Introduction To Mathematical Thinking And Inquiry

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This is the first draft of a textbook on Mathematical Thinking and Inquiry. It attempts to introduce students to discovering conjectures, coming up with proofs, defining objects and setting up systems of axioms. Please get in touch if you find the stuff in here interesting, have some feedback or are interested in working with ThinQ, a network committed to Inquiry Oriented Education in schools and colleges. The ideas in this book have been tested out on first year college engineers and 9th grade students, and they have worked very well in both instances. Videos of some of these sessions will soon be uploaded to YouTube. Please feel free to use the ideas in this book, and print and distribute the material from this book. However, do this on the condition that you drop me an email giving me feedback!

Table Of Contents

Introduction to Mathematical Thinking and Inquiry

Mathematics education, especially at the school level but also in undergraduate education, has largely focused on the following:

1. Understanding an existing body of mathematical knowledge
2. Learning some mathematical procedures and techniques of the form a computer could carry out
3. Applying the existing body of knowledge and the procedures/techniques

The abilities required by mathematical researchers are:

A. Coming up with conjectures
B. Validating/disproving those conjectures by searching for counter-examples
C. Coming up with proofs for those conjectures
D. Extending conjectures outside their domain of applicability to see if they have broader applicability
E. Setting up systems of axioms and defining objects
F. Evaluating proofs

So, the mathematical abilities acquired in educational institutions (if there are any) largely have no relationship to the mathematical abilities required by researchers. Notice that the abilities required by researchers all require creativity. Even F is not always formulaic. Otherwise, a computer would be validating proofs rather than a community of mathematicians. Creativity is largely absent in the case of education and in some cases is actively discouraged.

A few years ago, I setup an online learning/assessment platform, which would teach/test mathematical skills/procedures. The aim of this was that the computer could deal with teaching students these skills (which are required for examinations), while teachers (who otherwise waste a lot of time on grading and teaching these skills) could work on mathematical abilities. This was not happening. I soon realised that this was because there were no appropriate materials available to teach these abilities from. Books by people such as Polya, in their current form, are slightly out of reach of most middle and high school students.

This book is a start to a project aimed at addressing mathematical abilities required for research. Though written in a way accessible to fifteen year old students, this book can be useful for teacher education, individuals interested in mathematical thinking, college students studying related disciplines, as well as first year mathematics undergraduates. Since the book does not really require much pre knowledge, it may also be useful for some younger students with a keen interest in mathematics.

Mathematical thinking, though obviously useful to aspiring mathematicians, has wider applicability. My sessions with students, using the contents of this book, have brought out basic logical difficulties they face. For example:

- the relationship between P -> Q and Q -> P,
- the concept of 'only if', and
- when an example will suffice to prove/disprove a statement and when you require a justification

Also, tools such as generalising within a system, extending outside of a system, defining objects precisely, and clearly stating your assumptions/premises/axioms are all useful outside of mathematics.

As mentioned, this book is just a start, and there is a long way forward for all of us who are convinced of the need to include mathematical thinking and inquiry as an important component of school and college education. Some of the crucial parts of this are:

I. Making explicit intuitions and heuristics mathematicians use – This is not easy and might not always be possible. However, there might be certain pointers mathematicians can give students, after introspecting, on how to arrive at conjectures or proofs. It might be the case that some ways of proceeding work for some students while other ways work for others. Hence, it is important that a large number of mathematicians engage in this exercise. Some basic techniques of proceeding are outlined in this book such as when and how to generalise or where you should be looking to extend results or taking simple examples in order to find patterns.

II. Creating mathematical theories appropriate for students being introduced to mathematics – Things like Euclidean geometry and the study of the reals are not necessarily ideas which should be broached at the school level. Points being zero dimensional or the various counter intuitive results which result from real analysis are not appropriate for students. Even calculus as it is currently taught is not useful to our purpose since students will not be able to come up with many conjectures or come up with proofs in calculus. The mathematical theories, which need to be constructed for students, will largely not be interesting to mathematical researchers. They must be theories which students can construct right from axioms and definitions to proofs (with guidance). The theory this book is based on might reside on a Euclidean plane, but takes straight lines as primitives rather than points, and does not have the concepts of a curved line. (In fact, you do not need the entire continuum of the Euclidean plane to construct this geometry. The concluding chapter contains more about this). As far as possible these theories should address a question, which students can understand through their experience, such as Euler's bridges of Konigsberg problem.

III. Creating Learning Resources based on I and II – Once theories are developed, material has to be created to transfer mathematical abilities, along with possible heuristics, to students. These need to be developmentally appropriate as well as interesting. These materials include self-learning material for students such as online courses, books, video and audio, and material for teachers such as lesson plans, resource packs and textbooks. An example of such material outside of this book could be resources aimed at students constructing Graph Theory through examination of real world situations.

IV. Convincing schools, school systems and governments on the need for these materials – Once learning resources are created, they must be used in schools in order to convince people of their utility and usability. These sessions should be recorded and made available publicly so that support builds for their use. There are entrenched beliefs and interests which will protect the current system.

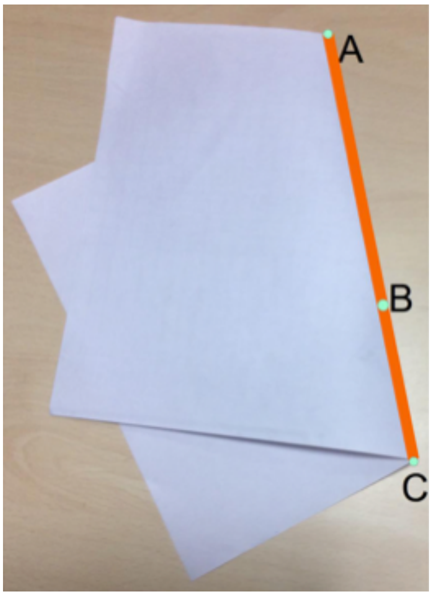
V. Creating teacher development programmes for using these materials – Once materials are created for students, the same materials can be used for teacher education coupled with pedagogical and assessment strategies.

To accomplish I-V we need mathematicians, educators, policy makers, and teachers to work together

Paper Folding - Right Angle

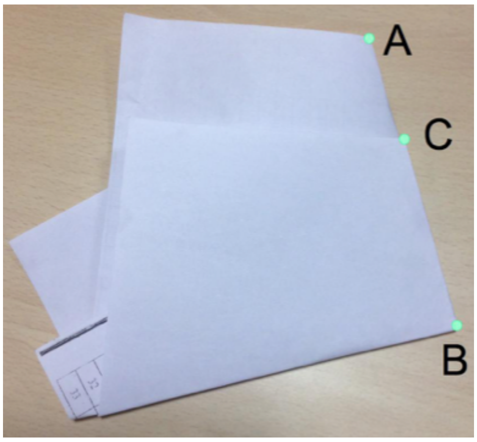
Take a blank piece of paper

Fold it over once. It will look something like:

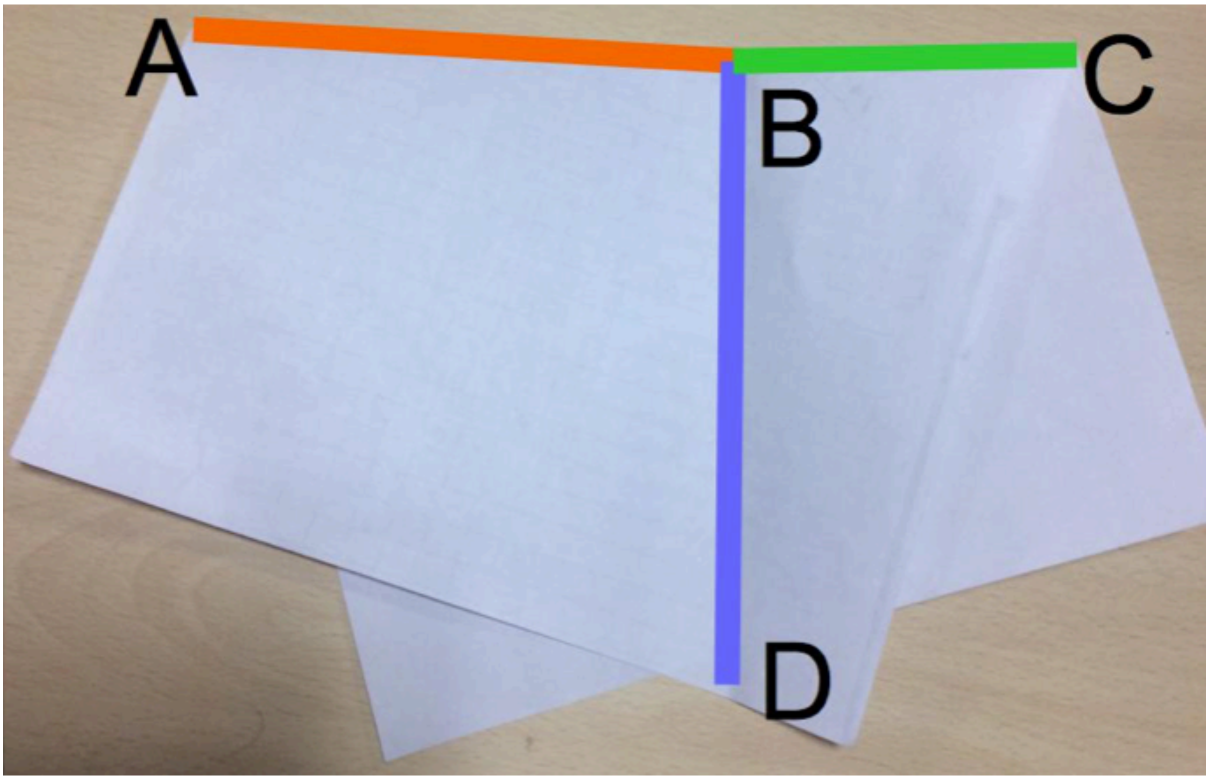


Now, pick a point on the edge you folded. The point you picked is represented by the letter B on the image below. A and C are the end points of the line you created with your first fold.

Fold the paper again such that line BC is on top of line AB



Open up the second fold. You will see that both the angles here are right angles as below:



Is this always the case or can you find an example where, by following the same procedure, you get a different result? If you can find such an example, it is called a counter-example. We will get back to that later.

The statement you consider to be true is called a conjecture. State what you consider to be true clearly.

If you fail to find a counter-example, and you are convinced that this must be true, the statement is called a plausible conjecture.

What You Did in This Section

1. Found a pattern in a procedure you completed
2. Stated what that pattern was
3. Introduction to the words conjecture, counter example, plausible conjecture

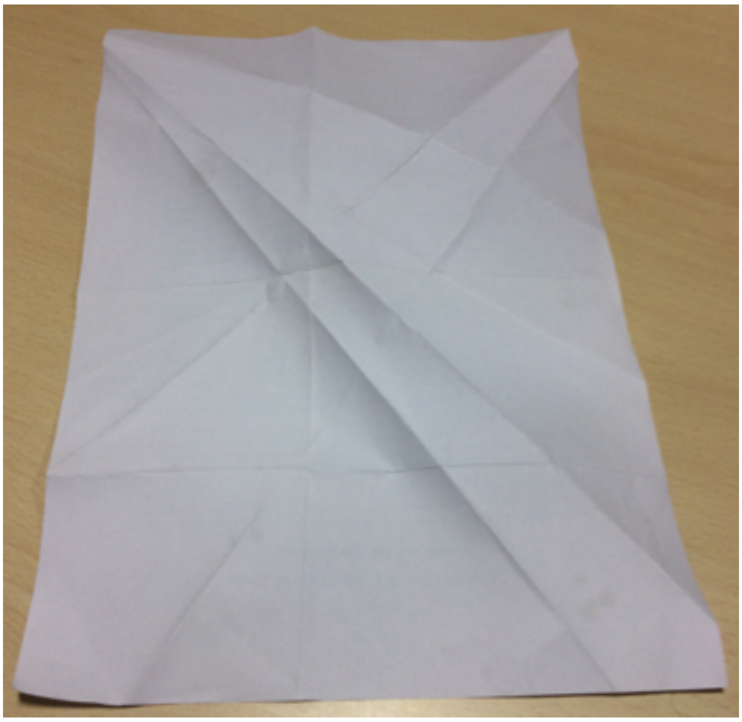
Some Help

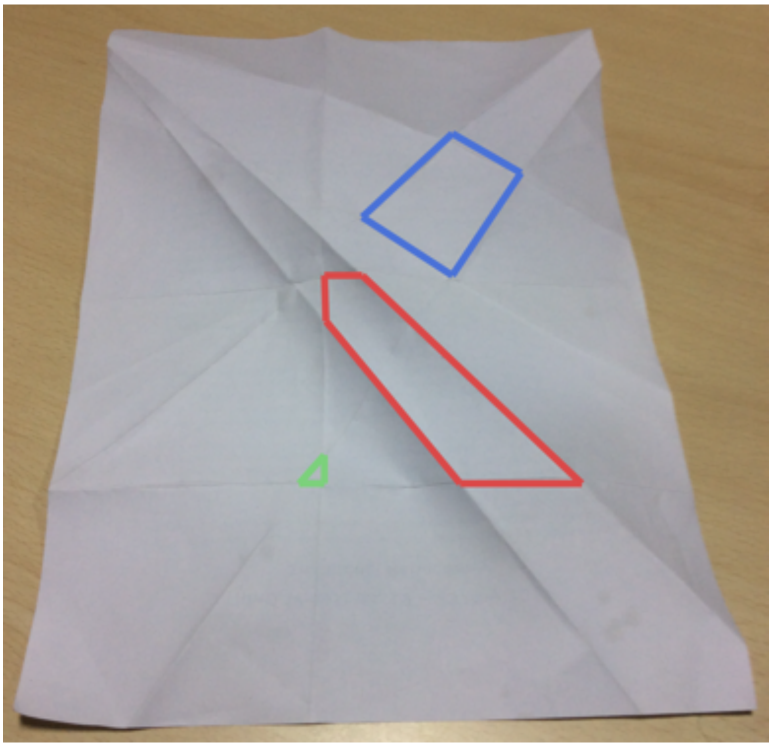
The conjecture:

If you follow the given folding procedure on a piece of paper, and open the paper, the creases formed by the folding are perpendicular to each other (intersect at right angles)

Paper Folding - Colouring

Take another sheet of paper. Fold it over and open it. Now fold it again in a different place and open it. Go crazy doing this again and again. Finally you will have a sheet of paper which looks something like:





You will see a large variety of shapes which have been created by these lines. We will call these shapes ‘regions’. Some examples are outlined in the image

The question we are asking is: Given a bunch of colours, what is the least number of colours required to colour the regions such that no two neighbouring regions share the same colour?

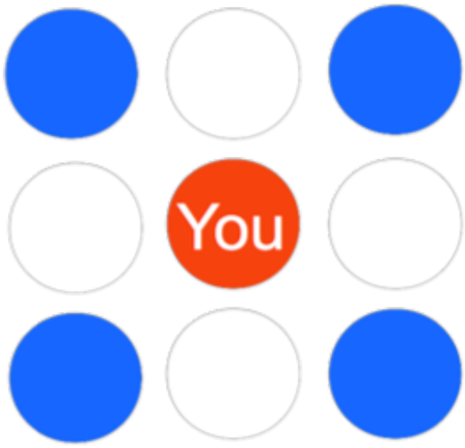
The first thing you must be clear about is the wording of the question. The one phrase in the question which might appear to be vague is 'neighbouring regions.' What are neighbouring regions?

Think about your classroom. Clearly, the people to your left and right are your neighbours. How about people in front and behind you? Are they your neighbours? See the image below:

Define neighbour such that only the coloured dots around you are your neighbours:

Now, lets look at the people diagonally situated to you. Are they your neighbours? See the image below:

Define neighbour such that only the coloured dots around you are your neighbours:



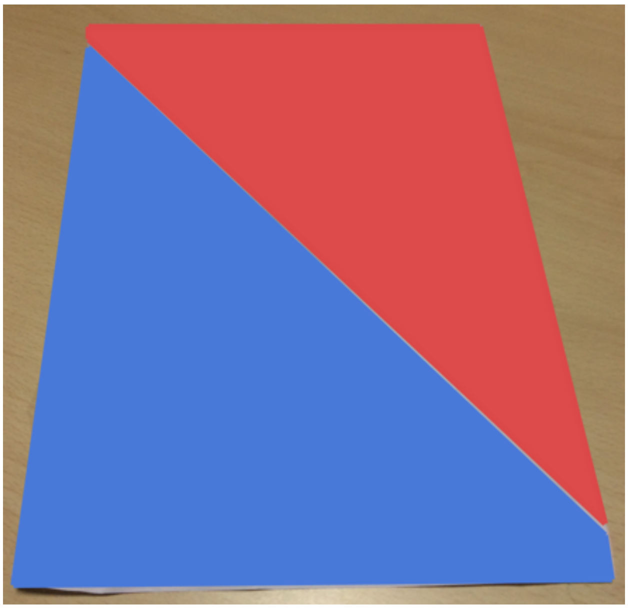
Now, lets take a third conception of neighbours:

Define neighbour such that all of the coloured dots around you are your neighbours:

Let’s be clear about what we mean by neighbouring regions. For now, let’s say that two regions are neighbours if they share an edge in common.

Go back to your sheet of paper. Try colouring it in so that you are using the least number of colours. This seems a little too hard, doesn't it?

The first thing a mathematician would do in this case would be to generalise the situation. Rather than looking at your specific configuration, a mathematician would ask the question: If you followed the procedure above, given n folds, what would be the minimum number of colours required to colour the sheet such that no two neighbouring regions have the same colour? Put down any ideas you might have:



One way a mathematician would proceed is to try a few simpler examples in order to find a pattern. Take another sheet of paper (hopefully, you are using scrap paper for this activity and not wasting brand new sheets) and fold it once. Now try and answer the above question for this configuration.

Now try two folds and so on. Have you come up with a possible answer in general? State this as a conjecture in precise language. We will get back to this later.

What You Did in This Section

1. Tried to answer a question by going through various complex examples
2. Clarified the meaning of some words in the question
3. Generalised the question
4. Tried out simple examples of the generalised question
5. Stated a conjecture in response to the question

Some Help

The conjecture could take the form:

If you follow the folding procedure above, the most number of colours required to colour the paper, such that no two neighbouring regions have the same colour, is _______.

My Maternal Grandmother

Here is a conversation which was overheard at a bank:

Jomo: "I'm Jomo. Its nice to meet you. While we are waiting in line, let me tell you a little bit about myself. I was born when my mother gave birth to me. When I was three, my brother was born. At the age of four, I started school and when I was five, my biological maternal grandmother was born. There was a lot of celebration on this occasion and there was a huge party..."

Miko: "Stop Lying!"

Jomo: "Are you saying I am lying about when my brother was born?"

Miko: "Maybe."

Jomo: "So then, how are you so sure I am a liar, if all you are saying is 'maybe'?"

Miko: "You are lying about your grandmother."

Jomo: "How can you be so sure? Anyways, I'm really insulted by what you are saying."

What is so different about the two claims made above, about Jomo's brother and Jomo's biological maternal grandmother? Is Miko right in saying that Jomo is lying about her grandmother with so much surety? Why?

The first thing we need to do here is to be very clear about the claim we are troubled by. Let's restate it:

"Jomo was five when her biological maternal grandmother was born"

Are we troubled by the 'five'? What if it was 'four' or 'three'? Would you be satisfied? Why or why not?

It wouldn't matter. Let's restate more clearly what is troubling us:

"Jomo was born before her biological maternal grandmother was born."
Now, as we did in the last section, we need to define certain words, namely 'biological maternal grandmother.' To do this, lets first try and define 'biological mother.'

"Jomo's biological mother is the person who gave birth to Jomo"
So, lets try defining biological grandmother:

"Jomo's biological grandmother is the person who gave birth to Jomo's mother."
We can also write this as:

"Jomo's biological maternal grandmother is Jomo's mother's mother."
Is this enough to prove that Jomo's claim is false? Why can't somebody be born before the person who gave birth to them is born?

It's very hard to answer why. So, lets just take that as an assumption that a person cannot be born before the person who gave birth to them is born. In mathematics, this sort of assumption is called an 'axiom'. Let us all it Axiom 1.

Axiom 1: If X gives birth to Y, then X was born before Y

So, what is our final argument? Lets try putting it down clearly:

1. Assume Jomo's maternal grandmother was born after he was.
2. We know that Jomo's maternal grandmother is Jomo's mother's mother, so Jomo's maternal grandmother gave birth to Jomo's mother, from our definition of mother.
3. From Axiom 1, Jomo's maternal grandmother must be born before Jomo's mother.
4. Similarly, Jomo's mother must be born before Jomo.
5. So, Jomo's grandmother must be born before Jomo, which contradicts statement 1.

The argument given above is a 'Proof by Contradiction' since we showed that assuming a certain statement is true, it must be false as well. Since a statement can't be both true and false, the initial statement must be false.

There were also some other axioms in our argument which we smuggled in implicitly. Can you figure out some of them? (Hint: One of them concerns relationships of birth order)

You have seen how a mathematician would approach a problem. Now, how would a scientist answer the question we posed in this section, and answered by just thinking about it? How is science different from mathematics in this regard? (Think about the concept of sampling)

Take a look at the following claims. Which of these could be true? Which of these are false for sure? Which of these are true for sure? Why? Articulate your reasons precisely from axioms and definitions. Compare your proofs with those of your friends and critique each others' work.

A. My younger brother was born after me.

B. My younger brother was born 2 years after me

C. My aunt was born before me

D. My aunt was born after me

E. Humans are descended from apes

F. My older brother was born after me

G. I was born before my mother gave birth to me

H. My older brother was born before me

What You Did in This Section

1. Compared the surety of truth of two different questions
2. Generalised a statement we were troubled by in order to bring out clearly what bothered us
3. Introduction to the concept of axiom
4. Clearly stated an axiom
5. Introduction to the concept of proof
6. Clarified and defined words in the statement precisely
7. Example of a proof from axioms and definitions
8. Considered the difference between scientific and mathematical proof
9. Tried to find hidden axioms in our proof
10. Went through examples of various statements to apply what we learnt in the rest of the exercise

Some Help

One possible hidden axiom in our proof was that if A is born before B and B is born before C, then A is born before C. Lets call this ‘transitivity of birth order’.

Creating Worlds

In the previous section, we attempted to set up a mathematical world, by defining objects precisely and setting up a system of axioms. What we then did was to try and figure out the logical consequences of these axioms on the objects.

Now, lets move away from mothers, grandmothers and brothers, and instead return to a similar situation to the first two sections.

The worlds we will be operating in from now on are similar to that of your folded piece of paper. We will be working on a flat surface. We will be a little more precise about it later. What you have in this world are points and straight lines, similar to those in your world of paper folding.

We have two possibilities of worlds based on their size:

1. a finite world with definite boundaries, like your sheet of paper.
2. an infinite flat world which goes on forever in all directions

We also have three possibilities, based on the length of the straight lines

A. only finite straight lines allowed (what are also called line segments)

B. only infinite straight lines allowed, which go on forever in both directions

C. both finite and infinite lines allowed

If not specified later, by default, assume we are working in a world which satisfies condition 2 and condition C. These are some of the axioms of this world

I am setting one other condition in this world: not more than two lines can intersect at a given point. This gives us another axiom.

Based on the conditions set so far, let me ask you a questions: Given 4575 straight lines and exactly 25 points of intersection per line, what are the total number of points of intersection? Write what comes to your mind:

This sounds like an exceptionally hard problem, just like the colouring problem in the second section. Lets do what we did there and generalise. Try and state a general version of the above question before you look below:

"Given n straight lines and exactly i points of intersection per line, how many points of intersection are there in total?" Is your phrasing of the question saying the same thing?

There are still a few ambiguous concepts in this phrasing. Lets start with the word 'intersection.' Lets define a pair of intersecting lines as a pair of lines which have a finite number of points in common.

The obvious question which this leads to is: Can two intersecting straight lines have two points in common or three points and so on? Do you think this is possible? Why or why not?

Intuitively, it is probably obvious to you that two straight lines cannot have 2 or 3 or any other finite number of points in common. They can only have one point in common. However, in mathematics, intuition can only point us towards a result. We need a rigorous proof from axioms and definitions in order to move forward.

You have some choices here. You could either assume this to be an axiom, or you could state some axioms from which you can prove this. In mathematics, one of the aims is to make the axioms as simple as possible. So, we should try creating an axiom to prove this.

Consider the axiom, "Given two distinct points, there is only one straight line which starts are one of the points and ends at the second one." Is this good enough or do you need to add in another few axioms?

Try and prove the conjecture using your axioms and definitions like we did in our proof in the previous section

(Hint: The first thing you will have to do here is to define straight line. This is a lot harder than you think. Consider the idea of 'shortest distance')

What You Did in This Section

1. Set up various worlds
2. Decided which world we would be operating in
3. Asked a question in that world
4. Generalised that question
5. Clarified the generalised question
6. Speculated on the answer to the question
7. Proved a statement we might have found intuitive
8. Defined an object precisely

Some Help

Proof to the Conjecture that a pair of distinct straight lines can have at most one point in common:

Axiom: Given two distinct points, there is only one distinct straight line starting at one point and ending at the other.

Assume a two straight lines intersect with each other at more than one point. If you start from one end of any of the lines moving towards the other end, the points of intersection will come one after the other.

Take the first two points of intersection (we can do this since there can only be a finite number of points of intersection by our definition of intersection). Let these points be called A and B.

Definition: A straight line is the shortest distance between any two points.

Let AB be the straight line between A and B. Since AB is the shortest distance between A and B, it must be a part of both of the lines. Hence, the two lines do not have a finite number of points in common. This contradicts our definition of intersection.

Hence, two lines can have at most one point of intersection in common.

Engaging with our Question

Lets return to the generalised version of our question:

"Given n straight lines and exactly i points of intersection per line, how many points of intersection are there in total?"

Lets continue with what we did in the colouring section, and take some simple examples. How about 1 straight line and 0 points of intersection per line? Thats easy, right? Take a straight line, finite or infinite, and place it on our flat surface. It will have no points of intersection. How many are the total points of intersection? Well they will also be 0.

Now, take the case of 1 straight line and exactly 1 point of intersection per line. Try drawing a representation of this case

I'm trying really hard but I just cannot create this configuration. Maybe we should try 1 straight line and exactly 2 points of intersection per line.

This also seems really hard. I just can't do it. Can you? Try it out.

There is something going on here. I'm suspicious. I'm beginning to think that if the number of intersection per straight line is greater than or equal to the number of straight lines, the configuration is impossible to create! Is that true? Can you prove it from a result we arrived to in the last section? (Hint: try proving first that given n straight lines, no one straight line can have more than (n-1) intersections)

So, maybe we need to re-look at our question. The question asks us how many points of intersection there are of a given configuration. However, if there are situations where the configuration is not possible, the question is meaningless.

So, lets ask a more fundamental question before we move on to the question above: "Given n straight lines and exactly i points of intersection per line, is it possible to create such a configuration?" Write your reflections on what you think the answer to this new question is.

What You Did in This Section

1. Explored simple examples of our generalised question
2. Realised that our question might not be valid
3. Created a new question which we will need to answer before we can answer our original question

Some Help

Proof that the number of points of intersection per line has to be less than the total number of lines:

Given n straight lines, take one of the lines and call it m. There are n-1 lines left. m can intersect with a single straight line at most at one point (from the theorem we proved in the last section). Hence, m can intersect with n-1 straight lines at most at n-1 points. Since, we cannot create a configuration where a single line can have more than n-1 points of intersection, we certainly cannot create a configuration where all the lines have n-1 points of intersection.

Testing Our Conjecture

In the last section, we found a situation when we could not construct a configuration and we proved that we could not. That was in the case that the number of points of intersection per line was more than or equal to the number of straight lines. This was kind of obvious. So, maybe in every other condition we will be able to create the configuration?

Lets state our new conjecture: "Given n straight lines and i points of intersection per line, you can create this configuration if and only if $n > i$."

There is a new term we have used above: 'if and only if.' What does this mean? I'm sure you know what each of the component words of this term means, but try and mull over what the entire term means and write down your thoughts before you proceed:

Whenever you see 'if and only if,' break up the entire statement into two parts, like:

1. Given n straight lines and i points of intersection per line, you can create this configuration if $n > i$
2. Given n straight lines and i points of intersection per line, you can create this configuration only if $n > i$

Does this make things clearer?

Lets try and rephrase 2:

3. If n is lesser than or equal to i, you cannot create a configuration of n straight lines and i points of intersection per line.

Are 2 and 3 logically equivalent? Why or why not?

For the original conjecture to be true, both 1 and 2 must be true. If either 1, 2 or both are false, the conjecture is false. We already proved 3 in the last section, which is equivalent to 2. Now, lets inquire into 1, and continue with examples.

Before we go forward, lets introduce some notation to make things less cumbersome. Let (n,i) represent a configuration of n straight lines and exactly i points of intersection per line. Then, we can re-state our question as:

"Is (n,i) a possible configuration?"

We can re-state 1 as: "(n,i) is a possible configuration if n > i"

The example we had stopped at was (1,1). Lets ignore any configuration where n <= i. Try the following: (2,1) and (3,1). Remember the condition we had set in a previous section that not more than two lines can intersect at a given point. (2,1) seems easy.

Can you prove it is possible? (Hint: To prove something exists, you just have to construct it).

So far so good. It seems that our conjecture is working. How about (3,1)? Can you construct (3,1)?

(3,1), which looks easy at first glance, might not be easy. Can you prove that (3,1) is not a possible configuration? (Hint: Make two lines intersect with each other and think about what becomes of the third line)

So, now what happens to our conjecture? We have found an example which contradicts 1 and hence contradicts the conjecture. This is another instance of a 'counter-example.' It requires just one counter example to show that a conjecture is false. Now, the conjecture has either to be amended or it must be thrown in the trash. Mathematicians have to be willing to discard their hard work in coming up with a conjecture even if there is just one counter-example. Lets throw it away for now!

Can you find Counter-examples to the following conjectures? If not, can you attempt to prove them:

1. (n,1) is always a possible configuration for all n

2. (n,1) is always a possible configuration if $n > 1$

3. (n,1) is always a possible configuration if $n > 3$

4. (2,i) is always a possible configuration for all i

5. (2,i) is always a possible configuration for 2 > i

6. (3,i) is never a possible configuration for any i

What You Did in This Section

1. Stated a conjecture to answer a question
2. Explored examples of that conjecture
3. Found a counter example to that conjecture
4. Explored the concept of counter example
5. Tried to explore various conjectures
6. Introduced notation to make statements less cumbersome

Some Help

2. Given n straight lines and i points of intersection per line, you can create this configuration only if n > i

3. If n <= i, you cannot create a configuration of n straight lines and i points of intersection per line.

Are 2 and 3 logically equivalent?

If a statement S is true only if another statement T is true, then S must be false if T is false.

So, if T is false S must be false.

	T True	**T False**
S True	Statement either True or False	Statement is False
S False	Statement either True or False	Statement is True

Is (3,1) a possible configuration?

Conjecture: (3,1) is not a possible configuration

Proof: Call the lines a,b and c. Take a and b. These two lines can at most intersect with each other at one point (from a previous theorem we proved). Now, we have two cases

1. a and b intersect at no points
2. a and b intersect at exactly one point

Lets take case 1 first. a and b do not intersect and we only have c left. We have 4 possible cases

I. c does not intersect with either a or b - this does not satisfy the conditions of (3,1) as each of the lines have no points of intersection

II. intersects with only a - this does not satisfy the conditions of (3,1) as b has no points of intersection

III. c intersects with only b - this does not satisfy the conditions of (3,1) as a has no points of intersection

IV. c intersects with a and b - this does not satisfy the conditions of (3,1) as c has two points of intersection

Now, lets take case 2. If a and b intersect exactly at one point, we have the same possibilities for c

I. c does not intersect with either a or b - this does not satisfy the conditions of (3,1) as c has no points of intersection

II. intersects with only a - this does not satisfy the conditions of (3,1) as a has two points of intersection

III. c intersects with only b - this does not satisfy the conditions of (3,1) as b has two points of intersection

IV. c intersects with a and b - this does not satisfy the conditions of (3,1) as a, b and c have two points of intersection.

Generalising Conjectures

Lets re-state our question: Given two Natural Numbers n and i, Is (n,i) a possible configuration? (Given the notation we developed in the previous section)

We have two variables in our question, n and i. One way to proceed in trying to find patterns is to keep one of the variables constant and change the other. We have kind of tried doing that with n. We tried (2,0), (2,1) and (2,2), and found some were possible but others were not.

Just for now, lets try keeping i constant. First lets try (n,1).

We know that (1,1) is not possible, but (2,1) is. We also know that (3,1) is not possible. How about (4,1) and (5,1)? While proving these, keep in mind the proofs of (2,1) and (3,1).

1. (4,1) is possible/impossible

2. (5,1) is possible/impossible

3. (6,1) is possible/impossible

4. (7,1) is possible/impossible

Do you see a pattern emerging? Can you state it as a conjecture? Try and state it as an 'if and only if' conjecture. (Hint: You will see (n,1) is possible for some values of n but impossible for others. Try and think about what sort of values of n it is possible for what values it is not possible for. Look at the proof for (2,1) existing and (3,1) not being a possible configuration and see how they can be generalised)

For all $n > 1$, (n,1) is possible if __________

For all $n > 1$, (n,1) is possible only if __________

For all $n > 1$, (n,1) is possible if and only if _________

Now, try and prove the conjecture:

Proof for: For all $n > 1$, $(n,1)$ is possible if __________

Proof for: For all $n > 1$, $(n,1)$ is possible only if __________

Proof for: $(n,1)$ is possible if and only if __________

For all $n > 1$, $(n,1)$ is possible if __________

And, For all $n > 1$, $(n,1)$ is possible only if __________

Hence, For all $n > 1$, $(n,1)$ is possible if and only if __________

What we did in this section was to take some examples, (2,1), (3,1), (4,1), (5,1) etc. For some we proved that the configuration was possible, while for others we proved that it was not. In the numbers for which it was possible, we found a pattern. We also saw a pattern in the numbers for which it was not possible. So, we were able to generalise from a few examples to all $n > 1$. This is somethings mathematicians do all the time. They find a bunch of patterns, and prove conjectures. Then they try to find patterns in those conjectures, and try to state a more general conjecture.

What You Did in This Section

1. We decided to explore a family of examples of our Conjecture
2. We came up with a conjecture on that family of examples
3. Introduction to ‘if and only if’
4. Proved our conjecture

Some Help

For all $n > 1$, $(n,1)$ is possible if n is even

For all $n > 1$, $(n,1)$ is possible only if n is even

For all $n > 1$, $(n,1)$ is possible if and only if n is even

Proof for: For all $n > 1$, $(n,1)$ is possible if n is even

If n is even, you can break the lines into pairs (from the definition of even). For each pair, make the two lines in that pair intersect. Now, each of the lines have exactly one point of intersection. Hence, $(n,1)$ is possible if n is even.

Proof for: For all $n > 1$, $(n,1)$ is possible only if n is even

We can translate this conjecture into: For all $n > 1$, if n is odd, $(n,1)$ is not a possible configuration

Take any of the n lines. It has to intersect with exactly one other line for the configuration to work. Once you make two lines intersect at one point, no other line can intersect with either of those two lines. Otherwise, the lines would have more than one point of intersection. You can keep doing this. You will see that what you are forced to do is to pair up lines. Finally you have one line left, which will cannot intersect with any other line since they already have one point of intersection. Hence, it is impossible to create $(n,1)$ if n is odd.

Proof for: $(n,1)$ is possible if and only if n is even

Since For all $n > 1$, $(n,1)$ is possible if n is even

And, For all $n > 1$, $(n,1)$ is possible only if n is even

Hence, For all $n > 1$, $(n,1)$ is possible if and only if n is even

Extending A Result

When a conjecture is proved, it is called a theorem. In the last section we proved that:

“if n > 1, (n,1) is possible iff n is even”

(iff stands for if and only if)

Maybe the same result could work for (n,2). What we are doing here is taking a result with one domain of application and applying it in a new situation. However, when you do this remember to be very careful and prove the result again. What was a theorem in one situation is now just a conjecture. Lets state our conjecture:

“if n > 2, (n,2) is possible iff n is even”

The first thing we need to do here is to break up the conjecture into it two components.

1. if n > 2, (n,2) is possible if n is even
2. if n > 2, (n,2) is possible only if n is even

Lets attack 1 first. For 1, we only have to consider even n. Start with 4. Since we are only asking for existence, all we have to do is construct a configuration of (4,2):

Now try (6,2). Try taking your construction of (4,2) and extend it to (6,2).

Can you now think of a general way to construct (n,2) when n is even? Write out your description of the construction.

So far, so good. We seem to have a proof for 1. How about for 2?

2: if n > 2, (n,2) is possible only if n is even

Lets re-state 2 in terms of odd numbers

2': if n > 2, (n,2) is not possible if n is odd

Note: This is only possible because if a number is not even, it must be odd. However, if rather than even and odd. It was leaves a remainder of 0 when divided by 4 and leaves a remainder of 1 when divided by 4, if the conjecture was: if n > 2, (n,2) is possible only if n is divisible by 4, we could not translate it to: if n > 2, (n,2) is not possible if n leaves a remainder of 1 when divided by 4. So, lets translate from 2 to 2' step by step:

2: if n > 2, (n,2) is possible only if n is even

if n > 2, (n,2) is not possible if not (n is even)

if n > 2, (n,2) is not possible if n is not even

all numbers which are not even are odd

2': if n > 2, (n,2) is not possible if n is odd

In this case, you are being asked to show something is not possible. You can prove the statement to be false by just providing a counter-example - which in this case would be a single construction of (n,2) for some odd n.

Try (3,2) and (5,2). Are any of them counter-examples? Try drawing them

What this shows us is that we need to be very careful when extending results outside their initial domain of applicability. We have shown that 2 is false and hence our initial conjecture in this section was false.

Maybe (n,2) is not just possible for n even. It might also be possible for n odd. However, take a look at your proof of 1 for n even. Can that proof be extended to n odd. Try stating your construction for all n precisely:

So, in some cases we can extend results without a problem. Trying to extend results to new domains is a great tool for mathematicians to quickly figure things out about their new world. However, it is also something fraught with danger if you aren't very careful.

What You Did in This Section

1. We attempted to extend a result outside its domain of application
2. We explored that result in our new domain of application and found a counter example
3. Our wrong conjecture, however, gave us a direction in which to inquire
4. We extended a proved conjecture outside its domain of application and found that it worked

Uniqueness

Lets look at some examples from the previous sections. Look at (4,2). We know we create it by just creating a square. However, is a square the only way to create (4,2)? Try finding another way:

You have probably realised that rectangles work as well as squares, and so do trapeziums and parallelograms. In fact (4,2) can be created by any quadrilateral. The question which comes to mind is: Can (4,2) only be created by a quadrilateral, or are there other ways to create (4,2)? How about (5,2)?

Now try (6,2)? Is there more than one way of constructing (6,2)? (Hint: Take a look at the construction of (3,2))

Can you state a general conjecture for when there is only one way to construct (n,2) and when there is more than one way?

Now, how about if I put forward this conjecture:

"(n,2) can only be constructed by a combination of one or more polygons." Is this true? Can you try and prove it? (Try defining 'polygon')

What You Did in This Section

1. We came up with a question as a result of a result related to another question
2. We explored examples of our new question
3. Introduction to the concept of uniqueness

Proceeding From Here

There is a lot left in order to completely answer the problem we set for ourselves. Till now, we have only been able to address a few examples. The rest of the problem is for you to engage with. However, let me give you some ways to think about how to proceed. Our generalised statement had two numbers, represented by n and i. If we proceed like we have so far, addressing specific values of n and i, we will not get very far. If we wish to completely answer the question, we need to completely classify possibilities for n and i in such a way that the number of classes is finite. For example, a finite classification could be one which has the following three classes:

A. Where n is divisible by 3
B. Where n leaves a remainder of 1 when divided by 3
C. Where n leaves a remainder of 2 when divided by 3.

Humans have finite lifespans, so this is the only way that we can make a serious dent in the problem. This classification must also tell us something about the problem, however, this is only something we will know after the fact. The above classification might be useful or it might not. We can take a look at the examples we have taken so far and see whether they indicate the usefulness of such a classification or not. An example of usefulness in the above classification would be if we find out: In all cases of A and B, the configuration is possible, while in C it is not. If this works, it completely answers our question.

An additional possibility is trying to return to our original question, which asked 'how many points of intersection are there?' Sometimes, answering related questions could give us an insight into the question we are dealing with. However, if you have never dealt with questions related to 'how many,' you may have to read a book/watch some videos on a discipline within mathematics called 'Combinatorics.'

Musings on Mathematics and Education

Rational Number Geometry

I have been involved in a lot of discussions recently on discrete geometries in education. I have come to the realisation that such geometries are a better way of introducing Mathematical Thinking, especially at a younger age. I have been exploring various different discrete geometries including Taxi Cab geometry and some Pixel Geometries. This got me thinking about this distinction between discrete and continuous, especially in the case of Rational Number Geometry. Euclidean Geometry is isomorphic to Real Number Geometry (on a Cartesian plane). Similarly, think of a Cartesian Plane with only rational coordinates. The question which first occurred to me was: Is Rational Number geometry continuous?

This leads us to unpacking the concept of continuity. There are at least two notions of continuity (I am deliberately not using the language of analysis here such as limit points etc). The first is in relation to the space we are operating in: a space is continuous if between any two points, there exists a point (The concept of 'between' deserves some clarification, but lets assume we understand what it means for now). By this, Rational Number Geometry is continuous. The second notion is in relation to objects in that space, that they do not have any 'holes.' To stay away from concepts like 'limits,' I would first like to establish a result about straight lines in Rational Geometry.

Take a straight line in Euclidean-Cartesian geometry with rational arguments: $ax+by=c$ where a, b and c are rational. $y = (c-ax)/b$ and $x = (c-by)/a$. We get the following: x is rational if and only if y is rational. Lets take all of the points of the form (rational, rational) and transport them to Rational Geometry. Every x coordinate in rational geometry has a corresponding y coordinate, and every y coordinate has a corresponding x coordinate. Also, between any two points in this set, there is another point. Another interesting result is that given a line with one point of the form (rational, rational), all points on the line in Euclidean-Cartesian Geometry will be of the form (rational, rational) or (irrational, irrational).

Lets say such objects (transported from Euclidean Geometry) in rational number geometry are continuous and are straight lines. Now, lets define continuity for any other line in rational geometry the following way:

A one dimensional object/set of points in rational geometry is continuous if and only if no straight line (I am including straight line segments as straight lines) can pass through it without intersecting with it (intersecting means having a point in common).

The only slightly ambiguous concept in the above definition is ‘pass through.’ A straight line passes through a one dimensional object if it has (x, y) values arbitrarily close to a point on that object from two directions without having a point in common.

Continuity:

I. Straight lines, as constructed above, are continuous

II. Any other one dimensional object is continuous if and only if no straight line passes through it without a point in common

Conjecture: There are no continuous one dimensional objects in rational geometry apart from, trivially, straight lines

If this conjecture is true, it is quite startling since continuous circles (as in the second notion of continuous; circles as in the set of points equidistant from a given point) can exist in some discrete geometries.

The reason I think this arises is in relation to the notion of distance in rational geometry. The only coherent metrics in Rational Geometry seem to be the same ones which are coherent in Euclidean Geometry. Hence, distances (between points) in Rational Geometry can probably not be confined to just rational numbers.

If you use the regular Euclidean-Cartesian notion of distance in Rational geometry, it is easy to see why circles cannot be continuous. Here’s why:

Lets confine ourselves to circles centred at the origin. Circle with radius r will be defined by the following: distance of every point on the circle = r. So, $(x^2 + y^2) = r^2$. If r is irrational, we can set x=0 (which is rational) and y will come out to be irrational. If r is rational, set x=r/2, which will also be rational. However, y^2 will equal 3r/4 and hence y will be irrational. So, no matter what, we will always get a point of the form (rational, irrational). Hence, in rational geometry, there will always be values of x for which there is no corresponding y value. Hence, the circle will not be continuous in rational geometry.

Repeating Decimals

I was looking at Middle School textbooks and stumbled across the notion of repeated decimals. I thought back to my seventh grade and remembered by discomfort with this notion. The following represents the ‘proof’ that 0.99..=1.

1. x=0.999…
2. Hence, 10x = 9.999…
3. Subtract 1 from 2
4. We get 9x=9
5. Therefore, x=1

Of course, once you arrive at step 4, 5 follows. However, there are two steps above which are not completely satisfactory. What allows us to subtract one repeated decimal from the other? We do not have an algorithm to do that, and even if we do that algorithm is not shared with seventh grade students. The retort I have got to that is: Even if we do not have an algorithm, we can subtract two things from each other which are equal. However, what tells us that they are equal. If we have two sets, of natural numbers, one of which has all the natural numbers and the other has all the natural numbers apart from 1. In one sense, they are equal, i.e. they have the same cardinality. However, if we take every element from the second set away from the first, we will be left with a set containing 1 and not an empty set. What axiom, definition or theorem tells us that this is not the case in the above proof?

I agree that the limit of a particular sequence of 9s after a decimal point is 1. If that is what we mean by 0.999…, we should make that explicit and clearly 7th graders should never be shown this. However, I see a deeper problem with infinite sums. What are the rules for manipulating infinite sums? When people say 1+2+…=-1/12, they mean a very different thing by '=' than the regular notion. However, you can arrive at that by relatively straightforward manipulations. Either we have to accept that 1+2+…=-1/12 with the regular notion of '=' or we have to give reasons/rules why some manipulations are permissible while others are not.

General Remarks

There are various areas such as Discrete Geometries, Knot Theory, Group Theory and Graph Theory, which would offer students a much better glimpse into the world of mathematics that Euclidean Geometry, Real Numbers and Calculus/Pre-calculus, which end up with students learning up rules/formulae and at best being able to solve tricky, Math GRE-type problems (However, as demonstrated by Michael Starbird's course for the Teaching Company, Calculus can be approached as a study of change largely ignoring computations). The next project I hope to take on would be related to Graph Theory.