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Notes for Teachers on
From Defining to Conjecturing and Proving

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A note from Tara, which will explain the nature (and limits) of my reflections (with her consent):

For a learning trigger designed for a classroom to get into a form that can be used as a learning trigger for (potential) teachers (or others), it needs to go through several stages. A first draft of the task needs to be tried out at least a couple of times in an actual classroom of the appropriate level, before it can be meaningfully revised. A second round of implementation and revision, either by the same person/team, or another, would improve it further. The revised versions could in principle turn out to be very different from the original draft in both substance and sequencing. Feedback on the revised version from other teachers, and even students, with further revisions, would yield an even more sound and decent learning task.

The learning trigger that you used as for discussion trigger 2 was still at the first-draft stage, so it isn't going to be as effective as it could potentially have been. Too late to think of that now. Fortunately, there has been some discussion of the tasks among participants. But for 'Reflections' on it, all you can do is base what you say on your experience of doing it in your class (unfortunate, again, that it's an undergrad class, but that can't be helped), and relate it as much as you can (if you have the time) to the discussion in the forum.
-tara

For the Teacher

Goal of the learning tasks of this unit (the intended learning outcomes)

To help learners:

- get a clearer, deeper *understanding* of the concepts of definition, conjecture, and proof;

and

- further strengthen the *capacity* to:
- think carefully through concepts;
- come up with definitions on the basis of (1), and formulate them;
- check for logical contradictions between the logical consequences of the definitions/axioms and the intuitive judgments they are based on;
- in case of conflict, modify either the definitions/axioms or the judgments;
- come up with conjectures and formulate them;
- prove the conjectures to establish them as theorems; and
- choose between alternative definitions on the basis of their potential to generate the widest range of non-trivial theorems from the smallest number of definitions.

The core of these abilities — with slight variations — is important beyond mathematical inquiry, for theoretical science, moral inquiry, and religious inquiry as well. Hence these trans-disciplinary abilities would constitute one of the foundations for any meaningful school/college education that aims to develop the capacity for thinking and inquiry.

The material in the notes 'For the Teacher' is not meant as an instruction manual that provides a set of pedagogical procedures to implement in a step-by-step manner. The design of the *curriculum*, and of the *learning materials* to implement the curriculum for a given grade, have to be guided by our estimate of the students' mental maturity and prior knowledge. The implementation of learning materials in a particular classroom depends on the teacher's estimate of the abilities and interests of that particular cohort, and most importantly, on the answers/solutions that the students come up with in response to the tasks, the doubts they raise (if any), and the questions they ask. For this, as we all know, there cannot be a recipe.

Part 1

Take a look at the circles, ellipses, and polygons in figure 1:

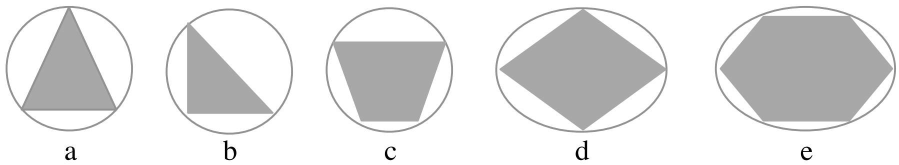


figure 1

As you already know, the diagrams in (1a) and (1b) are examples of a triangle inscribed in a circle. You also know that inscribing is the converse of circumscribing: in (1a) and (1b), the circle circumscribes the triangle. In (1c) a quadrilateral is inscribed in a circle (the circle circumscribes the quadrilateral). In (1d) an ellipse circumscribes a quadrilateral. In (1e) a hexagon is inscribed in an ellipse. To generalize, these are examples of polygons *inscribed* in circles or ellipses; alternatively, they are examples of circles or ellipses *circumscribing* polygons.

In figure 2, the relation is reversed: the polygons circumscribe the circles and ellipses:

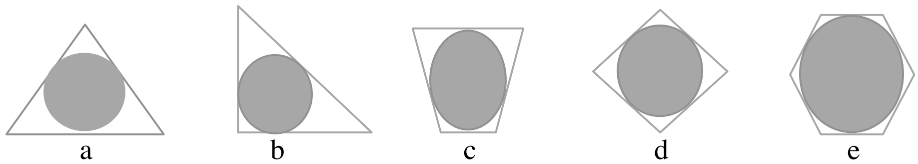


figure 2

To generalize further, we may say that the diagrams in (1) and (2) illustrate the relation of ***inscription-circumscription*** between circles and ellipses on the one hand and polygons on the other. In contrast, you would not consider the relation between the two shapes in each of the diagrams in figure 3 as one of inscription-circumscription:

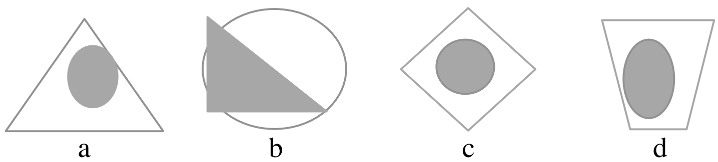


figure 3

Task 1: Define the terms *inscription* and *circumscription* such that it follows that the diagrams in (1) and (2) are instances of the inscription-circumscription relation, but not those in (3). (Note: Your task is not to find out how *textbooks* define the relation: it is for ***you*** to *construct* a definition that fits **your** judgments of the relation in the figures.]

Task 2: Consider the diagrams in fig. 4. Would you judge the relation between the two shapes in each of these diagrams as inscription-circumscription? Depending on your definition of the relation in Task 1, the answer could be yes or no.

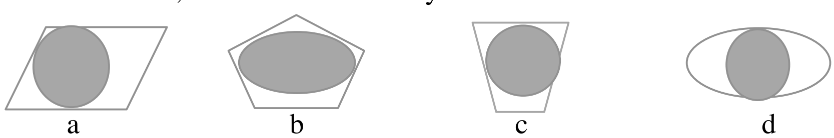


figure 4

Task 3: Suppose I judge the relations in the diagrams in fig. 4 as instances of inscription-circumscription. Does my judgment follow from your definition in Task 1? If it doesn't, say why, and modify the definition such that the judgment follows from the definition.

Task 4: Suppose I judge the relations in the diagrams in fig. 3 as not being cases of inscription-circumscription. Does my judgment follow from your definition in Task 1? If it doesn't, say why, and modify the definition such that the judgment follows from the definition.

On Part 1: For the Teacher

In inquiry-based pedagogy, it is next to impossible to anticipate student responses to most tasks. We can't predict, for instance, how they would define circumscription (task 1). Their responses would depend at least partly on level (e.g., high school vs. college). They would also vary across institutions, and even within the same institution across cohorts.

In the reflections below, I am simply thinking out loud what I might do if I were to implement this unit in a given classroom in a particular institution as an example. After designing the unit, I tried it out in a year 1 undergraduate class (part of my course on scientific inquiry at IISER-Pune). So these reflections are informed by that experience.

I will do this in two parts, the first dealing with curved shapes (circle, ellipse, etc.) circumscribing polygons, and the second dealing with polygons circumscribing curved shapes.

Curved shapes circumscribing polygons

Having tried out the material, I would modify the first task in the unit by drawing figs. (1a) and (1b) on the blackboard, and asking the students if they knew that these were examples of circumscription. I assume that year 1 undergraduate students in India would say yes. To contrast with (1a) and (1b), I would draw figs. (ia) and (ib), and ask if they were examples of circumscription:

figure i

I am guessing they would say no. At this point we can ask, "What is circumscription? How would you define it, such that (1a) and (1b) are examples of circumscription while (ia) and (ib) are not?"

This question is designed to introduce them to a specific inquiry tool: *to begin with a contrast (the asymmetry between their judgments on these two sets of diagrams) and to come up with a set of propositions (in this case a definition) such that the contrast follows as the logical consequence of the propositions (i.e., what they predict).*

If they do not know what to say, we might prompt them by writing the beginning of a possible answer on the blackboard, and ask them to complete it:

A circle X circumscribes a triangle Y iff:

[We may have to tell them "iff" means "if and only if".]

In a year 1 undergraduate class, chances are that the students would come up with the following:

Def 1a*: A circle X circumscribes a triangle Y iff every vertex of Y lies on X.*

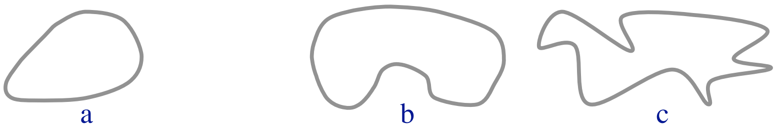
Definition 1 correctly predicts the desired results. We could now proceed to test the definition further by enlarging its scope. [I am borrowing the terms 'predict' and 'test' from scientific inquiry deliberately to signal what mathematical inquiry has in common with scientific inquiry.] For instance, we could draw a circumscribed rectangle, circumscribed pentagon, and so on, and ask how they would generalize the concept of circumscribing to include the new diagrams. At this point we might include an ellipse circumscribing a rectangle.

Given the new examples, students might revise Def 1a as follows:

Def 1b*: A circle/ellipse X circumscribes a polygon Y iff every vertex of Y lies on X.*

Such revision of earlier proposals is an important part of scientific inquiry.

To generalize further, we might introduce other *curved shapes* such as the following.



a b c ***figure ii***

Alternatively, we might focus on curved shapes without any concavity, such as that in fig. iia, leaving concavity for later. If we exclude concavity, they might modify Def 1b as Def 1c:

> ***Def 1c***: *A curved shape X circumscribes a polygon Y iff every vertex of Y lies on X.*

Polygons circumscribing curved shapes

Our definition of circumscription is based on our judgments on the diagrams in fig. 1 (and its extensions). We may now point to the asymmetry between figs. 1 and 2. We judge the diagrams in (2) as examples of circumscription; does Def 1c derive correct results for these diagrams?

The answer is no, because a circle has no vertex. Suppose we restrict our attention to the contrast between fig. 2 and fig. 4 (a, b, c), either of the following definitions would work:

> ***Def 2a***: *A polygon X circumscribes a curved shape Y iff every edge of X touches the periphery of Y.*

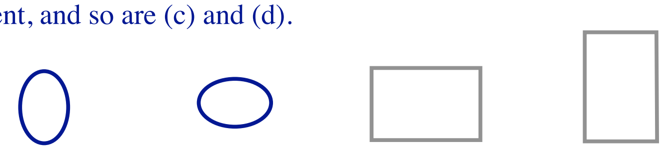
> ***Def 2b***: *A polygon X circumscribes a curved shape Y iff every edge of Y is a tangent on X.*

But Def 2a and Def 2b won't work if we consider non-square rectangles circumscribing circles (fig. iiia) or squares circumscribing non-circle ellipses:

a b ***figure iii***

In these diagrams, only two of the edges of the polygon touch (or is tangent to) the curved shape. By Def 2a and Def 2b, (iiia) and (iiib) are not instances of circumscription. One solution to this problem would be to accept that the diagrams are not instances of circumscription. If we judge them to be instances of circumscription, we need to modify the definition.

To resolve this dilemma, we might exploit the technical concepts of 'congruence' and 'similarity' in math. We say that two shapes are *congruent* if we can put one on top of the other such that the periphery of one lies exactly on top of the periphery of the other. In fig. iv, shapes (a) and (b) are congruent, and so are (c) and (d).



a b c d ***figure iv***

Now, if we enlarge or reduce (iva) and (ivc) using a photocopy machine, we can get (va) and (vb):

a (enlarged) b (reduced) ***figure v***

These shapes are not congruent with (iva) and (ivb), and their *sizes* are different; yet, they have the same *form*. Form is something that is invariant under enlargement/reduction. So we say that (iva) and (va) are similar; so are (ivc) and (vb). Appealing to this mathematical concept of *similarity* of shapes, we may propose the following definition:

Def 2c: A polygon X circumscribes a curved shape Y iff
(i) X completely contains Y, and
(ii) no enlargement of Y is completely contained by Y.

The intuition that (i) and (ii) above express is that of Y *snugly fitting into* X. We might now say that the concept of circumscription in definition 2c is that of one shape snugly fitting to another.

Two Distinct Concepts of Circumscription

Let us compare Def 1c and Def 2c:

Def 1c: A curved shape X circumscribes a polygon Y iff every vertex of Y lies on X.

Def 2c: A polygon X circumscribes a curved shape Y iff
(i) X completely contains Y, and
(ii) no enlargement of Y completely contains X.

These are two distinct definitions, one for curved shapes circumscribing polygons and the other for polygons circumscribing curved shapes. Can we generalize them such that a single definition applies to both cases? We could generalize (1c) as (1d), and (2c) as (2d):

Def 1d*: Shape X circumscribes shape Y iff every vertex of Y lies on X.*

Def 2d*: Shape X circumscribes shape Y iff*
(i) X completely contains Y, and
(ii) no enlargement of Y completely contains X.

Now consider the diagrams in fig. vi:

figure vi

Given Def 2d, a circle circumscribes a pentagon in diagram (vi.a). But given Def 1d, (vi.a) is not an instance of circumscription because the convex vertex of the pentagon does not lie on the circle. Conversely, given Def 1d, a closed curve circumscribes a rectangle in diagram (vi.b). But given Def 2d, (vi.b) is not an example of circumscription because the rectangle violates condition (i) of the definition. Such differences in the “predictions” (i.e., the logical consequences) of definitions, as we shall see, have important consequences for task 13, the task in part 6.

In each diagram in figs. 1 and 2, a pair of shapes illustrates *the relation between the two shapes*. Intuitively, that relation is the same in fig. 1 and fig. 2, regardless of the actual shapes. Yet, Def 1d and Def 2d treat the two figures as unrelated, and remain arbitrary, as they fail to capture the sameness of the relation.

If, on the other hand, we judge the relation to be distinct, intellectual hygiene demands that we use different terms for the different concepts. In short, we must *either* distinguish the concepts and how we refer to them, *or* look for a single unified definition to cover both. This is the most important learning point of Part 1.

Task 2 has to do with figs. 3 and 4. Problems arise only if we judge the relation in fig. 4 as one of inscription-circumscription.

Part 2

Now consider the following generalizations involving the inscription-circumscription relation between shapes:

(1) For every triangle, there is exactly one circle that can be inscribed in it.
(2) For every triangle, there is exactly one circle that can circumscribe it.
(3) For every circle, there is exactly one triangle that can be inscribed in it.
(4) For every circle, there is exactly one triangle that can circumscribe it.

[Note: "exactly one" means "one and only one".]

We will use the term ***conjecture*** to refer to statements of this kind. Do you think these conjectures are likely to be true? Think carefully. For each conjecture, if you think it is quite likely to be true, write "plausible" against it, then do the following tasks.

Task 5: For each conjecture in (1)-(4) that you judge as plausible, come up with a proof to demonstrate that is true.

Task 6: For each of the remaining conjectures, if you think it is false, come up with a proof that demonstrates that it is false.

On Part 2: For the Teacher

The goal of part 2 is to sensitize students to the concept of ***plausible*** and ***implausible*** conjectures, as distinct from conjectures that are neutral in their plausibility: [1]

Conjectures: Neutral; Plausible; Implausible

Theorems: Conjectures that have been proved to be true.

Part 3

In Tasks 5 and 6, we restricted ourselves to the inscription-circumscription relation between circles and triangles. Let us explore further.

Task 7: Think of other pairs of shapes, e.g., ellipses and triangles, circles and quadrilaterals, circles and squares, circles and pentagons, ellipses and squares, and so on. Try to come up with as many plausible and implausible conjectures as you can.

Task 8: Prove that the conjectures you judge to be plausible are true and those that you judge to be implausible are false.

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[1] G. Polya's two-volume book, *Plausible Reasoning in Mathematics*, is an invaluable resource for the inquiry processes that underlie the transition from neutral conjectures to plausible/implausible conjectures. The heuristics and modes of reasoning that Polya outlines for this step are identical to those in scientific inquiry. Daniel J Velleman's book, *How to Prove It: A Structured Approach*, deals with the strategies of establishing plausible conjectures as true (and implausible ones as false). The distinction between the two steps is an important thread in the process of developing the capacity to think like a mathematician.

It is equally important for students to see that:

Conjecture in mathematics is the counterpart of *hypothesis* in science: both are propositions that we think will turn out to be true, and proceed to look for ways to establish them as true.

Theorem in mathematics is the counterpart of *prediction* in science: the theorems of a mathematical theory are the logical consequences of the definitions and axioms of the theory, and the predictions of a scientific theory are the logical consequences of its definitions and general principles.

Despite the parallels, there is a subtle difference between math and science. In math, proof and explanation are not distinct. So, if we ask, "Why is this conjecture true?" the answer is of the form: "…because it follows logically from the definitions and axioms of the theory."

In science, when we ask, "Is this observational hypothesis true?" we respond by making observations, and support the claim of truth by showing that it follows (inductively) from the observations. But we must then ask: "why is this observational hypothesis true?" In response, we construct a theory, and show that the hypothesis follows as a prediction from the definitions and general principles of the theory. If now asked, "Why should we accept the general principles of the theory as true?" we defend them by pointing out that our explanation is the best one available, and hence must be taken to be true until we find either counter-evidence or a better explanation.

Part 4

In thinking about the inscription-circumscription relation so far, we have taken 'triangle' as a general category. Would the conjectures change if we specify the relation as being between circles/ellipses and right-angled triangles? Would they change if the relation is between circles/ellipses and equilateral triangles? Likewise, would it be different for the relation between circles/ellipses and parallelograms? How about circles/ellipses and rectangles? How about circles/ellipses and squares?

Task 9: Including special cases like the ones just mentioned, revise, or add to, the conjectures you have formulated so far.

Task 10: Needless to say, after discovering a conjecture, you need to prove it.

You probably already know that once a conjecture has been proved to be true, it is called a ***theorem.*** If you reflect on the theorems and conjectures that you had encountered before you engaged with tasks (1)-(10), you would realize that most of the conjectures and theorems that emerged from your engaging with tasks (1)-(10) were discovered by ***you*** (individually or collectively), on the basis of your own thinking, doing, observing, and reasoning. In other words, they were the results of your own inquiry.

On Part 4: For the Teacher

The goal of part 4 is to guide the students' attention to the strategy of ***specializing*** in the discovery of conjectures (as the converse of generalizing). Right-angled triangles and equilateral triangles, for instance, are special cases of triangles. The Pythagoras theorem expresses a relation in right-angled triangles that does not exist in triangles in general. Likewise, the generalization that all the angles of a triangle holds only in equilateral triangles (special) and not in all triangles (general).

Consider, for instance, the truth of the following conjectures:

Conjecture (i): For every circle, there is exactly one isosceles triangle that the circle circumscribes.

Conjecture (ii): For every circle, there is exactly one triangle that the circle circumscribes.

Now consider the following conjecture:

Conjecture (iii) For every triangle, there is exactly one circle that the triangle circumscribes.

Is this conjecture true under definition (1d)? Under definition (2d)?

Part 5

In tasks (3) and (4), you came up with two distinct definitions of inscription-circumscription. Which of these definitions did you go by in tasks (5)-(10?) What would happen if you adopted the other definition for tasks (5)-(10)?

Task 11: Go through tasks (5)-(10) on the basis of the alternative definition of inscription-circumscription.

Task 12: Compare the two sets of conjectures and theorems, one based on the definition you originally adopted for tasks (5)-(10), and the other you adopted for task 11. Reflect on which set of conjectures and theorems are more significant or interesting. On the basis of the comparison, say which definition you think is better, and why.

On Part 5: For the Teacher Anticipatory thoughts without classroom trial:

The patterns that we see in a collection of mathematical objects would depend on the particular objects we choose to populate the mathematical world with. The definitions we formulate would be geared towards generating these objects, and the choice of definitions will determine which patterns we see and formulate as conjectures. This means working backwards: depending on which pattern we wish to have in our theory, we will have to make choices among alternative definitions.

The goal of part 5 is to help students become aware of this facet of questions-answers/problems-solutions/puzzles-explanations.

This is not a point that a school student would understand readily, but it is worth understanding, because its counterparts are true for scientific inquiry and moral inquiry as well. In scientific inquiry, we formulate our definitions and general principles with an eye on which observations we wish to predict; in moral inquiry, we formulate our definitions and moral principles with an eye on which moral judgments we wish to predict. Understanding these trans-disciplinary strands is an important aspect of the integration of academic knowledge and inquiry.

Part 6

So far, we have restricted our attention to two-dimensional geometric shapes like circles, ellipses, and polygons. Let us expand the concept of inscription-circumscription to shapes that are non-geometric, like this one:

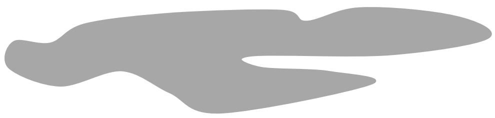

figure 5

Suppose you wish to include geometric shapes circumscribed by non-geometric shapes, non-geometric shapes circumscribed by geometric shapes, and non-geometric shapes circumscribed by non-geometric shapes?

There is an important problem in mathematics called the inscribed square problem: Does every plane of simple curve contain all four vertices of some square? The Wikipedia entry on the problem gives the following diagram by way of illustration:

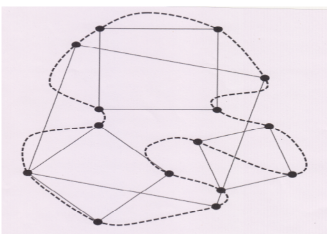

figure 6

The Wiki entry tells us that the problem was proposed by Otto Toeplitz in 1911, and that it remains an unsolved problem in mathematics. Notice that the formulation of the problem as an *inscribed* square problem requires a definition of a geometric shape inscribed in a non-geometric shape. If all the four squares in the diagram are inscribed in the non-geometric shape, what does "X is inscribed in Y" mean?

Task 13: Define "X is inscribed in Y" such that all four squares in fig. (6) are examples of the inscription-circumscription relation.

To make matters more daunting, we might think of generalizing the circumscription-inscription relation to three-dimensional geometric shapes (cones, spheres, ellipsoids, polyhedrons,…) and three-dimensional non-geometric shapes.

On Part 6: For the Teacher

The goal of Part 6 is to guide students' attention to two things. One is the strategy of ***generalizing*** in the discovery of conjectures (the opposite of what we did in part 4). The other is the need to ensure that the predictions of the definitions we choose are logically consistent with our judgments.

Notice that definition Def 1a, based on diagram a fig. 1, is the least general: it covers only the circumscription of a triangle by a circle. Def 1b and Def 1c attempt to generalize from circle to closed geometrical curves and from triangles to polygons. Even after such generalization, Def 1c is still too specific.

Def 2 is equally specific: it covers only those cases in which a polygon circumscribes a circle/ellipse. Part 1 seeks to generalize the definition of circumscription by expanding its scope to cover both sets of cases. What Part 6 tries to do is generalize even further, to cover (a) non-geometric shapes, and (b) three dimensional shapes.

Part 7

Right-angled triangles and equilateral triangles are special cases of triangles. What about ellipses and circles? Would you say that they are distinct entities (holding that a circle is not an ellipse) or would you say that an ellipse is a special case of a circle (holding that a circle is an ellipse)?

Likewise, would you say that rectangles and parallelograms are special cases of quadrilaterals and that squares are special cases of rectangles (taking the view, for instance, that a square is a rectangle, a rectangle is a parallelogram, and a parallelogram is a quadrilateral), or would you say that they are distinct (taking the view that that squares are not rectangles and that rectangles are not parallelograms)? You might be able to make a non-arbitrary decision if you ask yourself which of these two positions yield a better set of interesting conjectures and theorems.

Similar remarks apply to the circumscription-inscription relation as well. The best definition of the relation is one that lends itself to the formulation of the widest range of true theorems.

Task 14: On the basis of the true theorems that you can think of, come up with the best definition of "X is inscribed in Y."

Task 15: Does the definition you came up with in task 14 create problems for your judgment of what constitutes examples of inscription-circumscription in figures 1-4? If it does, consider the following options:

Option 1: We are faced with two distinct concepts. They are both valuable, but because they are distinct, it is impossible to unify them under a single "best" definition. So we should have two separate definitions for these two concepts.

Option 1: We are faced with two variants of the same abstract concept. We should look for a single unifying definition, and then spell out the differences in the special concepts, as in the case of polygon (general concept), triangles and quadrilaterals (special cases of polygons), and squares (special case of quadrilaterals.)

Which of these would you choose?

On Part 7: For the Teacher

The ability to investigate whether a given conjecture is true, and to establish its truth (or falsity) by offering a proof, is a relatively tangible ability that, given simple enough conjectures, school students should be able to develop, with some degree of expertise.

But the ability to evaluate the interestingness or significance of a conjecture or theorem, or the beauty or elegance of a proof, is an altogether different game. Acquiring the capacity to function like a mathematician requires one to develop these relatively intangible abilities. The purpose of part 7 is to nudge the students to be aware of these aspects of mathematical inquiry.