Paper Folding Geometry [*]

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CONTENTS

[*] This document has benefited immensely from the comments, criticisms and suggestions from Christina Castelli.

Note to the teacher

Traditional mathematics education (by which I mean the kind of math education I received in school and college) aims to get learners

understand mathematical concepts, procedures, and theorems and

develop the *skills* of *applying* equations and formulae to standard problems and *calculate* results.

In addition, some programs also expect learners to

understand and *reproduce* the *proofs* of the theorems.

What these enterprises often leave out is *mathematical thinking*:

inventing mathematical objects and procedures,

discovering patterns in the world of mathematical objects and procedures,

formulating these patterns as conjectures,

discovering proofs for conjectures,

constructing axioms and definitions,

creatively modeling novel phenomena in terms of mathematical objects and procedures, and

arriving at inferences from these models.

None of these is a mechanical *skill* that can be perfected merely through mere repeated practice. Nor can they be applied mechanically at high speed in standardized tests.

The purpose of the following teaching-learning material is to develop an intuitive understanding of Euclidean two dimensional geometry through paper folding activities, and to use it as the basis for providing first-hand experiences of discovering patterns, formulating conjectures and proving conjectures.

Section 1 is designed to get students used to the various possibilities of paper folding, so that the physical experience of paper folding prepares them for the more challenging tasks in the subsequent sections. Section 2 connects the shapes they observe on paper to the different categories of polygons in geometry. Section 3 introduces students to the concept of proof and provides the experience of discovering/inventing patterns, formulating them as conjectures, discovering proofs, and formulating proofs. It also briefly introduces students to the nature of mathematical knowledge. The activity of constructing axioms and definitions on the basis of which we can prove conjectures is reserved for the advanced treatment in Part II.

Some children (and some adults) may find the physical folding activities in section 1 too elementary. Some may find even section 2 too elementary. But then, for other children (and even adults), these activities are crucial for the activities in section 3. It is hard to tell ahead of time who needs the preparation and who does not. Feel free to skip some of the exercises, add more, or vary the pace depending on what you observe with your students and allow individuals to progress through the activities at their own speed.

Some of the tasks outlined in the following sections can be pursued in the classroom, for every individual student to engage with, or, if we design them as group activities, for every group. Others can be distributed across groups. Yet others can be given as homework. Some of the activities in the somewhat longish subsections 2.3, for instance, can be done in class, while others can be given as homework.

Part I: Introductory

1. Folding Paper

1.1. Single folds

1.1.1. Making shapes by folding paper

Take any rectangular sheet of paper, and fold it. You can fold it any way you like, from one side to the opposite side, from one side to the adjoining side, from a corner to a side, from a corner to the opposite corner, and so on. But only one fold.

Now unfold the paper, and look at the two shapes you have created by folding the paper. What shapes are they? Two rectangles? Two triangles? A rectangle and a triangle? Two circles?...

Take another rectangular sheet of paper and fold it in a different way. Unfold it, and identify the shapes you see. Repeat with a few more sheets.

Make a list of the shapes you have created through folding the paper.

Now, make a list of other possible shapes—shapes that you haven't made yet. Is it possible to create any of these shapes by folding the sheet just once?

If you are doing this in class with your friends doing the same paper folding exercise, compare the shapes you have with those of your friends. If you are doing this on your own outside the classroom, get a few of your friends to do this. Do they have shapes that you don't have? If so, do you need to revise your ideas about what kinds of shapes are possible just with one fold?

1.1.2. Getting used to the language of paper folding instructions

For each of the tasks (1)-(4) below, take a rectangular sheet of paper (separate sheets for each task) whose corners are marked A, B, C and D, as shown in the following diagram. Fold it once, unfold it, and identify the two shapes resulting from the folding.

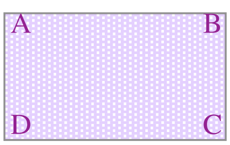
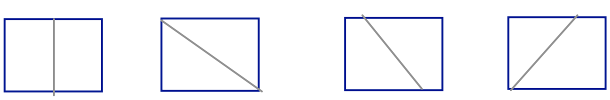


Figure 1

1) Fold the paper in such a way that BC lies exactly on top of AD.
2) Fold the paper in such a way that the crease runs through both A and C.
3) Fold the paper in such a way that corner B lies exactly on top of D.
4) Fold the paper such that A lies on DC (and a part of AB lies along edge DC.)

Note to the teacher: These tasks yield the following shapes.



Task 1 Task 2 Task 3 Task 4

(gray line: crease created by the fold.)

It might be useful to point out that the shapes resulting from task 1 are rectangles and those resulting from task 2 are two triangles. The two shapes in task 3 are called quadrilaterals. Task 4 has a triangle and a quadrilateral. (The names of the shapes are useful but not important for now. What matters at this stage is the *perception* of the different shapes resulting from paper folding.)

1.2. Multiple independent folds

Take one of the sheets you have folded in section 1.1, and fold it again in a different way, creating another crease. Unfold it, fold it again, with yet another crease. Repeat as many times as you wish. Repeat multiple independent folds on another sheet of paper, as many times as you wish.

Make a list of the types of shapes you see. What are the shapes you can create with multiple independent folds but not with a single fold?

As in 1.1.1., if you are doing this in class with your friends doing the same paper folding exercise, compare the shapes you have with those of your friends. If you are doing this on your own outside the classroom, get a few of your friends to do this. Do they have shapes that you don't have? If so, do you need to revise your ideas about what kinds of shapes are possible just with multiple independent folds?

1.3. Making creases by folding paper

As you must have figured out by now, a *crease* is a line that you create when you fold a piece of paper and then unfold it. In the following diagrams, creases are indicated by dotted lines.

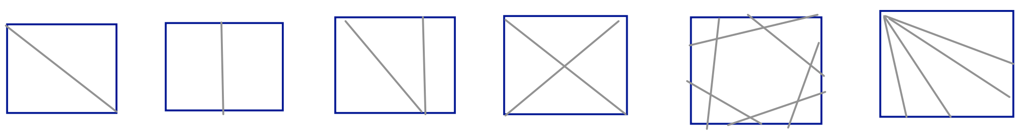


Figure 2

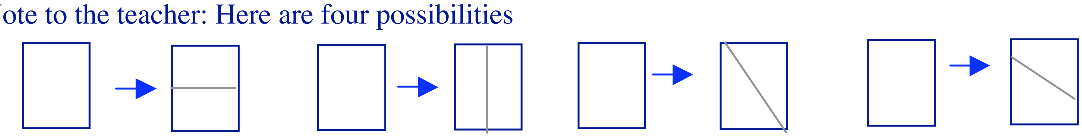
Create the creases in each of these diagrams using single or multiple independent folds.

2. Making Different Shapes

2.1. Dividing a rectangle into two equal halves

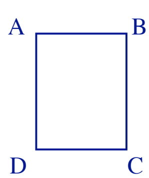
Take a rectangular sheet of paper, and fold it into two equal halves. What are the different ways in which you can do this?

Note to the teacher: Here are four possibilities



possibility 1 possibility 2 possibility 3 possibility 4

If the corners of the paper are marked ABCD with AB and DC as the short sides,



the folding in possibility (1) is done by placing AB on top of DC, and possibility 2 by placing AD on top of BC. Possibility C involves a fold from one of the corners to the opposite corner. And 4 requires a fold in which corner B is on top of corner D.

2.2. Shapes bounded by straight lines

The shapes we have been trying to create by folding paper have one common characteristic: they are all made with straight lines. Such shapes are called ***polygons***.

Given below are a few examples of polygons:

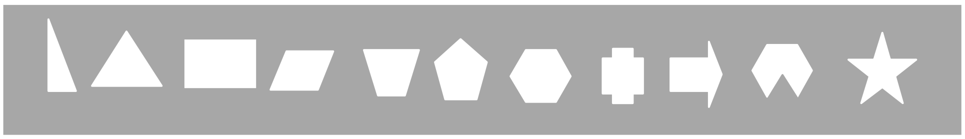
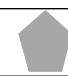
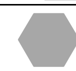
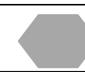
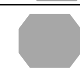


Figure 3: Polygons

As you can see from these examples, a ***polygon*** is a (two dimensional) shape bounded by straight lines. Triangles, squares, rectangles, parallelograms, pentagons, hexagons, etc., are polygons.

This is a three-sided polygon (= a *triangle*).	
This is four-sided polygon (= *a quadrilateral/tetragon*).	
This is a five-sided polygons (a *pentagon*).	
This is a six-sided polygon (= a *hexagon*)	
This is a seven-sided polygon (= a *heptagon*)	
This is an eight sided polygon (= an *octagon*.)	

Identify the type of polygons in figure 3.

To introduce a few more concepts:

Parallelogram: A quadrilateral in which each side is parallel to its opposite side.

Rectangle: A four-sided polygon in which all the angles are right angles. (A parallelogram in which all the angles are right angles.)

Square: A rectangle whose sides are equal,

Isosceles triangle: A triangle in which at least two sides are equal. (Alternatively: a triangle in which at least two angles are equal.)

Right angled triangle: A triangle in which one of the angles is a right angle.

Right angle: If the four angles created by two lines crossing each other are equal, they are right angles. (Alternatively: A line perpendicular to another is at right angle to it.)

These are *definitions* of geometric objects.

2.3. Creating polygons with paper folding

2.3.1. Rectangles of equal size

Task 1: Take any rectangular sheet of paper, and fold it in such a way that it is divided into two rectangles of equal size.

Task 2: Take any rectangular sheet of paper, and fold it in such a way that it is divided into four rectangles of equal size.

Task 3: Take any rectangular sheet of paper, and fold it in such a way that it is divided into eight rectangles of equal size.

Task 4: Take any rectangular sheet of paper, and fold it in such a way that one of the shapes is a triangle.

2.3.2. Triangles of equal size

Task 1: Take any rectangular sheet of paper, and fold it in such a way that it is divided into two triangles of equal size.

Task 2: Take any rectangular sheet of paper, and fold it in such a way that it is divided into four triangles of equal size.

Task 3: Take any rectangular sheet of paper, and fold it in such a way that it is divided into eight triangles of equal size.

2.3.3. Right angled triangles and isosceles triangles

Task 1: Take any rectangular sheet of paper, and fold it in such a way that one of the shapes is a right angled triangle.

Task 2: Take any rectangular sheet of paper, and fold it in such a way that one of the shapes is an isosceles triangle.

Task 3: Take any rectangular sheet of paper, and fold it in such a way that one of the shapes is an isosceles right angled triangle.

Task 4: Take any rectangular sheet of paper, and fold it in such a way that one of the shapes is an isosceles triangle whose area is half the area of the rectangle. 6

2.3.4 Squares

Task 1: Take any rectangular sheet of paper, and fold it in such a way that one of the shapes is a square.

Task 2: Take any rectangular sheet of paper, and fold it in such a way that four of the shapes are squares of equal size.

3. Creating Knowledge

3.1. Proving conjectures

A *conjecture* is a statement that we guess is true, but haven't actually proved. A mathematical *proof* is a set of logical steps that demonstrate that the conjecture we wish to prove follows logically from what we already know (or assume) and what we are given. Once a conjecture is proved, it becomes a *theorem*.

Suppose you are told that Zeno is a human being, and are now asked to prove that he is mortal.

We can offer the following proof:

(1) Zeno is a human being (given)
(2) All human beings are mortal (What we already know)
(3) It follows from (1) and (2) that Zeno is a mortal.

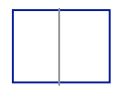
Suppose we are asked to prove that it is impossible to create a circle by folding paper. We can give a proof as follows:

(1) The crease created by any fold is a straight line. (What we already know)
(2) A circle is a curved shape. (What we already know: definition of a circle)
(3) No combination of straight lines can create a curved shape. (What we already know)
(4) It follows from (2) and (3) that no combination of straight lines can create a circle.
(5) It follows from (1) and (4) that it is impossible to create a circle by folding paper.

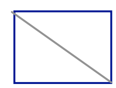
Let us go back to some of our earlier activities and see what we can prove. In section 1.2., we folded the paper ABCD in the ways indicated below:

Figure 1

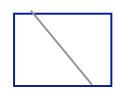
1) Fold the paper in such a way that BC lies exactly on top of AD.



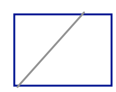
2) Fold the paper in such a way that the crease runs through both A and C.



3) Fold the paper in such a way that corner B lies exactly on top of D.



4) Fold the paper in such a way that corner A lies exactly on edge DC.



In section 2.3, we discovered that each of the folds described in (1)-(3) result in two equal halves. More than that, the folding creates two congruent shapes. (Two shapes are *congruent* when one shape can be placed exactly on top of the other.) We also discovered that the procedure in (4) yields a right angled triangle (in addition to a quadrilateral.)

But is it true that for *any* rectangular piece of paper (not just the one you happen to have picked) (1)- (3) yield two congruent shapes, and (4) yields a right angled triangle?

Try proving the following:

Task 1

Prove that for any rectangular sheet of paper, the folding procedure you used in task 1 above yields two congruent rectangles.

Task 2

Prove that for any rectangular sheet of paper, the folding procedure you used in task 2 above yields two congruent triangles.

Task 3

Prove that for any rectangular sheet of paper, the folding procedure you used in task 3 above yields two congruent quadrilaterals.

Task 4

Prove that for any rectangular sheet of paper, the folding procedure you used in task 4 section 1.2. yields a right angled triangle (in addition to a quadrilateral.)

Task 5

Prove that for any rectangular sheet of paper, the folding procedure you used in task 4 above yields an isosceles triangle.

Try the following proofs as well.

Task 6

In task 3 of section 2.3.1, you were asked to fold a rectangular sheet of paper into eight congruent rectangles. One possible way to do this is to do the procedure in task 1 above three times without unfolding. Prove that for any rectangle, this procedure divides the rectangle into eight congruent rectangles. Prove that for any rectangle, this procedure divides the rectangle into eight congruent rectangles.

Task 7

In task 3 of section 2.3.2, you were asked to fold a rectangular sheet of paper into eight congruent triangles. One possible way to do this is to take a rectangular sheet of paper, fold it into 4 congruent rectangles, and then fold it again diagonally. Prove that for any rectangular sheet of paper, this procedure divides the rectangle into eight congruent triangles.

Task 8

Prove that the two congruent rectangles in task 1 above, the two triangles in task 2 above, and then two quadrilaterals in task 3 above have the same area.

Note to the teacher: proofs for tasks 1-8

To be written

3.2. The Nature of Mathematical Knowledge

Knowledge is a body of statements that we believe to be true. The statement that the earth revolves around the sun is part of our knowledge today, because we believe it to be true. In contrast, the statement that the sun revolves around the earth is not part of our knowledge because we do not believe it to be true. A few hundred years ago, however, this statement was part of the knowledge of educated people.

Scientific knowledge is a body of statements that we believe to be true of *the world we live in*, consisting of inanimate objects as well as living organisms including humans. In contrast, mathematical knowledge is a body of statements that we believe to be true of *the objects and processes in the world of mathematics* – in any logically possible universe in which the axioms and definitions that create the world hold.

While scientists study such things as planets, stars, crystals, frogs, human illnesses and the human mind, mathematicians study such things as numbers, circles, sets, and so on. The statement that an apple is spherical in shape is a statement about the world we live in, but the statement that the volume of a sphere with radius r is $4/3\,\pi r^3$ is a statement about the world of mathematics – in any logically possible world.

Unlike an apple, a sphere has no colour (you can't see it), no weight (it doesn't fall), and no taste.

You can't touch it or smell it. The properties of the objects in the real world are identified through sense perception and contemplation, but those of the world of mathematics are identified through pure contemplation.

3.3. Axioms Definitions, Conjectures, Theorems, and Proofs

Let us revisit the concepts that shed light on mathematical justification.

As stated earlier, a claim that we think is correct, but has not been proved yet, is a *conjecture*. Once a conjecture is proved, it is a called a *theorem*. (Occasionally they may also be called a formula or an equation, as in the case of the formula that says that the area of a triangle is half its base multiplied by the height.)

When we offer a proof for a conjecture, we appeal to a set of *grounds* – what we already know and assume to be true – and show that the conjecture follows as an inescapable conclusion from the grounds. If a ground statement is questioned, we need to treat it as a conjecture and prove that statement first.

Not all ground statements, however, can be proved. There are two specific classes of grounds that are themselves not provable by appealing to independent grounds. One of them, as we have seen, is the set of ground statements that are offered as *definitions*. Definitions are neither true nor false: they are statements that clearly identify what it is that we are talking about. For instance, in section 2.2 we defined parallelograms and rectangles as follows:

> *Parallelogram:* A quadrilateral in which each side is parallel to its opposite side.
>
> *Rectangle:* A four-sided polygon in which all the angles are right angles. (A parallelogram in which all the angles are right angles.)

An alternative would have been to define them as follows:

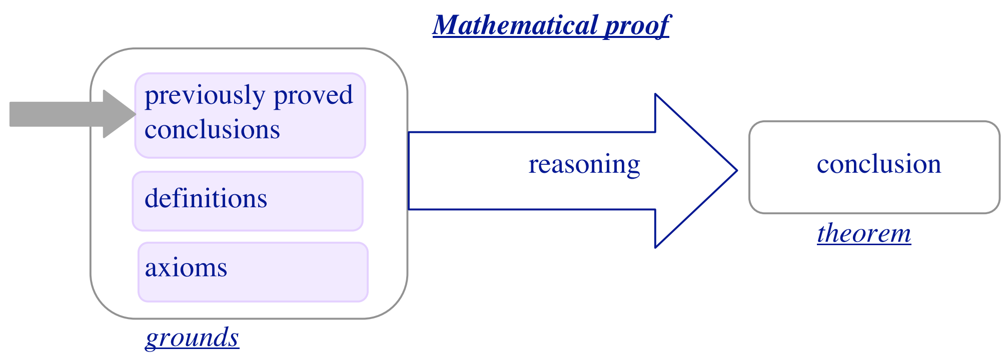
> *Parallelogram:* A quadrilateral in which *there are no right angles* and each side is parallel to its opposite side, without any of the angles being right angles.
>
> *Rectangle:* A four-sided polygon in which all the angles are right angles but *the adjacent sides are not equal*.

Given the first set of definitions, a square is a specific kind of rectangle and a rectangle is a specific kind of parallelogram. Given the second set of definitions, a square is not a rectangle and a rectangle is not a parallelogram. Which of them is the "true" definition? That question doesn't make sense. All we can ask is, "Which of them is (most) useful for our purpose of discovering patterns?"

The other class of ground statements are *axioms*. Axioms are statements that cannot be proved on independent grounds. Typically, they are statements whose correctness is so obvious that we don't demand that they be proved. Such axioms are regarded as "self-evident".

Here is an example of a self-evident axiom in paper folding: *the crease created by a fold is a straight line*. We appealed to this axiom in section 3.1 to prove that it is impossible to create a circle through paper folding.

Given the points we have made about the different kinds of grounds – axioms, definitions, and previously proved statements – we may view the structure of mathematical proofs as follows:



3.4. Proof Revisited

Knowledge, as we said, is a body of statements that we believe to be true. Now, when we are faced with a *knowledge claim* – a statement that someone advances as a candidate to be included in the existing body of knowledge – it is important to ask why we should accept it as true. When someone says that the earth revolves around the sun, for instance, we should ask why we should accept that statement as true. The response to that question, the reasons for accepting a statement as true, is *justification*. *Proof* is the term that mathematicians use to refer to the form of justification characteristic of mathematics.

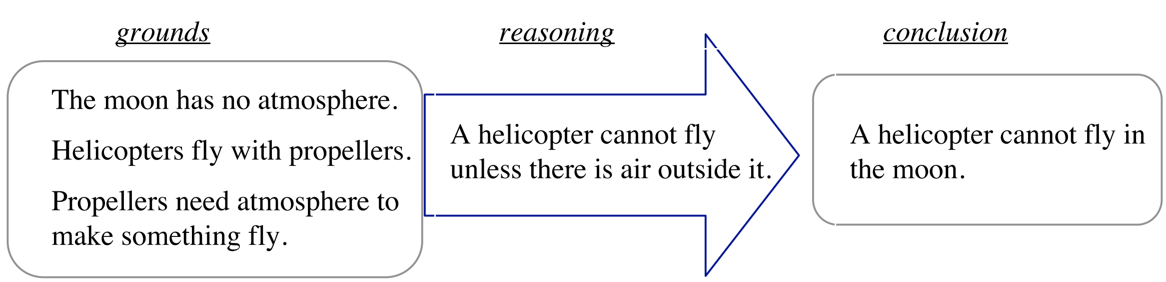
Here is an example of a math-type proof for the claim that a helicopter cannot fly on the moon.

Claim (what is to be proved): *A helicopter cannot fly on the moon*.

Proof:

1. We know that the moon has no atmosphere (air).
2. We also know that helicopters (unlike jets) fly with propellers.
3. We know that propellers need atmosphere (air) to make something fly. (Like birds, they fly by pushing against air.)
4. From 2 and 3, it follows that a helicopter cannot fly unless there is air outside it.
5. From 1 and 4 it follows that a helicopter cannot fly in the moon. (*QED*)

A proof is a set of steps that demonstrates the claim follows as conclusion. Step 5 in this proof is the final *conclusion*. (QED is an abbreviation of the Latin phrase *quod erat demonstrandum* meaning "that which was to be demonstrated", i.e, the claim.) Statements 1-3 constitute the *grounds* for our proof. We use *reasoning* to move from the grounds to the conclusion. Statement 4 is one of the intermediate steps in our reasoning.

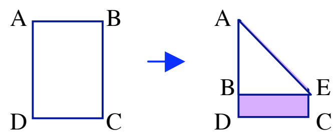


If we accept every statement given in the grounds box, the statement in the conclusion box is inescapable. This is the essence of the mathematical proof. However, we may at this point ask why we should accept as true one of the ground statements, say, the statement that the moon has no atmosphere.

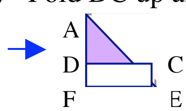
When a ground statement is questioned, we have to change its status and treat it as a claim to be proved (a statement that is offered as a candidate seeking admission to knowledgehood, not something that is already admitted) and offer justification to establish it as knowledge. We won't go into the issues of scientific evidence to conclude that the moon has no atmosphere (even though this is indeed an important matter.) At this point, all that we want to do is point out that a statement that is offered as the grounds to prove some other statement can itself be questioned.

Take for instance, the following claim that emerges from task 1 in section 2.3.4:

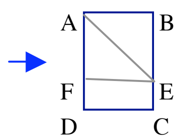
> If ABCD is a rectangular piece of paper in which AB and CD are the short sides, we can create a square shape on it with the following procedure:
>
> 1) Fold AB over AD, creating a fold that begins at A. Mark the other edge of the fold as E.



> 2) Fold DC up and over such that BE lies on the new fold. Mark the edge opposite E as F.



> 3) Unfold. ABEF is a square.



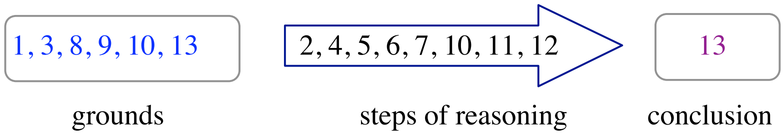
Here is a proof:

<u>Claim:</u> ABEF is a square.

<u>Proof:</u>

1) (We know that) AFE is an isosceles right angled triangle, with F as the right angle.
2) It follows from (1) that AF = FE.
3) ABE was placed exactly on AFE,
4) It follows from (3) that ABE is congruent with AFE.
5) It follows from (1) and (4) that ABE is an isosceles right angled triangle, with B as the right angle.
6) It follows from (5) that AB = BE.
7) It follows from (2), (4) and (6) that AB = BE = AF = FE.
8) ABCD is a rectangle.
9) (We know that) a rectangle is a quadrilateral in which all angles are right angles.
10) (We know that) the sum of angles in a quadrilateral is four right angles (i.e., 360 degrees)
11) It follows from (8) that A is a right angle.
12) It follows from (1), (5), (10) and (11) that E is also a right angle.
13) (We know that) a square is a quadrilateral in which all sides are equal and the all the angles are right angles.
14) It follows from (1), (5), (11),(13) and (7) that ABEF is a square (QED).

Here is the structure of the proof:



Why should we accept the grounds as correct? Let us take a look.

(9) and (13) are *definitions* of rectangles and squares respectively. Definitions are not true or false, but they are needed to tell us what the words in our proof mean, or what mathematical objects the words refer to. (3) and (8) are part of what is *given* to us, what we are told: (3) is part of the given procedure, and (8) is the object that we are given. These ground statements, therefore, are not subject to questioning.

What is subject to questioning is ground statement (1). How do we know that AFE is an isosceles right angled triangle, with F as the right angle? This statement needs to be proved.
What this means is that before attempting to prove that the procedure outlined above yields a square from a rectangle, we should have proved the claims implicit in tasks 4 and 5 in section 8.

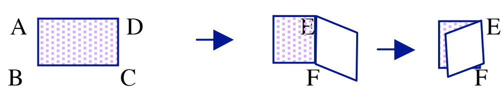
PART II: ADVANCED

4. Origami and Mathematics

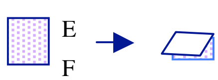
The preceding activities have been concerned with *crease patterns* on paper, that is, those patterns created by folding paper and then unfolding it. As a result, the shapes we have been concerned with are those that exist on a flat surface such as a sheet of paper or the surface of a table.

Now instead of unfolding the folds we have created, we can go on to create three dimensional shapes by paper folding. For instance, here is a way of creating a conical shape out of a rectangular sheet of paper:

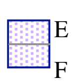
Step 1: Fold the paper lengthwise so that one part (DC) lies exactly on top of the other (AB).



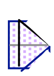
Step 2: Without unfolding, fold it again such that one part of the earlier fold lies exactly on top of the other.



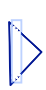
Step 3: Unfold the fold in step 2.



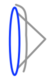
Step 4: Make two folds such that E and F lie top of the crease (E and F touch.)



Step 5: Fold the flaps on both sides over the triangular shape.



Step 6: Pull out the two parts of the base and shape it into a circular shape: you have a cone.



Note: These pictures may have to be replaced by photographs or a video.

Folding paper to make three dimensional shapes of this kind is the art and craft of *origami*. Making cones out of paper is probably the least interesting of origami activities. If you want to find out how to make far more interesting shapes (frogs, penguins, dinosaurs, elephants, eagles, turtles, …) take a look at the following sites:

Wikipedia entry on Origami (http://en.wikipedia.org/wiki/Origami)

Eric's Origami page (http://www.paperfolding.com/)

Tammy Yee's Origami page (http://www.tammyyee.com/origami.html)

In recent years, mathematicians have discovered that underlying the art of paper folding is a rich body of mathematics of considerable value not only to art but also to science and technology. Watch Robert Lang's TED talk on Idea + square = origami at

http://www.ted.com/index.php/talks/robert_lang_folds_way_new_origami.html)

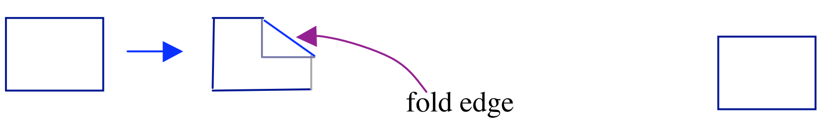
We are not going to do origami per se, but look at the crease patterns resulting from paper folding, as we did in part I. An origamist would make a penguin out of a sheet of paper, but we would unfold the penguin back into a sheet of paper and look at the geometry of the creases.

5. Investigating Creases

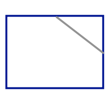
5.1. Definitions

In part I, we were concerned primarily with the shapes created by paper folding. We will now proceed to the investigation of creases (corresponding to lines), and then return to shapes. One of the central ingredients of rigour in mathematical inquiry is that of the definition of the mathematical objects we are investigating. In our paper folding adventure, the objects we have been investigating have been the shapes and lines created by the folds. In section 2.2., we gave the definitions of some of the shapes. Since we are now turning to the investigation of lines, we need appropriate definitions as the starting points of our inquiry.

Def 1: A *fold edge* is an edge created by folding a sheet of paper.

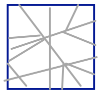
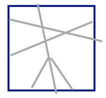


If we unfold the fold in the above diagram, we get the crease in the diagram below:



Def 2: A *crease* is a line left by a fold edge when the fold is unfolded.

Creases in paper folding geometry correspond to *lines* in Euclidean geometry. If we fold the paper many times and unfold it (and fold and unfold again), we get many creases on paper, some of them crisscrossing, as illustrated below:

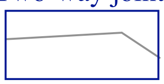



Def 5: A *crease complex* is the configuration of creases on a sheet of paper when the paper is completely unfolded.

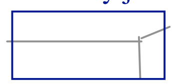
The *joints* where two creases intersect or touch each other correspond to *points* in Euclidean geometry.

Def 3: A *joint i*s the junction between two or more lines.

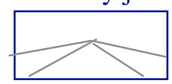
Two-way joint



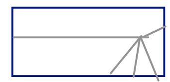
three way joint



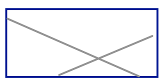
four way joint



five-way joint



Def 4: A *cross-joint* is a four way joint created by the intersection of two straight line segments.



Def 5: A *crease complex* is the configuration of creases on a sheet of paper when the paper is completely unfolded.

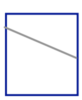
5.2. Discovering Generalizations on Creases

Let us go back to paper folding in section 1.1.1, but a different observational task.

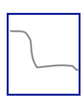
<u>Task 1: Observing patterns in crease complexes</u>

Take any rectangular sheet of paper, and fold it. You can fold it any way you like, from one side to the opposite side, from one side to the adjoining side, from a corner to a side, from a corner to the opposite corner, and so on. But only one fold.

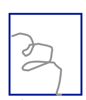
Now unfold the paper, and look at the two patterns of creases you have created by folding the paper. You would see straight line creases but not curved line creases.



Observed



not observed



not observed

Let us state what we discovered in the above task as a generalization we think is true.

G(eneralization)1: The crease produced by folding a sheet of paper is a straight line segment.

We used this generalization earlier in our proof for the impossibility of creating circles through paper folding (section 3.1)

Now let us look at patterns of joints.

<u>Task 2: Observing patterns in joints</u>

Take a rectangular piece of paper, fold it, and fold it again without unfolding (such that both layers get folded.) Fold again (this time you are folding four layers). Fold as many times as you wish (fold on fold). Unfold it, and repeat the above process, as many times as you wish with the same sheet of paper. Then look at the joints. Pick four or five joints randomly, and count the number of lines from each joint. Do you see any patterns/regularities? Write down each pattern you observe as a generalization. Look at a few more joints and check if there are counterexamples to the generalizations you have observed are indeed correct.

Note to the Teacher: Chances are that students would come up with something like G2 and G3:

G2: No crease complex has odd-numbered joints—i.e three-way joint, five-way joint, etc., as shown in Section 5.1, definition 3.

G3: No crease complex has a two-way joint—i.e. one in which two creases meet at an angle, as shown in Section 5.1, definition 3.

Task 3: Folds and folds

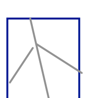
In task 1, we unfolded the paper before folding it again. In task 2, we folded the paper without unfolding. Compare the crease patterns in

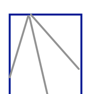
A. Creases created without folds on folds (i.e., always unfolding the paper before folding it again)

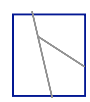
B. Creases created with folds on folds–in other words, without unfolding before folding again

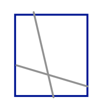
Do you see any interesting differences?

Consider the following crease patterns. For each type of folding (A and B), say whether the pattern is possible or impossible


configuration 1


configuration 2


configuration 3


configuration 4

Note to the Teacher: Chances are that students would come up with something like G4 and G5:

G4: If no folds are refolded, all creases are straight line segments extending from one edge of the paper to another.

G5: When a fold edge is folded again, the two creases of the second fold are mirror images of each other (reflection) with the first crease as the axis.

Task 4: Aligning fold edges

Make a fold. Now make another fold which breaks the first fold edge and also lines up with the fold edge.

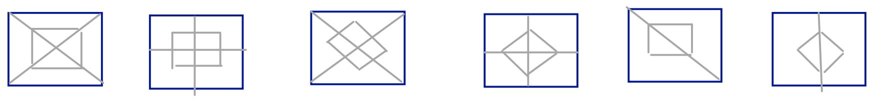
Do this a few times, and observe the angles. Do you see a pattern here?

Note to the teacher: Chances are that students would come up with something like G6

G6: When a fold edge is folded to lie over itself, the new fold edge is perpendicular to the previous fold edge.

Task 5: Limits of crease patters through paper folding

Which of the following crease patterns are possible through paper folding that allows both folds and folds and unfolding and refolding:



State the generalization behind these observations.

5.3. Checking if the Generalizations have Counterexamples.

Check if the following generalizations are true. For each generalization you think is false, show a counterexample.

G7: Given any two points p_1 and p_2 on a sheet of paper, it is possible to fold it such that the crease passes through both points.

G8: Given any two points p_1 and p_2 on a sheet of paper, it is possible to fold it such that p_1 lies on top of p_2.

G9: Given any two lines l_1 and l_2, we can fold l_1 over l_2.

G10: Given any line l_1 and a point p_1, we can make a crease perpendicular to l_1 passing through p_1.

G11: Given any line l_1 and a point p_1, we can make a fold such that p_1 lies on l_1.

G12: Given any line l_1 and any two points p_1 and p_2, we can make a fold that passes through p_1 and places p_2 on l_1.

G13: When a folded sheet is folded again, the two creases intersect.

G14: When a folded sheet is folded again, the two creases touch each other.

G15: It is possible to make any polygon with creases.

G16: It is possible to make a circle with creases.

G17: Given any two points p_1 and p_2 on a sheet of paper, crease c_1 that passes through p_1 and p_2 is perpendicular to crease c_2 that results from a fold that places p_1 on top of p_2.

> _Note to the teacher:_ These exercises are designed to develop the capacity to discover patterns and test the patterns on the basis of examples. This is very similar to the *hypothesis testing* model of research in science in which we arrive at test observable generalizations (hypotheses). The next step involves moving from scientific hypothesis testing to mathematical *conjecture proving* model.

5.4. Separating Axioms from Conjectures

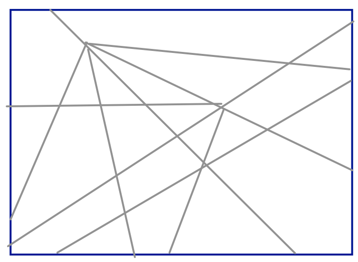
Some of the generalizations you discovered in the previous section might view as so obvious that you don't need to prove them (= demonstrate that they are true.) These you can take as *axioms*, that is, statements that do not require proof. Some of the other generalizations may be such that you think they are true, but would need to prove that they are true. These are *conjectures*. Once a conjecture is proved, it becomes a *theorem*.

Try to prove as many of the generalizations listed above as true (or false, as the case may be) on the basis of the other generalizations. Once they are proved, they become the theorems of origami math. Of the unproven statements, identify the axioms that you can take as the foundations of origami math without proof. The remaining propositions would be conjectures which you have not been able to prove yet.

Central to the enterprise of mathematics is to minimize the number of axioms we need to postulate as self-evident and maximize the number of conjectures that we can prove and establish as theorems.

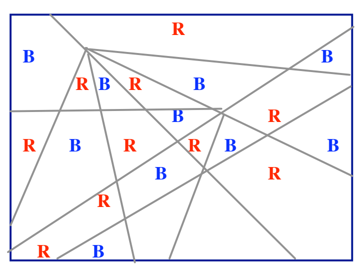
6. Two Colour and Four Colour Theorems

Take a rectangular piece of paper, fold it many times (fold on fold), unfold it, fold it many more times and unfold, and repeat. You have a number lines on the paper outlining a number of shapes. Let us call the result a "crease map", a crease map being one created by folding the paper. Here is a possible crease map as an example:



Crease map 1

Suppose you were asked to colour the shapes on your crease map such that no two adjacent shapes have the same colour ("adjacent" shapes being defined as two shapes sharing a single line segment as their boundary. Your task is to use the minimum number of colours. Let us try the colouring task on crease map 1, using **B** for blue and **R** for red.

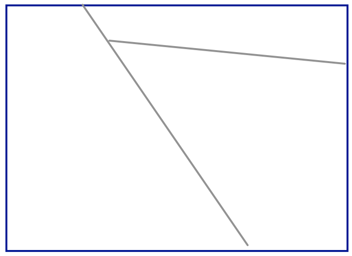


Crease map 2

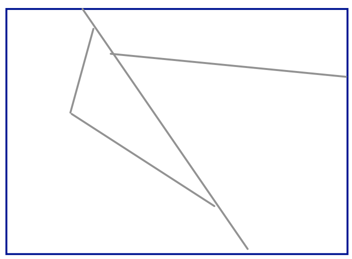
For this particular task, we have been able to use only two colours to accomplish the task of colouring the paper without shapes of the same colour touching each other. Try to do this with the crease maps you have made, and see if you really need more than two colours. Get some of your friends to make crease maps and find out if they need more than two.

<u>*Note to the teacher*</u>: They won't need more than two colours. If any student comes up with a crease map in which (s)he has used three or more colours, you can easily demonstrate that it was unnecessary to have used more than two. In the literature on origami math, this is called the Two Colour Theorem.

Suppose you are given the following map:



It is easy to see that two colours are not sufficient to colour this map such that no two adjacent shapes have the same colour. And in the following map we need at least four:



So why is it that in crease maps created by paper folding two colours are sufficient? If you think deeply about this question you might be able to come up with a proof of the two colour theorem for crease maps. (Clue: observe patterns of crease joints.)

After you have done this, read up on the famous four colour theorem in Wikipedia. (http://en.wikipedia.org/wiki/Four_color_theorem)