The Product of Two Negative Numbers[1]

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2nd March 2009

1. The Story

1.1 Plus and minus as locations

When my daughter Ammu was seven years old, I introduced her to the concept of negative numbers using an analogy of a vertical ladder that goes infinitely upward above ground level and infinitely downward below ground level. Taking ground level as ZERO, the steps above were positive numbers, and those below were negative numbers.

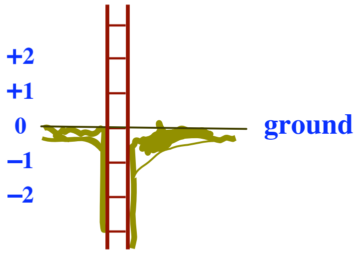


Figure 1

This is a variant of the standard number line in textbooks where the numbers to the left of ZERO are negative and those to its right are positive:

Figure 2

The ladder analogy helped her understand the concept of addition and subtraction involving negative numbers. But when it came to multiplication with negative numbers, we were stuck. I was unable to explain to her why the result of multiplying two negative numbers was a positive number. And she refused to budge without understanding.

1.2 Doing math without understanding

The reason I couldn't explain the concept was that I didn't understand it myself. My math teacher in school had told me that the product of two negative numbers is a positive number. Taking his word for it, I learnt to make appropriate calculations. It didn't occur to me to ask *why* two negative numbers when multiplied yielded a positive number, but when added, they yielded a negative number. I had learnt to go through the motions of mathematical calculation without understanding. But Ammu insisted on understanding.

Had I demanded conceptual and critical understanding in primary school, my teachers would probably have been in the same position that I was in as a parent: they may not have been able to give me a reasonable answer, having accepted, without understanding, what their own teachers had told them. It occurred to me that, had Ammu asked the question in a classroom, a typical teacher might have told her to just sit down and learn to calculate instead of asking stupid questions. She might have come to conclude that the reason for her failure to understand was her lack of mathematical ability, and that would have led to a permanent mental block against math. I was able to survive my primary school math because I didn't demand understanding; Ammu, for demanding understanding, would have paid a price: she would have grown up with the belief that she was mathematically deficient.

[1] This write-up owes to Eight people: Ammu, Adam, Uttam, Walter, Chandra, Debashree, Chi Tat, and Tara.

Many linguists have a strong interest in mathematics, and some are even trained in mathematics. Being a linguist, I sent out an email to colleagues in my department asking for help in explaining to a seven-year old why the product of two negative numbers is a positive number. Some colleagues responded saying that they didn't know either, and would I please tell them when I found an answer. A couple of colleagues offered esoteric reasons that would have been beyond a seven-year old.

1.3 Plus and minus as directions

Another analogy then presented itself, and seemed to work: the +/– signs to the left of a numeral can be viewed as indicating *direction*, and the number itself expressing *quantity*. In this view, positive and negative numbers are vectors. The + sign means "continue in the same direction," while the – sign means "reverse direction." When there are two negative numbers, we need to reverse the direction, then reverse the direction again, which takes us back in the original direction.

Given this perspective, Ammu understood the concept — or at least, we both thought we understood. Little did I realize that the explanation I had given her was conceptually confused and inconsistent.

Several years later, when Ammu was an undergraduate student, a conversation at a friend's home turned to the issue of the product of two negative numbers being a positive number. I recounted my explanation in terms of the +/– signs as directions, and the – sign as an instruction to reverse direction. After some thought, Adam, also an undergraduate student, exclaimed, "Wait a minute. If the minus sign is an instruction to reverse direction, and two reversals in multiplication get us back to the original direction, how come the same doesn't happen in the case of addition? Why is the sum of two negative numbers not a positive?"

The edifice I had built up collapsed. Adam's question pointed to a deep inconsistency in my understanding of addition and multiplication. I didn't really understand why, given two negative numbers, their sum was a negative number, but their product a positive number.

When I got home, I asked a couple of colleagues in the math department for a conceptual explanation that a primary school student could understand. One colleague replied that there was no conceptual answer: this was just a formal convention. The other looked for a conceptual answer but couldn't come up with a satisfactory one.

1.4 Pluses and Minuses as assets and debts

A few weeks ago, I was reading Ian Stewart's *Letters to a Young Mathematician* (2006), and to my delight I found a discussion of the plus-minus problem:

> "At some point early in your teaching career, one of your precalculus students is going to ask you why "minus times minus gives plus …
>
> … it is a convention. It may be the only choice that makes sense, but mathematicians could, if they wished, have insisted that (-3) x (-5) = -15. The concept of multiplication would then have been different, and the usual laws of algebra would have been torn to shreds and thrown out the window …
>
> There are two reasons why the standard convention is a good one: an external one, to do with how mathematics models reality, and an internal one, to do with elegance.
>
> The external reason convinces a lot of students. Think of numbers as representing money in the bank, with positive numbers being money you possess and negative numbers being debts to the bank. Thus -5 is a debt of $5, so 3 x (-5) is three debts of $5, which clearly amount to a total debt of $15. So 3 x (-5) = -15, and no one seems bothered much about that. But what of (-3) x (-5)? This is what you get when the bank *forgives* 3 debts of $5. If it does that, you gain $15. So (-3) x (-5) = + 15." (pp. 162-3)

Stewart's "external" interpretation of plus and minus as saving and debt doesn't quite make sense from a mathematical perspective. Possessing no money and having no debt adds up to zero. If I now borrow five dollars from the bank, I have five dollars in my pocket, but I also have a debt. This means that my borrowing resulted in both (+5) and (-5). There is no actual mathematical operation of this kind. Forgiving is forgiving of the debt created by borrowing. Given the (+5) + (-5) resulting from borrowing, forgiving erases the (-5), keeping the (+5). This too has no mathematical analog.

Stewart's internal account, on the other hand, makes eminent sense, even to a layman. The point he makes is that the convention of the product of two negative numbers being a positive number is consistent with the conventions of algebra that we are familiar with. Following some other convention — say, two negatives making a negative, or for that matter, two positives making a negative — would result in an inconsistency that would force us to throw away a great deal of what is valuable in algebra (pp.163-4). This internal account would be appropriate for an older child, but not quite so for a seven-year old.

> When he saw an earlier draft of this write up, Chong Chi Tat, a colleague of mine, emailed me the following internal argument, probably easier for a child to understand:
>
> The product of (–3) x (–5) is either (–15) or (+15).
>
> If it is (–15), then (–5) x (–3) = (–5) x (+3).
>
> Canceling (–5) from both sides, we get (–3) = (+3).
>
> This is an absurd result so it couldn't be that the product is (–15). Hence the product of (–3) x (–5) is (+15)

As Stewart says,

> "The important thing is not to say to the student, "That's how it is. Don't question it. Just learn it." But to my mind it would be even worse to leave them with the impression that there was never any choice to be made, that it is somehow *ordained* that minus times minus makes plus. All of those concepts — plus, minus, times — are human inventions." (p. 164)

1.5 The meaning of multiplication with negative numbers

A few days ago, Chandra, Uttam, Walter, Cindy and I were talking about science and math education, when the issue of the product of two negative numbers came up yet again. I recounted its history beginning with what happened twenty years ago, went on to Adam's question, and ended with Ian Stewart's unsatisfactory solution. And Uttam said, "But doesn't it follow from the textbook idea of multiplication as repeated addition?" "No," I said, "We can view five times three as adding three five times, but what does it mean to say "minus five times" in minus five times minus three?"

After a minute's pause, Uttam said, "We can view multiplication with negative numbers as repeated subtraction."

What follows is the working out of this idea.

2. The Explanation

2.1 Ambiguity in the meanings of plus and minus

To understand why the product of two negative numbers is a positive number, we have to begin with the meanings of the symbols + and – in mathematics. Depending on the context, they may refer either to the categories of positive and negative numbers, or to the operations of addition and subtraction.

When there is a + or – to the left of a numeral by itself, these signs refer to the ***categories*** of positive and negative numbers. Thus, (+3) means "positive number three," and (-5) means "negative number five." To use the number line (as in figure 2), + refers to a location to the right of zero, while – refers to a location to the left of zero:

Figure 3

However, when they occur between two numbers, the same signs, + and –, refer to the ***operations*** of addition and subtraction respectively. Thus,

3 + 5	(add five to three)	3 – 5	(subtract five from three)
3 x 5	(multiply three by five)	3 ÷ 5	(divide three by five)

To avoid the ambiguity, it might be useful to adopt the following conventions when referring to positive and negative numbers:

i. Use the + sign whenever referring to a positive number.
ii. Enclose the number, along with the +/– sign to its left, in brackets.

Examples of the use of these conventions are given below:

(+3) + (+5)	(add plus five to plus three)
(-3) – (+5)	(subtract plus five from minus three)
(+3) x (-5)	(multiply plus three by minus five)
(-3) ÷ (-5)	(divide minus three by minus five)

2.2 Meanings of addition and subtraction

When the symbols + and – indicate the concepts of positive and negative numbers, figure 3 visualizes them as locations to the right and to the left of zero. How would figure 3 visualize the operations of addition and subtraction expressed by the same symbols?

The addition of a positive number to a positive number is easy: we can interpret the expression (+3) + (+5) as: "Take three steps to the right of zero, then take five steps also to the right." The resultant location (+8) is exactly the result we want. What about adding a negative number to a negative number? We can interpret (-3) + (-5) as: "Take three steps to the left of zero, then take five steps also to the left." Now we can combine the two as follows:

(+3) + (-5)	(Take three steps to the right of zero, then take five steps to the left.)
(-3) + (+5)	(Take three steps to the left of zero, then take five steps to the right.)

We understand subtraction as the reverse of addition. That is, the subtraction of a positive number is the addition of a negative number, and the subtraction of a negative number is the addition of a positive number. From this, we derive the following results:

$$(+3)-(+5) = (+3)+(-5)$$
$$(-3)-(-5) = (-3)+(+5)$$
$$(+3)-(-5) = (+3)+(+5)$$
$$(-3)-(+5) = (-3)+(-5)$$

We already have interpretations of the expressions in the right-hand column.

To communicate these concepts to children, it might help to use the analogy of a leaping frog. (+3) + (5), for instance, can be expressed as follows:

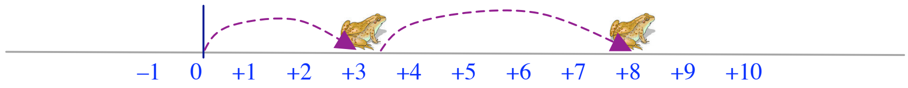


Figure 4a

And (+3) – (+5) would be as follows, with the frog changing direction after the first leap. (I couldn't find a reversible clip art frog on the web.)

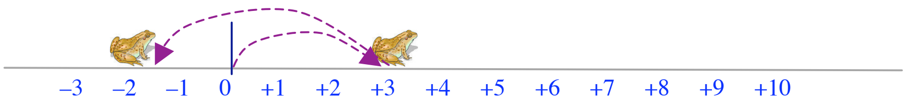


Figure 4b

2.3 The meaning of multiplication by positive and negative numbers

The next step would be to ask what multiplication by a positive number means, and then proceed to make sense of the notion of multiplication by a negative number. We have seen that multiplication by a positive number can be thought of as repeated addition. For instance, when we interpret the expression (+6) x (+4) as "four ***times*** six", we are interpreting it as:

(+6) + (+6) + (+6) + (+6) = (+24)

(+6) +	o	o	o	o	o	o
(+6)	o	o	o	o	o	o
(+6)	o	o	o	o	o	o
(+6)	o	o	o	o	o	o

This, in fact, is the result of adding six to zero four times:

0 + (+6) + (+6) + (+6) + (+6) = (+24)

In this example, we multiplied a positive number by a positive number. The idea of repeated addition works for the multiplication of negative numbers by positive numbers as well. Thus, the expression (-6) x (+4) means "four times *minus* six". It is the result of adding minus six to zero four times:

0 + (-6) + (-6) + (-6) + (-6) = (-24)

A difficulty arises when we ask what the multiplication by a negative number is. What does the expression (+6) x (-4) mean?

We already know that addition of a negative number is the same as subtraction of a positive number. Thus the expression (+6) + (-4) is the same as (+6) – (+4). Extending this result, we may view repeated addition of a negative number to zero as ***repeated subtraction*** from zero.

One way to think of this is to view subtraction as the addition of a negative number. If so, (+6) x (-4) means: "subtract six from zero four times." That is

0 – (+6) – (+6) – (+6) – (+6) = (-24)

Extending this idea further, we find that the expression (-6) x (- 4) means "subtract minus six from zero four times," which is the same as "add plus six to zero four times."

0 – (-6) – (-6) – (-6) – (-6) = 0 + (+ 6) + (+6) + (+6) + (+6) = (+24)

Nothing further needs to be said.

The analogy of a leaping frog can cover this situation as well. Given below, for instance, is (+3) x (4).

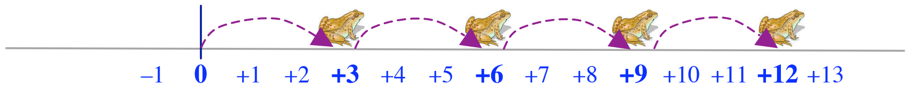


Figure 5

When multiplying by a negative number, the frog leaps in the opposite direction.

2.4 The meaning of division by a negative number

This line of thinking naturally leads to the question of what division by a negative number (e.g., six divided by minus four) means, and what happens when we divide a negative number by a negative number. I leave it to you to look for an answer. But here are some clues:

Just as subtraction is the reversal of addition, division is the reversal of multiplication:
(+4) x (+6) is "the result of adding four to zero six times"
(+24) ÷ (+6) is "the number that, when subtracted six times from 24, results in zero."

Mental animation:

(+4) x (+6): A frog begins at zero, leaps to the right six times, each time taking four steps. It arrives at 24.

(+24) ÷ (+6): The frog's return journey: it begins at 24, leaps to the left six times, and arrives at zero. The number of steps for each leap is four.

3. Concepts, metaphors, and reasoning in mathematics

I would like to spell out some of the lessons to draw from the above discussion:

A) Young learners need help to understand ***why*** we accept certain beliefs and practices, and reject others. For instance, it is important that children, when they are exposed to the belief that the earth spins on a tilted axis and revolves around the sun, understand why we believe this. Likewise, it is important for young learners to understand why the product of two negative numbers is positive, though the sum of two negative numbers is negative. Conventional education fails to provide such understanding.

B) Mathematical inquiry includes *formal reasoning* as well as *conceptual reasoning*. Solving algebraic equations like '$x^2 + 4x + 9 = 21$' requires formal reasoning, while seeing that in spherical geometry, two straight lines can intersect at more than one point requires conceptual reasoning. Conceptual reasoning is itself enhanced by *visual* reasoning (e.g., the diagrams in sections 1 and 2), and *analogical* reasoning (the ideas of ladder, debts, and leaping frogs).

C) A great deal of the math that we teach our children focuses on the skills of making mathematical *calculations*, with no attention to the modes of *mathematical thinking*, mathematical *intuitions*, and mathematical *concepts*. It is important that children *understand* the concepts of numbers, addition and subtraction, multiplication and division, and use that understanding in their *reasoning*.

In exploring the basic arithmetic concepts in the preceding section, we used the number line as the crucial image to systematically pursue the concepts of positive and negative numbers, addition, subtraction, multiplication, and division. A more common metaphor used in classrooms to understand arithmetic operations is that of objects in boxes and baskets. While the number line metaphor conceptualizes numbers as *locations ordered along a scale* (the line) as in figure 6a, the box/basket metaphor conceptualizes them as *unordered dots* in two dimensional space, as in figure 6b:

number line

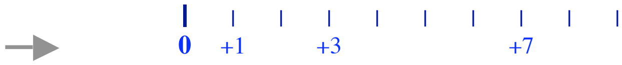


Figure 6a

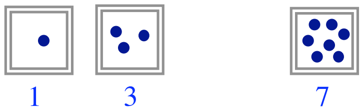
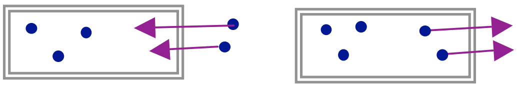


Figure 6b

See how the idea of the leaping-frog-on-the-number-line changes in figure 6b. We could think of the dots in the boxes in figure 6b as frogs. Addition and subtraction would then not be leaping from one location to another, but leaping into and out of boxes.

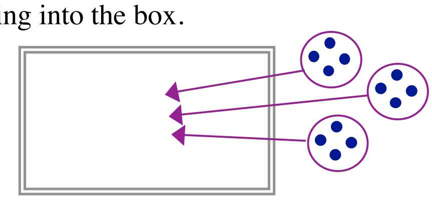


(+3) + (+2) (+5) – (+2)

Figure 7

The idea of multiplication as repeated addition can now be expressed as:

a number of baskets (circles), each with the same number of frogs (dots);
an empty box (rectangle); and
all the frogs leaping into the box.

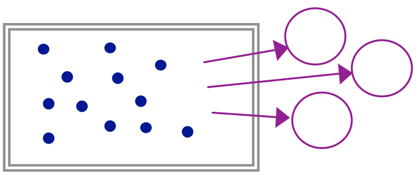


4 x 3 (three times four)

Figure 8

And division can be expressed as:

a number of frogs in a box;
a number of empty baskets; and
the frogs leaping into the baskets such that every basket has the same number of frogs.



12 ÷ 3 (twelve divided by three)

Figure 9

The conceptualization and visualization of numbers as counting the frogs in boxes and baskets, represented visually as dots in rectangles and circles, rather than in terms of a number line, would perhaps make it easier for younger children to understand arithmetic operations on natural numbers (positive integers). However, moving on to operations on negative integers, we would need the more abstract number line. Dots, rectangles, and circles do not lend themselves to a representation of negative numbers such that the representation tells us why the product of two negative numbers is a positive number.