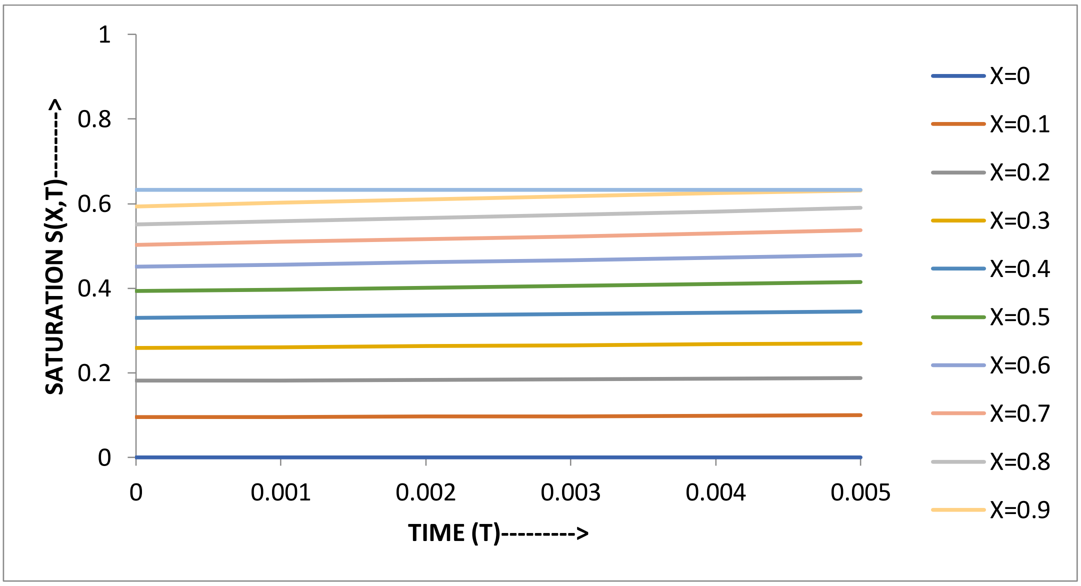


FIGURE 5.4: Graph $S\ (X, T)$ versus T for a fixed distance X for $\theta = 10^{\circ}$

5.4.3 Inclination with porous medium $(\boldsymbol{\theta} = 0^{\circ})$.

The numerical values of saturation of injected water for $\theta = 0^{\circ}$ shows in table (5.3). The graph of solution $S(X, T)$ versus distance X for fixed time T= 0, 0.001, 0.002, 0.003, 0.004, 0.005.

X/T	0	0.001	0.002	0.003	0.004	0.005
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0952	0.0959	0.0966	0.0974	0.0982	0.0989
0.2	0.1813	0.1824	0.1836	0.1848	0.1860	0.1873
0.3	0.2592	0.2611	0.2630	0.2649	0.2669	0.2689
0.4	0.3297	0.3325	0.3354	0.3383	0.3413	0.3443
0.5	0.3935	0.3974	0.4013	0.4054	0.4096	0.4138
0.6	0.4512	0.4562	0.4614	0.4668	0.4722	0.4778
0.7	0.5034	0.5097	0.5161	0.5228	0.5296	0.5365
0.8	0.5507	0.5582	0.5659	0.5737	0.5816	0.5895
0.9	0.5934	0.6017	0.6096	0.6171	0.6242	0.6310
1	0.6300	0.6300	0.6300	0.6300	0.6300	0.6300

TABLE 5.3: Numerical value of $S(X, T)$ in homogeneous porous media for cocurrent imbibition phenomenon with $\theta = 0^{\circ}$

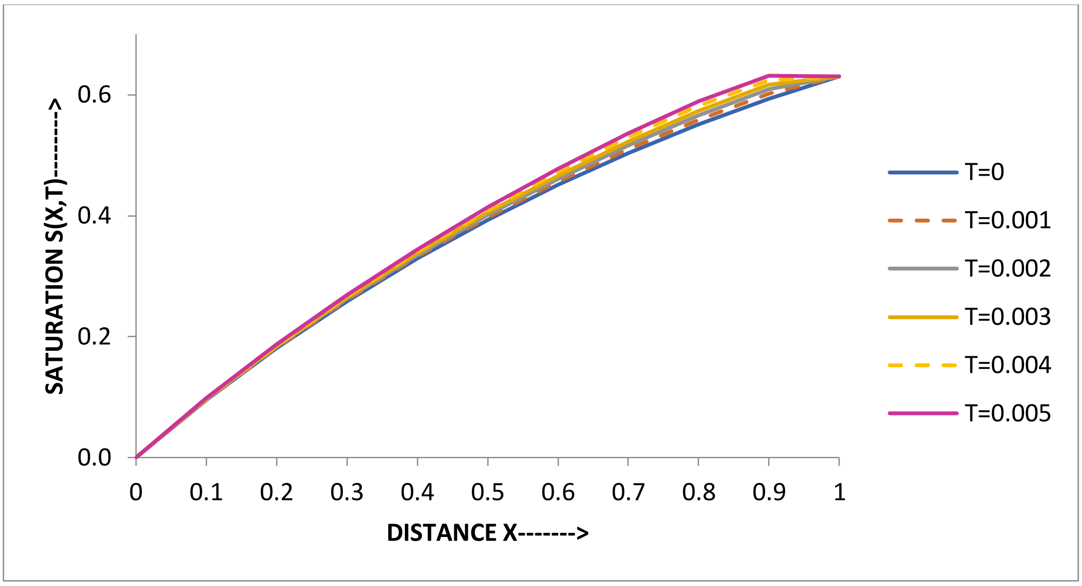


FIGURE 5.5: Graph $S\ (X,T)$ versus distance X for a fixed time T for $\theta = 0^{\circ}$

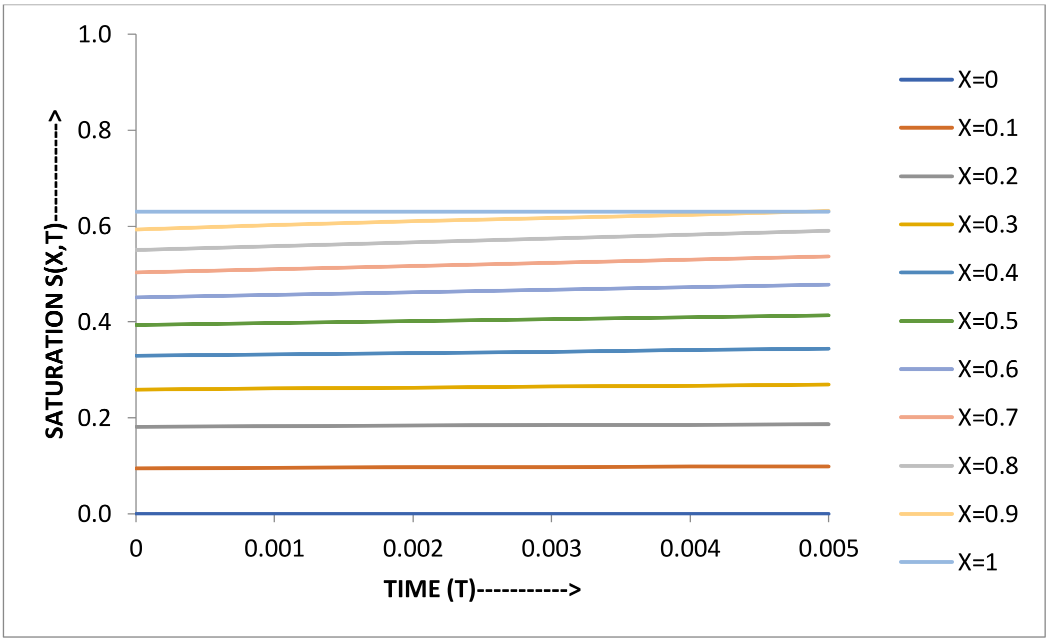


FIGURE 5.6: Graph $S\ (X,T)$ versus time T for a fixed distance X for $\theta = 0^{\circ}$

5.5 Conclusion:

In this chapter, cocurrent imbibition phenomenon is solved by finite difference Crank-Nicolson scheme which is well known numerical method. The graph and tabular values show that the saturation of injected water is increasing as distance X and time T are increase.

CHAPTER 6

Mathematical Investigation of Counter-Current Imbibition Phenomenon in Homogeneous Porous Media

Chapter 6

Mathematical Investigation of Counter-Current Imbibition Phenomenon in Homogeneous Porous Media

6.1 Introduction:

In this chapter the researcher has discussed about counter current imbibition phenomenon in homogeneous porous matrix.

The spontaneous displacement of non-wetting fluid (oil) by a wetting fluid (water) in a water wet reservoir. It is called an imbibition process. This phenomenon arises due to the difference of wetting capacity of water and oil. Further, it is well known physical fact that when a porous medium filled with some fluid is brought contact with other fluid which preferentially wets the medium then there is a spontaneous flow of the resident fluid from the medium. When two immiscible fluids like water and oil flow in the opposite direction that type of flow called as Counter current flow. This phenomenon arising due to the difference in the wetting abilities of the fluid is called counter- current imbibition phenomenon.

Many researchers have studied the counter current imbibition phenomenon with different point of views [95, 49, 68, 1, 125, 14]. The numerical solution for countercurrent imbibition in porous rocks has been presented by Blair [95]. Graham and Richardson [68], Scheidegger [1], Bokserman et. al. [31] has described the physics of oil-water motion in a cracked porous medium. Joshi et al. [81] have studied the countercurrent imbibition phenomenon by product method. The mathematical model of countercurrent imbibition phenomenon in vertical porous matrix has been studied by Parikh et al [12]. Pradhan and Verma have achieved the numerical solution of a specific imbibition phenomenon using Crank-Nicolson Scheme for finite differences.

Throughout the chapter, the researcher has tried to focus on the counter current imbibition phenomenon in the inclined homogeneous porous media. The goal of study to this chapter is to calculate the saturation of water for the inclined porous medium($\theta = 5^{\circ}, \theta = 10^{\circ}, \theta = 15^{\circ}$ and $\theta = 20^{\circ}$). This nonlinear partial differential equation has been solved using Crank Nicolson scheme for finite differences with appropriate initial and boundary conditions.

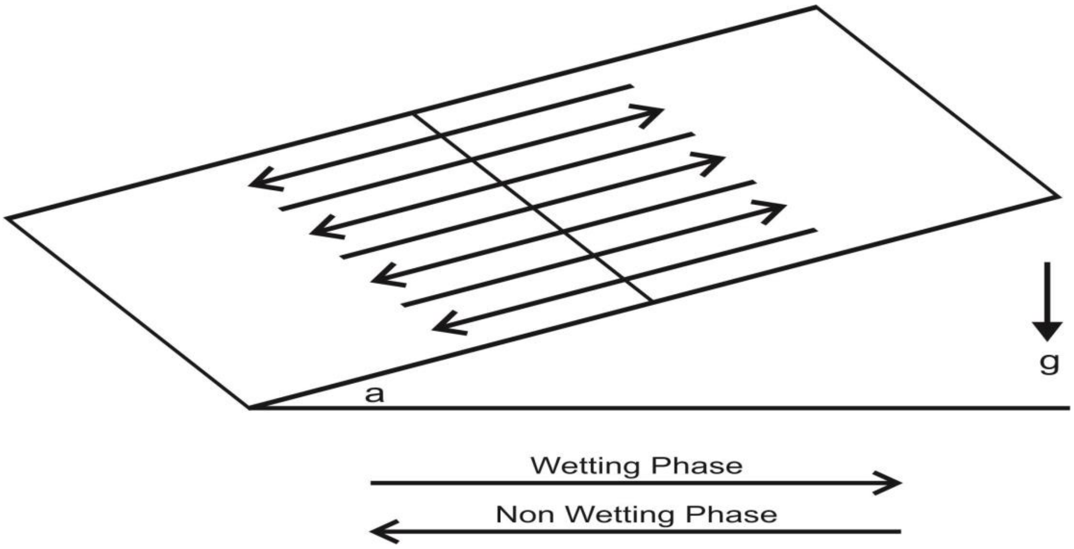


FIGURE 6.1: Counter current imbibition phenomenon.

Take a cylindrical piece of homogeneous porous matrix having length L and its three sides are surrounded by an impermeable surface has been taken whose one open end which is labeled as imbibition face $x = 0$ and it is inclined at an angle θ the ground surface.

When water injects into an oil saturated porous matrix at imbibition face $x = 0$. The native oil take the place through a small space $x = l$ due to in phase difference. Since water take place at common interface in inclined homogeneous porous matrix satisfying oil that will displace by injected water.

6.2 Formulation of the Mathematical Modeling of the Problem:

Assuming the validity of Darcy's law, which is governing law in the case being investigated the equations of seepage velocity of following fluids may be written as; [24, 63, 75, 1]

$$V_w = -\frac{k_w}{\delta_w} K \left[\frac{\partial P_w}{\partial x} + \rho_w g \sin\theta\right] \tag{6.1}$$

$$V_o = -\frac{k_o}{\delta_o} K \left[\frac{\partial P_o}{\partial x} - \rho_o g \sin\theta\right] \tag{6.2}$$

The equation of continuity describe as,

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{6.3}$$

The capillary pressure (P_c) is expressed as,

$$P_c = P_o - P_w \tag{6.4}$$

The capillary pressure (P_c) is assumed as a function of saturation of water [76];

$$P_c = -\beta S_w \tag{6.5}$$

Where β is constant.

The relation between relative permeability and phase saturation defined as, [1,5]

$$k_w = S_w \text{ and } k_0 = 1 - \alpha S_w \tag{6.6}$$

Where α is constant.

In countercurrent imbibition phenomenon of relation between velocity of oil and velocity of water can be expressed as;

$$V_o + V_w = 0 \tag{6.7}$$

Using equations (6.1) and (6.2) in (6.7), we get

$$\frac{k_w}{\delta_w} K \left[\frac{\partial P_w}{\partial x} + \rho_w g \sin\theta\right] + \frac{k_o}{\delta_o} K \left[\frac{\partial P_o}{\partial x} - \rho_o g \sin\theta\right] = 0 \tag{6.8}$$

Using the relation (6.4) in equation (6.8), we get

$$\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\mathrm{K}\left[\frac{\partial}{\partial \mathrm{x}}\,(\mathrm{P_o}-\ \mathrm{P_c})+\rho_\mathrm{w}\mathrm{g}\sin\theta\right]+\ \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\ \mathrm{K}\left[\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}}-\rho_\mathrm{o}\mathrm{g}\sin\theta\right]=0$$

$$\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\mathrm{K}\left[\frac{\partial P_o}{\partial x}\ +\rho_\mathrm{w}\mathrm{g}\sin\theta\right]-\ \frac{\mathrm{k_w}}{\delta_\mathrm{w}}\ \mathrm{K}\left[\frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}+\rho_\mathrm{w}\mathrm{g}\sin\theta\right]+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}\ \mathrm{K}\left[\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}}-\rho_\mathrm{o}\mathrm{g}\sin\theta\right]\ =0$$

$$\left(\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}\right)\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}}-\ \frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}=\ -\left(\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\ \rho_\mathrm{w}-\ \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\ \rho_\mathrm{o}\right)\ \mathrm{g}\ \sin\theta \qquad (6.9)$$

Solve the equation (4.9) for $\frac{\partial P_o}{\partial x}$

$$\frac{\partial \mathrm{P_o}}{\partial \mathrm{x}}=\ -\left[\frac{\left(\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\,\rho_\mathrm{w}-\frac{\mathrm{k_o}}{\delta_\mathrm{o}}\,\rho_\mathrm{o}\right)\mathrm{g}\ \sin\theta-\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\frac{\partial \mathrm{P_c}}{\partial x}}{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}\right] \qquad (6.10)$$

From equation (6.2) we get,

$$\mathrm{V_o}=\ \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\ \mathrm{K}\left[\frac{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}(\rho_\mathrm{w}+\rho_\mathrm{o})\ \mathrm{g}\ \sin\theta-\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\frac{\partial \mathrm{P_c}}{\partial x}}{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}\right] \qquad (6.11)$$

Using (6.7) and (6.11)

$$\mathrm{V_w}=\ -\ \frac{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}\ \mathrm{K}\left[(\rho_\mathrm{w}+\ \rho_\mathrm{o})\ \mathrm{g}\ \sin\theta-\ \frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}\right] \qquad (6.12)$$

Substituting the value of V_w in equation (4.3), we get

$$\mathrm{P}\ \frac{\partial \mathrm{S_w}}{\partial \mathrm{t}}-\ \frac{\partial}{\partial \mathrm{x}}\left[\frac{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}\ \mathrm{K}\,(\rho_\mathrm{w}+\ \rho_\mathrm{o})\ \mathrm{g}\ \sin\theta\right]+\frac{\partial}{\partial \mathrm{x}}\left[\frac{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}\ \frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}\right]=0 \qquad (6.13)$$

According to the scheidegger of flow system [1], we have

$$\frac{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}{\frac{\mathrm{k_w}}{\delta_\mathrm{w}}+\frac{\mathrm{k_o}}{\delta_\mathrm{o}}}\ \approx\ \frac{k_o}{\delta_o} \qquad (6.14)$$

Substituting the values from (6.5), (6.6) and (6.14) into (6.13), we have

$$\mathrm{P}\ \frac{\partial \mathrm{S_w}}{\partial \mathrm{t}}-\ \frac{\partial}{\partial \mathrm{x}}\left[\ \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\ \mathrm{K}\,(\rho_\mathrm{w}+\ \rho_\mathrm{o})\ \mathrm{g}\ \sin\theta\right]+\frac{\partial}{\partial \mathrm{x}}\left[\ \frac{\mathrm{k_o}}{\delta_\mathrm{o}}\ \mathrm{K}\ \ \frac{\partial \mathrm{P_c}}{\partial \mathrm{x}}\right]=0$$

$$\mathrm{P}\ \frac{\partial \mathrm{S_w}}{\partial \mathrm{t}}=\ \frac{\mathrm{K}\,(\rho_\mathrm{w}+\rho_\mathrm{o})\ \mathrm{g}\ \sin\theta}{\delta_\mathrm{o}}\ \ \frac{\partial}{\partial \mathrm{x}}\ (1-\alpha\mathrm{S_w})+\ \frac{\mathrm{K}\beta}{\delta_\mathrm{o}}\,\frac{\partial}{\partial \mathrm{x}}\left[(1-\alpha\mathrm{S_w})\ \frac{\partial \mathrm{S_w}}{\partial \mathrm{x}}\right] \qquad (6.15)$$

Using dimensionless variables,

$$X = \frac{x}{L}, T = \frac{\beta t K}{\delta_o L^2 P}$$

Equation (6.15) reduce to

$$\frac{\partial S_w}{\partial T} = \frac{\partial^2 S_w}{\partial X^2} - \alpha S_w \frac{\partial^2 S_w}{\partial X^2} - \alpha \left(\frac{\partial S_w}{\partial X}\right)^2 - \alpha C \frac{\partial S_w}{\partial X} \tag{6.16}$$

where $C = \frac{L(\rho_w + \rho_o)\, g \sin\theta}{\beta}$ and $S_w(x, t) = S_w(X, T)$

Suppose that$S_w(X, T) = S(X, T)$.

Rewriting the equation (6.16), we got

$$\frac{\partial S}{\partial T} = \frac{\partial^2 S}{\partial X^2} - \alpha S \frac{\partial^2 S}{\partial X^2} - \alpha \left(\frac{\partial S}{\partial X}\right)^2 - \alpha C \frac{\partial S}{\partial X} \tag{6.17}$$

The equation (6.17) can be described as governing equation for countercurrent imbibition phenomenon.

Equation (6.17) solved with appropriate initial and boundary conditions using finite difference method.

Choose initial condition

$$S(X, 0) = (1 - X)^2, \ 0 < X \leq 1 \tag{6.18}$$

Boundary conditions are,

$$S(0, T) = S_0 = 1, \ 0 < T \leq 1$$

$$S(1, T) = S_1 = 0, \ , \ 0 \leq T < 1 \tag{6.19}$$

6.3 Solution by Finite Difference Method:

The equation (6.17) can be solved by finite difference Crank-Nicolson scheme [109, 32] with appropriate initial and boundary conditions.

$$\frac{\partial S}{\partial T} = \frac{S_{i,n+1} - S_{i,n}}{\Delta T} \tag{6.20}$$

$$\frac{\partial^2 S}{\partial X^2} = \frac{1}{2}\left[\frac{S_{i+1,n+1}-2S_{i,n+1}+S_{i-1,n+1}}{(\Delta X)^2} + \frac{S_{i+1,n}-2S_{i,n}+S_{i-1,n}}{(\Delta X)^2}\right] \tag{6.21}$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}\right]^2$$

$$\left(\frac{\partial S}{\partial X}\right)^2 = \left[\frac{\left(\frac{S_{i+1,n}+S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n}+S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]^2 \tag{6.22}$$

$$\frac{\partial S}{\partial X} = \frac{S_{i+1,n+1/2} - S_{i-1,n+1/2}}{2(\Delta X)}$$

$$= \left[\frac{\left(\frac{S_{i+1,n}+S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n}+S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right] \tag{6.23}$$

Equations (6.20), (6.21), (6.22) and (6.23) in using equation (6.17), we have

$$\begin{aligned}
\left[\frac{S_{i,n+1} - S_{i,n}}{\Delta T}\right]
&= \left[\frac{1}{2}\left(\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right) + \left(\frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right)\right] \\
&\quad - \alpha\left(S_{i,n+\frac{1}{2}}\right)\left[\frac{1}{2}\left(\frac{S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}}{(\Delta X)^2}\right)\right. \\
&\quad \left. + \left(\frac{S_{i+1,n} - 2S_{i,n} + S_{i-1,n}}{(\Delta X)^2}\right)\right] - \alpha\left[\frac{S_{i+1,n+\frac{1}{2}} - S_{i-1,n+\frac{1}{2}}}{2(\Delta X)}\right]^2 \\
&\quad - \alpha C\left[\frac{\left(\frac{S_{i+1,n} + S_{i+1,n+1}}{2}\right) - \left(\frac{S_{i-1,n} + S_{i-1,n+1}}{2}\right)}{2(\Delta X)}\right]
\end{aligned}$$

$$= \left[\frac{1}{2\,(\Delta X)^2}\left(\mathrm{S_{i+1,n+1}} - 2\mathrm{S_{i,n+1}} + \mathrm{S_{i-1,n+1}}\right) + \left(\mathrm{S_{i+1,n}} - 2\mathrm{S_{i,n}} + \mathrm{S_{i-1,n}}\right)\right]$$

$$- \alpha\left(\frac{S_{i,n} + S_{i,n+1}}{2}\right)\left[\frac{1}{2\,(\Delta X)^2}\left(\mathrm{S_{i+1,n+1}} - 2\mathrm{S_{i,n+1}} + \mathrm{S_{i-1,n+1}}\right)\right.$$

$$\left. + \left(\mathrm{S_{i+1,n}} - 2\mathrm{S_{i,n}} + \mathrm{S_{i-1,n}}\right)\right]$$

$$- \alpha\left[\frac{\left(\frac{\mathrm{S_{i+1,n}} + \mathrm{S_{i+1,n+1}}}{2}\right) - \left(\frac{\mathrm{S_{i-1,n}} + \mathrm{S_{i-1,n+1}}}{2}\right)}{2(\Delta \mathrm{X})}\right]^2$$

$$- \alpha C\left[\frac{\left(\frac{\mathrm{S_{i+1,n}} + \mathrm{S_{i+1,n+1}}}{2}\right) - \left(\frac{\mathrm{S_{i-1,n}} + \mathrm{S_{i-1,n+1}}}{2}\right)}{2(\Delta \mathrm{X})}\right]$$

$$= \left[\frac{1}{2\,(\Delta \mathrm{X})^2}\left(\mathrm{S_{i+1,n+1}} - 2\mathrm{S_{i,n+1}} + \mathrm{S_{i-1,n+1}}\right) + \left(\mathrm{S_{i+1,n}} - 2\mathrm{S_{i,n}} + \mathrm{S_{i-1,n}}\right)\right]$$

$$- \frac{\alpha}{4(\Delta \mathrm{X})^2}\left(\mathrm{S_{i,n}} + \mathrm{S_{i,n+1}}\right)\left[\left(\mathrm{S_{i+1,n+1}} - 2\mathrm{S_{i,n+1}} + \mathrm{S_{i-1,n+1}}\right)\right.$$

$$\left. + \left(\mathrm{S_{i+1,n}} - 2\mathrm{S_{i,n}} + \mathrm{S_{i-1,n}}\right)\right]$$

$$- \frac{\alpha}{16(\Delta \mathrm{X})^2}\left[\left(\mathrm{S_{i+1,n}} + \mathrm{S_{i+1,n+1}}\right) - \left(\mathrm{S_{i-1,n}} + \mathrm{S_{i-1,n+1}}\right)\right]^2$$

$$- \frac{\alpha \mathrm{C}}{4(\Delta \mathrm{X})}\left[\left(\mathrm{S_{i+1,n}} + \mathrm{S_{i+1,n+1}}\right) - \left(\mathrm{S_{i-1,n}} + \mathrm{S_{i-1,n+1}}\right)\right]$$

$$= \left[\frac{1}{2\,(\Delta \mathrm{X})^2}\left(\mathrm{S_{i+1,n+1}} - 2\mathrm{S_{i,n+1}} + \mathrm{S_{i-1,n+1}}\right) + \left(\mathrm{S_{i+1,n}} - 2\mathrm{S_{i,n}} + \mathrm{S_{i-1,n}}\right)\right]$$

$$- \frac{\alpha}{4(\Delta \mathrm{X})^2}\left(\mathrm{S_{i,n}} + \mathrm{S_{i,n+1}}\right)\left[\left(\mathrm{S_{i+1,n+1}} - 2\mathrm{S_{i,n+1}} + \mathrm{S_{i-1,n+1}}\right)\right.$$

$$\left. + \left(\mathrm{S_{i+1,n}} - 2\mathrm{S_{i,n}} + \mathrm{S_{i-1,n}}\right)\right]$$

$$- \frac{\alpha}{16\,(\Delta X)^2}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]^2$$

$$- \frac{\alpha C}{4(\Delta X)}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]$$

$$= \left[\frac{1}{2\,(\Delta X)^2}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right]$$

$$- \frac{\alpha}{4(\Delta X)^2}\left(S_{i,n} + S_{i,n+1}\right)\left[\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right)\right.$$

$$\left. + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right]$$

$$- \frac{\alpha}{16(\Delta X)^2}\left[\left(S_{i+1,n} - S_{i-1,n}\right)^2 + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)^2\right.$$

$$\left. + 2\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]$$

$$- \frac{\alpha C}{4(\Delta X)}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]$$

$$= \left[\frac{1}{2\,(\Delta X)^2}\left(S_{i+1,n+1} - 2S_{I,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{I,n} + S_{i-1,n}\right)\right] -$$

$$\frac{\alpha}{4(\Delta X)^2}\left(S_{I,n} + S_{I,n+1}\right)\left[\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right)\right]$$

$$- \frac{\alpha}{16(\Delta X)^2}\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix}$$

$$- \frac{\alpha C}{4(\Delta X)}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]$$

$$= \left[\frac{1}{2\,(\Delta X)^2}\left(S_{i+1,n+1} - 2S_{I,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{I,n} + S_{i-1,n}\right)\right]$$

$$- \frac{\alpha}{4(\Delta X)^2}\begin{bmatrix} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) + \\ S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{bmatrix}$$

$$- \frac{\alpha}{16(\Delta X)^2}\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix}$$

$$- \frac{\alpha C}{4(\Delta X)}\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right]$$

Where $C_0 = \frac{1}{16}\frac{\Delta T}{(\Delta X)^2}$

$$S_{i,n+1} - S_{i,n} = 8C_0\left[\left(S_{i+1,n+1} - 2S_{I,n+1} + S_{i-1,n+1}\right) + \left(S_{i+1,n} - 2S_{I,n} + S_{i-1,n}\right)\right] -$$

$$4\alpha C_0\begin{bmatrix} S_{i,n}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) + \\ S_{i,n+1}\left(S_{i+1,n+1} - 2S_{i,n+1} + S_{i-1,n+1}\right) + S_{i,n+1}\left(S_{i+1,n} - 2S_{i,n} + S_{i-1,n}\right) \end{bmatrix} -$$

$$\alpha C_0\begin{bmatrix} \left(S_{i+1,n}\right)^2 + \left(S_{i-1,n}\right)^2 - 2\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \\ \left(S_{i+1,n+1}\right)^2 + \left(S_{i-1,n+1}\right)^2 - 2\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \\ +2\left[\left(S_{i+1,n} - S_{i-1,n}\right)\left(S_{i+1,n+1}\right) + \left(S_{i+1,n} - S_{i-1,n}\right)\left(-S_{i-1,n+1}\right)\right] \end{bmatrix} -$$

$$4\alpha CC_0\Delta X\left[\left(S_{i+1,n} - S_{i-1,n}\right) + \left(S_{i+1,n+1} - S_{i-1,n+1}\right)\right] \tag{6.24}$$

From equation (6.24) different out $n + 1$ and n terms, we get

Only $n + 1$ terms,

1) $+\left(S_{i+1,n+1}\right)^2 \rightarrow \{-\alpha C_0\}$

2) $+\left(S_{i-1,n+1}\right)^2 \rightarrow \{-\alpha C_0\}$

3) $+\left(S_{i,n+1}\right)^2 \rightarrow \{8\alpha C_0\}$

4) $+\left(S_{i+1,n+1}\right)\left(S_{i-1,n+1}\right) \rightarrow \{2\alpha C_0\}$

5) $+\left(S_{i+1,n+1}\right)\left(S_{i,n+1}\right) \rightarrow \{-4\alpha C_0\}$

6) $+\left(S_{i,n+1}\right)\left(S_{i-1,n+1}\right) \rightarrow \{-4\alpha C_0\}$

7) $+\left(S_{i+1,n+1}\right) \rightarrow \left\{8C_0 - 4\alpha C_0 S_{i,n} - 2\alpha C_0 S_{i+1,n} + 2\alpha C_0 S_{i-1,n} - 4\alpha CC_0\Delta X\right\}$

8) $+\left(S_{i,n+1}\right) \rightarrow \left\{-16C_0 - 4\alpha C_0 S_{i+1,n} - 4\alpha C_0 S_{i-1,n} + 8\alpha C_0 S_{i,n} - 1\right\}$

9) $+\left(S_{i-1,n+1}\right) \rightarrow \left\{8C_0 - 4\alpha C_0 S_{i,n} + 2\alpha C_0 S_{i+1,n} - 2\alpha C_0 S_{i-1,n} + 4\alpha CC_0\Delta X\right\}$

Only n terms, (overall –negative sign)

1) $+\left(S_{i+1,n}\right)^2 \rightarrow \{-\alpha C_0\}$

2) $+\left(S_{i-1,n}\right)^2 \rightarrow \{-\alpha C_0\}$

3) $+\left(S_{i,n}\right)^2 \rightarrow \{8\alpha C_0\}$

4) $+\left(S_{i+1,n}\right)\left(S_{i-1,n}\right) \rightarrow \{2\alpha C_0\}$

5) $+\left(S_{i+1,n}\right)\left(S_{i,n}\right) \rightarrow \{-4\alpha C_0\}$

6) $+\left(S_{i-1,n}\right)\left(S_{i,n}\right) \rightarrow \{-4\alpha C_0\}$

7) $+\left(S_{i+1,n}\right) \rightarrow \{8C_0 - 4\alpha C C_0 \Delta X\}$

8) $+\left(S_{i,n}\right) \rightarrow \{1 - 16C_0\}$

9) $+\left(S_{i-1,n}\right) \rightarrow \{8C_0 + 4\alpha C C_0 \Delta X\}$

is the solution of counter-current imbibition phenomenon for homogeneous porous medium.

6.4 Numerical Solution and Graphical Interpretation:

An approximate numerical solution has been obtained for the equation (6.17) with suitable initial and boundary conditions by Crank-Nicolson scheme of finite difference. Here we discuss saturation rate of wetting phase in homogeneous porous media with the effect of inclination for $\theta = 5^\circ, 10^\circ, 15^\circ, 20^\circ$.

The constant parameters are assumed as $\rho_w = 0.1\ {}^{kg}/_{m^3}$, $\rho_o = 0.3\ {}^{kg}/_{m^3}$, $g = 9.8\ {}^{m}/_{s^2}$, $\alpha = 1.11$, $\beta = 0.1\ {}^{N}/_{m^2}$.

6.4.1 Inclination with porous medium$(\boldsymbol{\theta = 5^{\circ}})$

Table (6.1) show that the numerical values of saturation of injected water for different distance X and time T.

X/T	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.1	0.99	0.679038	0.596545	0.560874	0.542698	0.532574
0.2	0.96	0.469394	0.389104	0.365984	0.358854	0.35699
0.3	0.91	0.326033	0.272967	0.266946	0.268765	0.271221
0.4	0.84	0.23292	0.208759	0.213127	0.217579	0.219959
0.5	0.75	0.172704	0.17114	0.178771	0.181911	0.182663
0.6	0.64	0.131235	0.144915	0.151515	0.152244	0.151663
0.7	0.51	0.097753	0.120372	0.124758	0.123803	0.122769
0.8	0.36	0.065095	0.090322	0.093319	0.092155	0.091482
0.9	0.19	0.031461	0.050137	0.052374	0.052032	0.052036
1	0	0	0	0	0	0

TABLE 6.1: Numerical values of saturation of injected water in homogeneous porous media for counter current imbibition phenomenon $(\theta = 5^{\circ})$

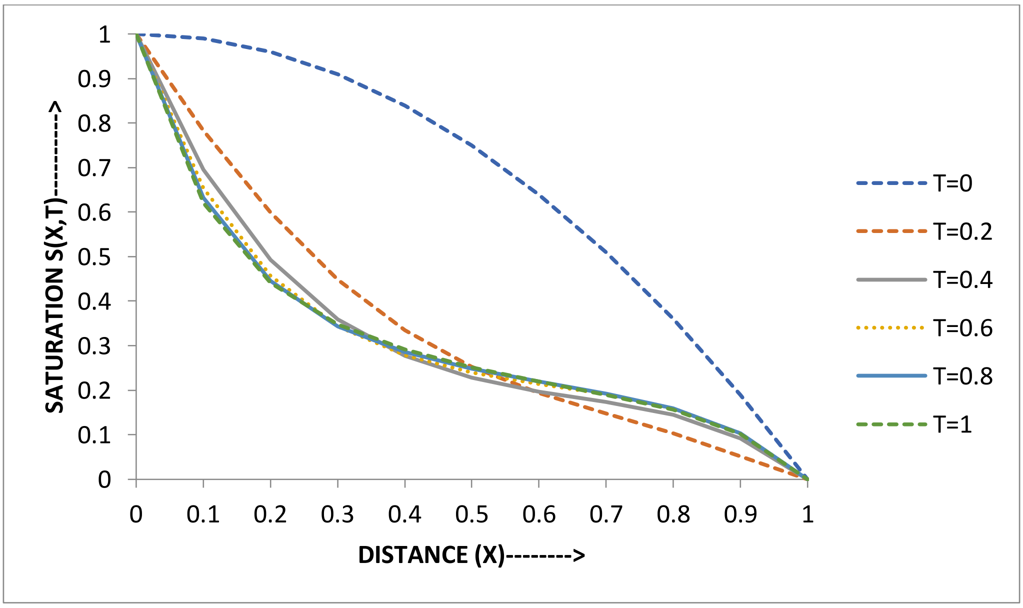


FIGURE 6.2: S (X, T) versus distance X for a fixed time T

6.4.2 Inclination with porous medium$(\boldsymbol{\theta} = 10^{\circ})$.

X/T	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.1	0.99	0.78365	0.69599	0.65424	0.63262	0.62082
0.2	0.96	0.59876	0.49266	0.4569	0.44495	0.44164
0.3	0.91	0.44806	0.35939	0.34314	0.34379	0.34737
0.4	0.84	0.33462	0.27713	0.27862	0.28621	0.2913
0.5	0.75	0.25264	0.2276	0.24006	0.24856	0.25168
0.6	0.64	0.19349	0.19682	0.21353	0.21922	0.21944
0.7	0.51	0.14727	0.17357	0.19007	0.1919	0.18997
0.8	0.36	0.10261	0.14486	0.1596	0.15897	0.15654
0.9	0.19	0.05081	0.09169	0.1034	0.10317	0.10216
1	0	0	0	0	0	0

TABLE 6.2: Numerical values of saturation of injected water in homogeneous porous media for counter current imbibition phenomenon $(\theta = 10^{\circ})$

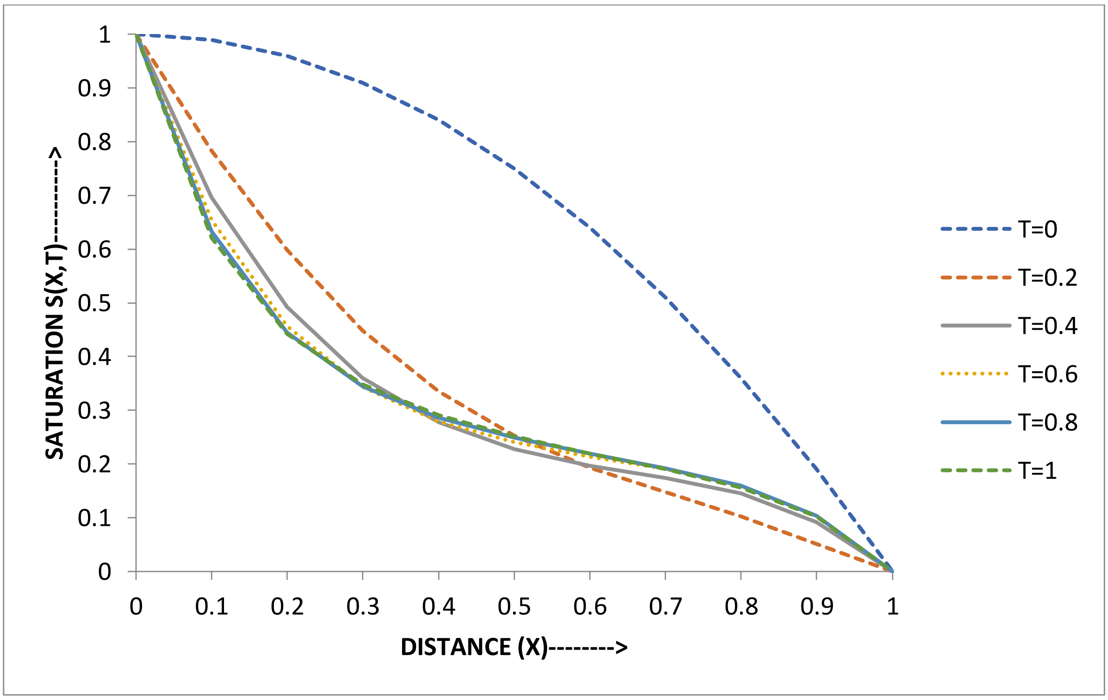


FIGURE 6.3: $S\ (X, T)$ versus distance X for a fixed time T.

6.4.3 Inclination with porous medium$(\boldsymbol{\theta} = \ \mathbf{15}^{\circ})$.

X/T	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.1	0.99	0.85465	0.77629	0.73188	0.70652	0.69195
0.2	0.96	0.69888	0.58581	0.53779	0.51846	0.51186
0.3	0.91	0.55389	0.44252	0.41116	0.40668	0.41011
0.4	0.84	0.43085	0.34313	0.33346	0.34131	0.34981
0.5	0.75	0.33181	0.27744	0.28652	0.30067	0.30899
0.6	0.64	0.25406	0.23506	0.25707	0.27186	0.27681
0.7	0.51	0.19231	0.20709	0.23622	0.24779	0.24845
0.8	0.36	0.13762	0.1833	0.21517	0.22197	0.21914
0.9	0.19	0.0761	0.13721	0.16601	0.16917	0.16557
1	0	0	0	0	0	0

TABLE 6.3: Numerical values of saturation of injected water in homogeneous porous media for counter current imbibition phenomenon $(\theta = \ 15^{\circ})$.

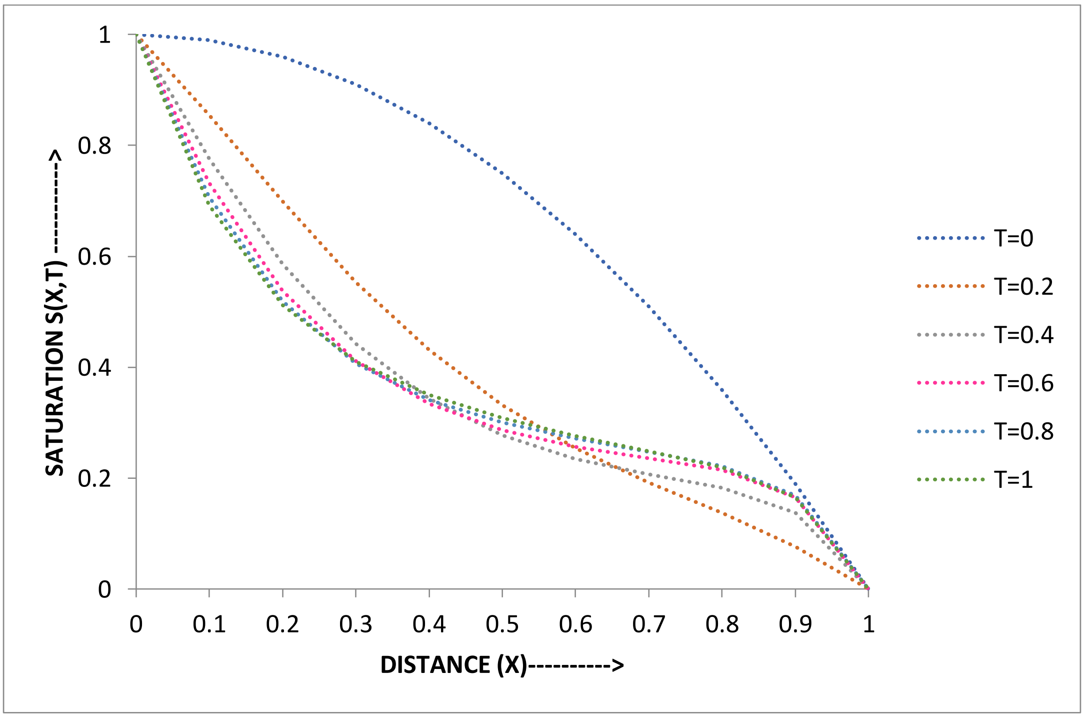


FIGURE 6.4: S (X, T) versus distance X for a fixed time T

6.4.4 Inclination with porous medium $(\theta = 20^{\circ})$.

X/T	0	0.2	0.4	0.6	0.8	1
0	1.00	1.00	1.00	1.00	1.00	1.00
0.1	0.99	0.90	0.84	0.79	0.77	0.75
0.2	0.96	0.77	0.66	0.61	0.58	0.57
0.3	0.91	0.64	0.52	0.47	0.46	0.46
0.4	0.84	0.51	0.41	0.40	0.39	0.40
0.5	0.75	0.40	0.33	0.36	0.34	0.31
0.6	0.64	0.31	0.27	0.35	0.31	0.31
0.7	0.51	0.23	0.23	0.30	0.30	0.30
0.8	0.36	0.17	0.22	0.27	0.26	0.28
0.9	0.19	0.11	0.18	0.20	0.21	0.17
1	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 6.4: Numerical values of saturation of injected water in homogeneous porous media for counter current imbibition phenomenon $(\theta = 20^{\circ})$.

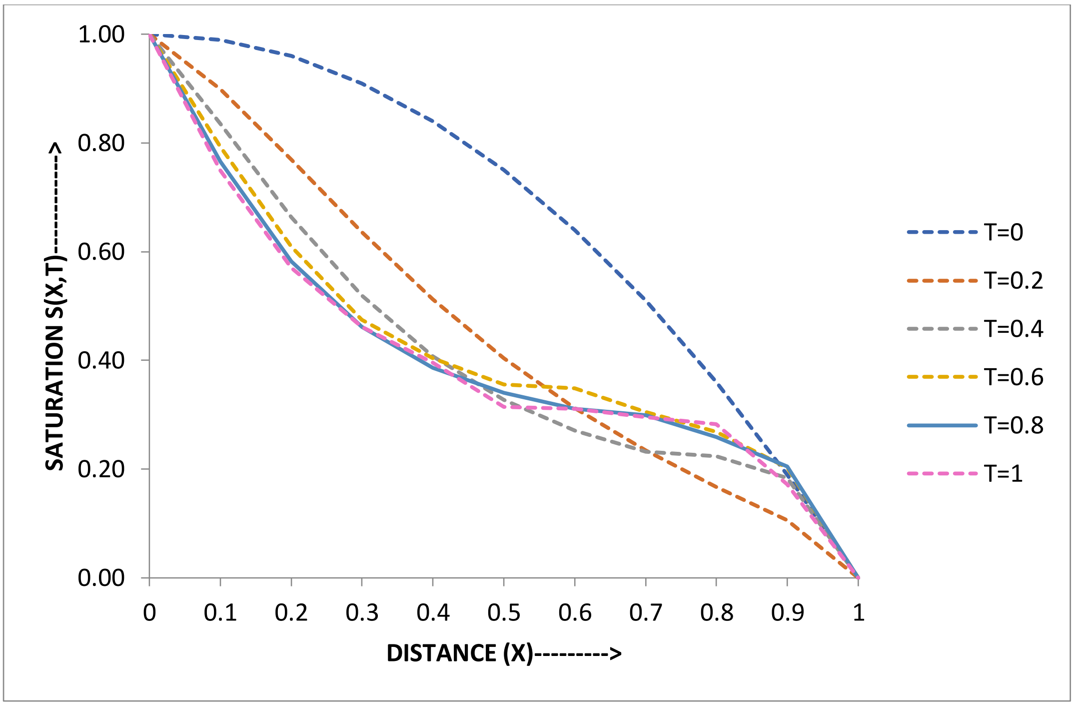


FIGURE 6.5: S (X, T) versus distance X for time T= 0, 0.2, 0.4, 0.6, 0.8…,1

6.5 Conclusion:

Equation (6.17) is a numerical solution of counter-current imbibition phenomenon obtained by finite difference Crank-Nicolson scheme with angle $\theta = 5^{\circ}, 10^{\circ}, 15^{\circ}, 20^{\circ}$. From the tabular values and graphs it is concluded that as distance X and time T increases the saturation of water gradually decreases. The saturation of water decrease as the distance X increase for the given time T > 0. Here the initial saturation of water at X=0 is highest and it decrease at distance X increase for given time T > 0.

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LIST OF PUBLICATIONS

List of Publications

1. An Analytical Solution of Porous Medium Equation by Similarity Transformation, Journal of Engineering and Technology, **3(5)**, (2016), 2395-0056.

2. Study of Imbibition Phenomenon in Homogenous Porous Medium Using Finite Difference Method, Journal of Applied Science and Computations, **5(9),** (2018), 1076-5131.

3. Instability Phenomenon Arising in Homogeneous Porous Media by Crank-Nicolson Finite Difference Method, Journal of Emerging Technologies and Innovative Research **6(2)**, (2019), 2349-5162.

Details of the Work Presented in Conference

1. The paper entitled as “Numerical solution of nonlinear partial differential equation arising in immiscible flow by finite difference method” presented at Mehsana Urban Institute of Sciences, Ganpat University, Kherva, Mehsana Advances in Pure and Applied Mathematics, December 2017.