A Soft Intro to Mean Field Games

Barnabé Monnot

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What we are looking at

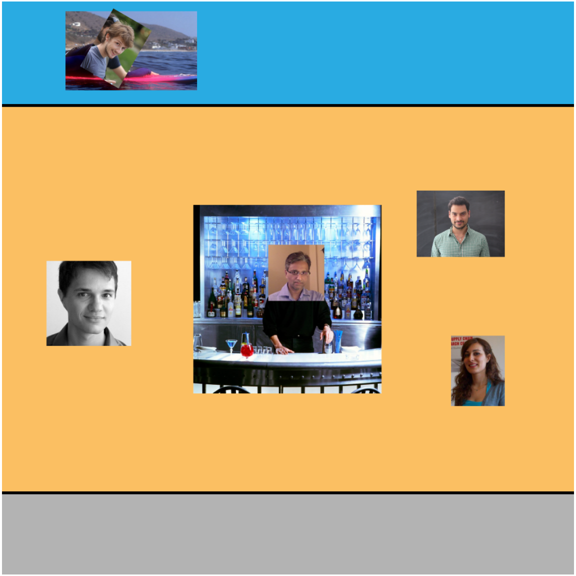
Game theory is complicated enough with only two players, but when the number explodes...

- Use 'population' arguments (congestion games...)
- Use 'algorithmic' arguments (learning...)
- Play on networks (local effects...)

Many connections between one another.

Today we are looking at way to move to the continuous using PDEs and mean field arguments.

If you like piña coladas...



The Beast

MFG "basic" system

$$\begin{cases} \frac{\partial u}{\partial t} + \nu \Delta u + H(x, \nabla u) + V[m] = 0 \\ \frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}\left(m \nabla_p H(x, \nabla u)\right) = 0 \\ m\big|_{t=0} = m_0, \, u\big|_{t=T} = g[m_T] \end{cases}$$

Why it is worth looking at

Very general model, extends far beyond economics. The previous system admits as particular cases famous equations such as the Euler equations or the Vlasov ones.

Piggybacks on very active fields of research, such as optimal transport or stochastic control.

Has found a good deal of numerical methods for solving these types of games, taking it out of the theoretical forest (pedestrian crossing animation).

Derivation of the MFG system

We will derive The Beast from a simple model. Two steps:

- Agents optimize their choices given the density of other players (our m) $\Rightarrow$ control in feedback, gives backward equation.
- The densities are transported by these choices $\Rightarrow$ forward equation.

Plan

- The optimizing agent
- The transported density
- Symmetric N-player games

The agent's problem

We have a nice space Y (take $\mathbb{R}^d$, or call it *the beach*), with a distance d. Look at time interval $[0, T]$.

For all $t \in [0, T]$, suppose the density of the players on Y, m_t is given.

The agent controls (using $\alpha \in A$) the stochastic equation

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$$

where B_t is a Brownian motion in $\mathbb{R}^d$.

The agent's problem (cont.)

The agent solves the following stochastic control problem

$$\begin{cases} \inf_\alpha \mathbb{E}\Big[\int_0^T \Big(L(X_t, \alpha_t) + V[m_t](X_t) \Big) dt + g[m_T](X_T) \Big] \\ dX_t = \alpha_t dt + \sqrt{2\nu} dB_t \\ X_0 = x \end{cases}$$

Define Hamiltonian H

$$H(x, p) = \min_{\alpha \in A} \{ L(x, \alpha) + \alpha \cdot p \}$$

Hamilton-Jacobi-Bellman

The value function u satisfies

$$\begin{cases} \frac{\partial u}{\partial t} + \nu \Delta u + H(x, \nabla u) + V[m] = 0 \\ u\big|_{t=T}(x) = g[m_T](x) \end{cases}$$

A closer look at the V

V is the cost incurred from the density of other players. Can give a general example:

$$V[m](x) = \underbrace{V_0(x)}_{\text{I like the bar}} + \overbrace{f(x, m(x))}^{\text{I don't like people}} + \underbrace{\int_Y W(x,y)m(dy)}_{\text{But not too alone either}}$$

BUT

- With congestion term $f(x, m(x))$, no existence of equilibrium (need $V[m] \in C(Y)$)
- With distance term $\int_Y W(x,y)m(dy)$, no uniqueness (need $V[m]$ to be monotonous in m, e/g $V[m](x, m(x)) \geq V[m](y, m(y))$ if $m(x) \geq m(y)$)

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Movement of agents

So far we have considered the measure m as given, but if agents move, m does too.

Agents find an optimal control $\bar{\alpha}(x)$ such that

$$\nabla_p H(x, p) = \bar{\alpha}(x)$$

This drift is their motion.

Transport of the measure

Now, for all t, $m_t = \mathcal{L}(X_t)$, with $m_0 = \mathcal{L}(X_0)$ given. So m_t is the solution of

$$\begin{cases} \int_Y \phi \, dm_t = \int_Y \mathbb{E}[\phi(X_t^x)] \, dm_0(x) \\ dX_t = \alpha_t dt + \sqrt{2\nu} dB_t \\ m_0 = \mathcal{L}(X_0) \end{cases}$$

ϕ is a test function ($\in C_c^\infty(Y)$)

Fokker-Planck equation

m_t is governed by

$$\begin{cases} \frac{\partial m}{\partial t} - \nu \Delta m + \mathrm{div}\,(m\bar{\alpha}) = 0 \\ m|_{t=0} = m_0 \end{cases}$$

The Beast, dissected

Clear where this system is coming from?

$$\begin{cases} \frac{\partial u}{\partial t} + \nu \Delta u + H(x, \nabla u) + V[m] = 0 \\ \frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}\left(m \nabla_p H(x, \nabla u)\right) = 0 \\ m\big|_{t=0} = m_0, \, u\big|_{t=T} = g[m_T] \end{cases}$$

The linear-quadratic case

Let $L(x, \alpha) = \frac{|\alpha|^2}{2}$, then

$$H(x, p) = \min_{\alpha} \frac{|\alpha|^2}{2} + p \cdot \alpha = -\frac{|p|^2}{2}$$

The MFG system now reads

$$\begin{cases} \frac{\partial u}{\partial t} + \nu \Delta u - \frac{|\nabla u|^2}{2} + V[m] = 0 \\ \frac{\partial m}{\partial t} - \nu \Delta m - \operatorname{div}(m \nabla u) = 0 \\ m|_{t=0} = m_0, u|_{t=T} = g[m_T] \end{cases}$$

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Symmetric N-player games

We can look at games with N players and see what happens when $N \to +\infty$.

We assume all players are identical, choose an action $x_i \in Y$. Cost function for player i is

$$J_i(x_i, x_{-i}) = F(x_i, x_{-i})$$

F is symmetric in the sense that for all permutations σ on $N-1$, $F(x_i, x_{-i}) = F(x_i, (x_{\sigma(j)})_{j \neq i})$.

If $N \to +\infty$, then the number of variables in F also goes to infinity...

Ashes to ashes, points to measures

The trick: map positions $x = (x_1, ..., x_N) \subset Y$ of N players to the empirical measure $m_x^N \in \mathcal{P}(Y)$

$$m_x^N = \frac{1}{N} \sum_{i \in N} \delta_{x_i}$$

Then if $x = y$ up to a permutation, $m_x^N = m_y^N$.

(Some like to think of x and y as belonging to the quotient space $Y/\mathfrak{G}_N$)

Distances on $\mathcal{P}(Y)$

Who says limits says distances, and we have a couple options for our space of measures $\mathcal{P}(Y)$.

Let $m, \tilde{m}$ in $\mathcal{P}(Y)$. Define the 1-Wasserstein distance by

$$W_1(m, \tilde{m}) = \begin{cases} \inf \int_{Y \times Y} d(x, y)\, \gamma(dx, dy) \\ \gamma \in \Pi(m, \tilde{m}) \end{cases}$$

$\Pi(m, \tilde{m})$ is the set of couplings between m and $\tilde{m}$, i/e joint distributions whose marginals are m and $\tilde{m}$.

Connections with optimal transport here...

Déblais et remblais

Problem: X and Y have the same volume, find the cheapest transportation plan between them.

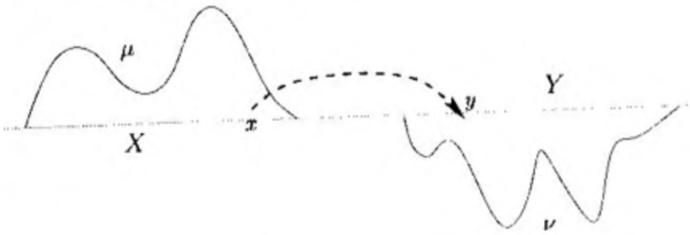


Figure: Pulled directly from *Topics on Optimal Transportation*, C. Villani

W_1 for empirical measures

To nail our transition from Y to $\mathcal{P}(Y)$, we show our distances translate naturally.

Take x and y in Y^N, define $\bar{d}$ a distance such that

$$\bar{d}(x,y) = \min_{\sigma \in \mathfrak{S}_N} \frac{1}{N} \sum_{i \in N} d(x_i, y_{\sigma(i)})$$

Then $W_1(m_x^N, m_y^N) = \bar{d}(x,y)$ (uses Birkhoff's theorem on bistochastic matrices).

Looking back

- We started from a space Y of players' positions.
- We moved from Y to $\mathcal{P}(Y)$ by using empirical measures.
- We defined a distance on the space $\mathcal{P}(Y)$, the 1-Wasserstein distance.
- This distance metrizes the weak topology on $\mathcal{P}(Y)$, i/e if $m_n \rightharpoonup^* m \Rightarrow W_1(m_n, m) \to 0$.

$\Rightarrow$ We can take limits in $\mathcal{P}(Y)$!

The limit of functions with N variables

Let $F_N : Y^N \longrightarrow \mathbb{R}$ symmetric, $F_N(x) = F_N(m_x^N)$. Under nice assumptions, there is a function $F \in C(\mathcal{P}(Y))$ such that:

$$\sup_{x \in Y^N} |F_N(m_x^N) - F(m_x^N)| \longrightarrow_{N \to \infty} 0$$

Back to games

We now have a strong conceptual background to look at limits of games.

Assume each player has cost function of the form $F_i^N(x_1, ..., x_N)$. Our previous theorem says we can send N to infinity.

Asymptotically, players' costs will be close to

$$F\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right)$$

for some $F : Y \times \mathcal{P}(Y) \to \mathbb{R}$.

Nash equilibrium in mean field

We arrive at our mean field theorem.

Mean field equation

Assume that for all N, $X^N = (x_1^N, ..., x_N^N)$ is a Nash equilibrium for the game $(F_1^N, ... F_N^N)$. Then, up to a subsequence, the sequence of empirical measures (m^N) converges to a measure $\bar{m}$ such that

$$\int_Y F(y, \bar{m})\, d\bar{m}(y) = \inf_{m \in \mathcal{P}(Y)} \int_Y F(y, \bar{m})\, dm(y)$$

We did the reasoning for pure strategies, can extend to mixed
$\Rightarrow$ guarantees existence of NE.

Where to go

Once we have all this, we can tie up the two parts (derivation of the MFG system and symmetric N-player games).
Assume N players, each one controls system in the form

$$\int_0^T L_i(x_1(t), ..., x_N(t), \alpha_i(t))dt + g_i(x_1(T), ..., x_N(T))$$

Can assume

- $L_i(x_1(t), ..., x_N(t), \alpha_i(t)) = \frac{|\alpha|^2}{2} + F\left(\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right)$
- $g_i(x_1(T), ..., x_N(T)) = g\left(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right)$

Nash equilibrium

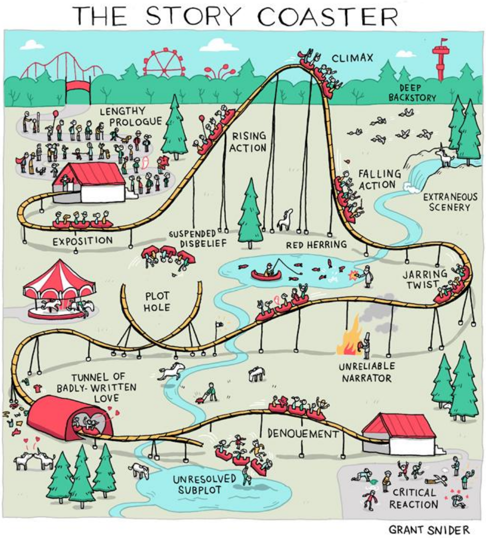
We assume there is a map U_i^N such that

$$U_i^N(x_i, t, (x_j)_{j\neq i}) = U^N(x_i, t, (x_j)_{j\neq i})$$

and it satisfies a bunch of HJB equations

$$\begin{cases} \frac{\partial U_i^N}{\partial t} + \frac{1}{2}\left|\nabla_{x_i} U_i^N\right|^2 - F\left(\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right) + \sum_{j\neq i}\nabla_{x_j} U_j^N \cdot \nabla_{x_j} U_i^N = 0 \\ U_i^N = g_i,\ t = T \end{cases}$$

In other words, we have a direction for the player to follow, and the optimal control $\bar{\alpha}_i = -\nabla_{x_i} U_i^N$ is a Nash equilibrium (can prove that).


THE STORY COASTER
CLIMAX
DEEP BACKSTORY
LENGTHY PROLOGUE
RISING ACTION
FALLING ACTION
EXTRANEOUS SCENERY
EXPOSITION
SUSPENDED DISBELIEF
RED HERRING
JARRING TWIST
PLOT HOLE
UNRELIABLE NARRATOR
TUNNEL OF BADLY-WRITTEN LOVE
DENOUEMENT
UNRESOLVED SUBPLOT
CRITICAL REACTION
GRANT SNIDER

Suspended disbelief

We know we can push N to infinity and the symmetric function U^N will have a limit U, but we need to do it in the space of probabilities $P(Y)$.

Using estimates on $\nabla_{x_i} U^N$, we can switch to that space (actually using the 2-Wasserstein distance...)

We then give ourselves the initial measure m_0 and push it forward using the obtained U (and its derivatives).

Resulting system

With a bit more work we obtain:

$$\begin{cases} -\frac{\partial u}{\partial t} + \frac{1}{2}|Du|^2 = F(m) \\ \frac{\partial m}{\partial t} - \operatorname{div}(mDu) = 0 \\ m\big|_{t=0} = m_0, u(x,T) = g(x, m(T)) \end{cases}$$

References

Good references out there, I used:

- P.L Lions videos at the Collège de France (some are in English)
- P. Cardaliaguet's notes (online)
- Lecture notes from class taught by G. Carlier in Dauphine

Cocorico!

Thank you!