Introduction to Conceptual Inquiry: Notes

K P Mohanan, IISER Pune (mohanan@iiserpune.ac.in)

The Video

What follows is a set of notes on an educational video on Conceptual Inquiry (Parts 1 and 2 at http://www.youtube.com/channel/UCApqaCflm3mVTnfh9VBfAPg). The video was made from a recording of a classroom session in a five-day workshop (17-21 June 2012) on scientific inquiry at IISER Pune, for 10th grade scholarship holders of the NCERT National Talent Search scheme in India.

If you are a school or college student, I would recommend that you watch the video first, and then read this .pdf file. If you are an educator interested in Inquiry-Oriented Education, it may perhaps be a good idea to read these notes, watch the video, and then come back and read it again.

If you are a math educator, I would be grateful for your comments on the material covered in the video and the notes, especially the discussion on Part 1, and the tasks in the final section.

Educational Goals

The goal of the five-day workshop was to initiate secondary school students to *scientific*, *mathematical*, *conceptual*, and *historical inquiries*, by providing a *first hand experience* of a variety of *activities* that call for these modes of inquiry. As part of this enterprise, the goal of the session on conceptual inquiry was:

- to help students gain a broad *understanding of the nature of conceptual inquiry:* how, on the one hand, it is distinct from mathematical and scientific inquiries, and on the other, the foundational concepts in math, science, and the 'humanities' call for conceptual inquiry; and
- to give them the *experience* of engaging in conceptual inquiry, to help them become *sensitive to the strategies of conceptual inquiry*, as the first step towards developing the *capacity to do conceptual inquiry*.

Though designed for 10th grade students, college students and even PhD students will benefit from the text discussion and the video if they are interested in inquiry/research.

What is Conceptual Inquiry?

Conceptual inquiry is the investigation of abstract concepts like parallel lines, straight lines, circumscription, entropy, time, space, species, inheritance, homology, life, community, social identity, language, consciousness, free will, knowledge, truth, science, democracy, fairness, and justice. Since such concepts are present in every discipline, conceptual inquiry cuts across disciplinary boundaries.

Why Conceptual Inquiry?

What do we need conceptual inquiry for? What function does it serve in mathematics, science, and other pursuits?

One answer is that the investigation of many important questions in all disciplines requires the prior investigation of conceptual questions. "Do fruit flies have consciousness?" is a scientific question, but in order to answer this question, we have to first address the question, "What is consciousness?" This calls for conceptual inquiry. "For every triangle, there is exactly one circle that circumscribes it, and exactly one circle that it circumscribes," is a mathematical conjecture, but to prove this conjecture, we need to answer the question, "What is circumscription?" This also calls for conceptual inquiry. "Is the justice system of imprisonment morally justified?" is a philosophical question, but to answer this question, we need to answer the question, "What is moral?" Again, this calls for conceptual inquiry.

Another answer is that there are many concepts and propositions that we tend to take for granted, but when we investigate them closely, they reveal themselves to be flawed or ill-understood. These foundations need to be rebuilt before we can build reliable knowledge on them. Biology textbooks, for instance, assert that acquired traits cannot be inherited. To critically evaluate this claim, we need to know what the terms 'acquired traits' and 'inherited traits' mean. When we ask, "What is an acquired trait?" and "What is an inherited trait?" we discover that we don't clearly understand these concepts. When we push further, and demand clear and precise definitions, we discover that given the definitions and given what we know in biology today, the claim that acquired traits cannot be inherited is false, and hence should be removed from biology textbooks.

In sum, conceptual inquiry is a powerful tool for *quality control* in every discipline: it sheds light on *ignorance clothed in the terminologies taken for granted, and unexamined beliefs that on close examination reveal themselves to be false*.

Article 51A(h) of the Indian Constitution holds that Indian citizens have a fundamental duty "to develop the scientific temper, humanism and the spirit of inquiry and reform." One of the central pillars of the rational temper that underlies the scientific temper is the ***doubting and questioning*** of unexamined concepts and beliefs. If we want to fulfill our constitutional obligation, it is crucial that our future citizens develop the habit of doubting questioning both themselves and the 'experts'. For students to learn to use the tools of conceptual inquiry to doubt and question the conclusions presented to them as 'knowledge' in their textbooks is an important step towards developing the rational temper and the spirit of inquiry.

Components of Conceptual Inquiry

The core strands of conceptual inquiry exemplified in the video are:

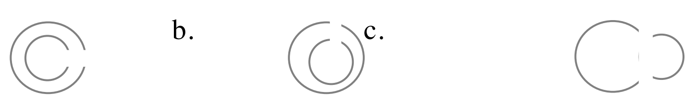
A. *Problematizing a concept* that is often taken for granted, typically by asking a question of the form, *"What is X*?" where X is the concept we wish to investigate.
B. Answering the question with a preliminary *definition* of the concept.
C. *Testing the definition*, by deducing its logical consequences (= predictions), and checking if the predictions agree with our judgments on a range of examples.
D. *Modifying either the definition or the judgments on the examples*, such that the logical consequences of the definitions match the judgments.

This list does not include other tools of conceptual inquiry such as concept amalgamation, comparison with related concepts, analogical thinking, and drawing distinctions. A-D includes only those strands that are used in the introductory video.

Examples

What are parallel lines? (step A). The definition "Two lines are parallel if and only if they are equidistant," is a plausible response to the question (step B). Consider the following examples:

(1) a. b. c.



The two lines in (1a) are equidistant, and those in (1b, c) are not, so it follows from our definition that the lines in (1a) are parallel, while those in (1b, c) are not (step C). If we judge the lines in (1a) to be not parallel, it is inconsistent with the first of these predictions. Given the requirement of logical consistency in rational inquiry, one possible option is to abandon our earlier judgment and accept (1a) as an instance of parallel lines (step D). An alternative would be to change our definition to: “Two lines are parallel if and only if they are straight lines and they are equidistant.” Given that circles are not straight lines, it follows from the revised definition that the concentric circular lines in (1a) are not parallel lines.

What is circumscription? (step A). The typical answer is: “A circumscribes B if and only if all the vertices of B lie on the circumference of A.” (step B). Consider a triangle and a circle inside touching all the three sides of the triangle, without intersecting. We judge this to be an example of a triangle circumscribing the circle, but given that a circle does not have vertices, it follows from our definition that this is not an example of circumscription. (Step C). If we wish to take it an instance of circumscription, we must modify our definition of circumscription (step D).

What is a species? (step A) The standard textbook definition is: a population of living organisms constitutes a species if and only if the members of the population can mate to produce fertile offspring (step B). Are there species of bacteria? Given that bacteria don’t mate, it follows from our definition that there cannot be species among bacteria. Yet the general consensus is that there are (step C). We must either look for a different definition of species consistent with our judgments on bacterial species, or take the position that every bacterial organism constitutes an independent species.

Given our species concept, *can an organism simultaneously belong to more than one species?* Most biologists would say no. If an organism belongs to species X, it cannot belong to a distinct species Y. Now, the well-known textbook phenomenon of ‘ring species’ (http://en.wikipedia.org/wiki/Ring_species) creates a problem for this view. The phenomenon can be described as follows. Populations A and B can interbreed to produce fertile offspring, and so can populations B and C. Populations A and C, however, cannot interbreed. Given the definition of species, it follows that A and B belong to the same species, and so do B and C, but A and C belong to two distinct species. It follows, therefore, that B belongs to two distinct species, one which includes A, and the other which includes C. This conclusion logically contradicts our judgment that no organism can simultaneously belong to more than one species: given the phenomenon of ring species, we must reject this judgment, and accept the conclusion that it is possible for an organism to belong to multiple distinct species. Since species are categories, it points to a system of categories (including genus, family, order, …kingdom, and domain) in which an organism can belong to multiple categories and a category can belong to multiple higher categories. This conclusion has far reaching consequences for the methodology of constructing phylogenetic trees based on similarities and differences at the molecular level.

In all these examples, we have appealed to the *prohibition of logical contradictions*. For a telling example of this pattern of arguments, see the discussion of the two concepts of ‘equal’ in the coursera course Introduction to Mathematical Philosophy at https://class.coursera.org/mathphil-001/lecture/index. (25 minutes, lectures 1-5 to 1-7)

Learning Outcomes: Awareness, Sensitivity, and Abilities

The most immediate learning outcome of the practice of conceptual inquiry is the realization that *we do not really understand many concepts that we think we understand*. Most high school and college students feel confident about their knowing the concepts of straight lines, parallel lines, circumscription, volume, and science because they ‘learnt’ these concepts in elementary school. Their realization that they don’t understand these concepts comes with a shock. *Clarification of what we do not know* is a fertile ground for learning to happen.

Another important learning outcome of this session on conceptual inquiry is that of becoming *sensitive to the art and craft of constructing definitions*. A one-hour session in conceptual inquiry is hardly sufficient to develop *the capacity to construct clear and precise definitions,* but that capacity can develop with further practice.

Likewise, this session can help students become sensitive to *the capacity for rigorous reasoning without the use of symbols, formulae, and equations* (step C), such that with further practice they develop this capacity. While students in mathematics and the physical sciences do follow mathematical proofs in terms of formal symbols and learn to make calculations using symbolically expressed formulae and equations, they typically don't extend this to *the rigour of reasoning without formalisms,* needed both within and outside math.

The session can also help *develop the capacity to look for and identify examples and scenarios relevant for the testing of definitions.*

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Video Part 1: Conceptual Inquiry in Mathematics

Learning Outcomes of Understanding

Empirical vs. mathematical concepts of line and point

Most students think of the concepts of point and line in terms of dots and lines they can draw on paper, which they can see, and in the case of a line, measure its length with a ruler. These are the scientific/empirical concepts of points and lines; they are different from the abstract mathematical concepts of points and lines.

Euclidean plane geometry conceptualizes a ***point*** as an entity with zero magnitude (one that does not occupy any space even though it has a location), and a ***line*** as an entity that has length, but no breadth. This view implies that points and lines must be judged to be invisible to the human eye, even though we represent them physically on paper with a visible dot (which occupies an area, however small, and hence has both length and breadth) and a visible line (which also occupies an area). The degree of abstraction needed for conceptualizing invisible points and lines doesn't come naturally, and needs nurturing at the secondary school level. Chances are that younger children may not have the intellectual maturity for such abstraction.

The Euclidean view also implies that a line cannot be thought of as a string of points next to each other. No matter how many zero length points we put along a given dimension, they don't add up to anything more than zero length. (Euclid treats both lines and points as undefined primitives, instead of defining lines in terms of points.)

The video session discusses the possibility of *defining a line* as a set of points with exactly two neighbours, the starting point and the end point having only one neighbour each. But this definition doesn't work in Euclidean geometry. To understand why, let us look at a difference between integers and fractions.

Integers have neighbours on both sides, antecedents that precede them, and successors that follow them. Thus, 5 has an antecedent, 4, and a successor, 6. No integer intervenes between them. In contrast, a fraction (which is the sane as decimals) has no neighbour (e.g., between 5 and 6, we have decimals like 5.1 and 5.2, and between them, an unending array of decimals like 5.11, 5.111, and 5.11111111). When numbers are integers, the so-called 'number line' is a set of discrete points. But when they are fractions, they form a continuum in which no point has a neighbour. The line in Euclidean geometry is like that of decimals, not integers, because points have zero magnitude, and hence there can be an infinite number of points between any two points on a line. As a result, no point on a line in Euclidean geometry can have a neighbour.

As a result, Euclidean geometry has no way of specifying the ***length*** of a line in terms of the number of points it is composed of, because any line, however short, has an infinite set of points. It can specify when two straight lines have the same length by putting one on top of the other. If they coincide (if they are 'congruent'), they have the same length. This doesn't work if the lines are not straight. For instance, if we take a straight line and an arc, or an arc and a wavy line, we can't tell if they have the same length.

What is a straight line?

One question raised in the video is: "Can two parallel lines meet?" The answer is, "Two parallel straight lines cannot meet, but two parallel curved lines can." This answer prompts the question, "What is a straight line?" Here is a possible answer:

> A line between two points A and B is straight iff it is the shortest path between them.

Given this definition, to decide whether or not a line AB is straight, we need to know if it is the shortest path between A and B. But distance (length) has no interpretation in Euclidean geometry. Hence, the concept of straightness of lines is undefinable in terms of the shortness (length) of the path. [Notice that this problem also plagues the definition of parallel lines as equidistant lines. What is 'distance' such that two lines are equidistant?]

Instead of the 'shortest path' idea, suppose we propose the following definition:

> A line is straight iff its slope is constant throughout.

The idea seems eminently reasonable. But slope presupposes coordinates. When we say that a straight line has constant slope, we are probably thinking about coordinates which are themselves straight lines. Suppose we have a wavy line as one of the coordinates and an arc as another coordinate. Would a line with constant slope in such a case be judged as 'straight'?

So, neither of these definitions work. The moral of the story is: *we do not know what a straight line is*. And because we don't know what a straight line is, given the definition of parallel lines as equidistant lines, *we cannot tell whether the conjecture that two parallel straight lines cannot meet is true*.

Likewise, take the conjecture: *No two straight lines meet at more than one point*. We take this to be true in Euclidean geometry. However, since we don't what a straight line is, we can't tell if this conjecture if true. Similarly, we think of a polygon as a closed shape bounded by straight lines. Again, since we don't know what a straight line is, we can't know what a polygon is.

From Conceptual Inquiry to Mathematical Inquiry

Once we know within the paradigm of mathematical inquiry that we don't know what lines, straight lines, and parallel lines are, how do move towards a solution that forms the basis for better understanding? The video doesn't address this question, as it stops with ***critical thinking*** that demonstrates ignorance. But it is indeed possible to explore possible solutions in a series of sessions that begin with ***theory building*** at a broad conceptual level, going on to theory building in mathematics proper.

The starting point for a solution lies in the concept of a line as an ordered set of points such that every point except the starting point has a(n immediate) **antecedent**, and every point except the end point has a(n immediate) **successor**, exactly like in the integer line. If we abstract out ***adjacency*** as the property shared between a number and its antecedent or successor, we see that this line of thinking connects us to topology, and its concepts of ***neighbour*** and neighbourhood. (See the Wikipedia entry on neighbourhood in mathematics at http://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Adjacency is also found in *graph theory* where nodes (also called vertices) correspond to points, and arcs (also called edges) signal adjacency between two points. In this geometry, a ***path*** is a set of connected points, corresponding to *line* in Euclidean geometry. In a graph theoretic geometry of this kind, the length of a line is defined as the number of points in a path, and the shortest path between two points is the path that contains the smallest number of points.

Adjacency is also found in *taxi-cab geometry*, where an intersection between streets is the counterpart of a point. (For an introduction, see http://taxicabgeometry.net/) In this geometry, the path connecting two adjacent intersections has length. The length of the path between two non-adjacent intersections, then, is the sum of the lengths of the paths between adjacent intersections in that path.

We may also play with a *graph paper geometry* as a special case of taxi-cab geometry in which the paths between any two adjacent intersections has the same length, and every point has exactly four neighbours. Suppose we change the number of neighbours a point can have, say, to two or three or five. If every point has exactly two neighbours, we get a one-dimensional graph theoretic geometry. If extended to a graph theoretic geometry in which every point has infinitely many neighbours on a single flat surface, we have the ***discrete*** counterpart of (the ***continuous)*** Euclidean geometry that allows for the definition of length.

Finally, we could develop a geometry in which a point in a one-dimensional geometry has ***infinitesimally small*** (instead of zero) length, and in a two dimensional geometry has infinitesimally small area. The length of a line will then be the number of points it contains.

Each of the concepts we have talked about can act as a seed for the practice of rigorous conceptual inquiry, leading to a particular theory of geometry. If we then proceed to look for plausible conjectures and proofs within these geometries, we will be practicing mathematical inquiry. What we discover need not develop into a publishable paper: the purpose of these activities is to help us develop the capacity for conceptual and mathematical inquiries.

The investigation of such multiple geometries leads to a deeper understanding of the *nature of mathematical truths*. On a Euclidean surface, there is exactly one straight line between two points; but in taxi-cab and graph theoretic geometries, there can be more than one, and on a spherical surface, there are infinitely many straight lines between two polar points. If parallel lines are defined as 'equidistant lines', then parallel lines can meet on a Euclidean surface. On the other hand, if they are defined as 'equidistant straight lines', they cannot meet.

The lesson to take away from this is discussion is that *the truth of a mathematical conjecture depends on the definitions and axioms of the theory: definitions generate the mathematical objects, relations, and operations of the theory; and the axioms govern the world in which the objects, relations, and operations exist*.

Video Part 2: Conceptual Inquiry in 'Philosophy'

Part 1 illustrated the strategy of beginning with:

- a *question* of the form "What is X?";
- constructing a clear and precise *definition* of X in response;
- critically evaluating the definition by deducing its *logical consequences* and checking how well these consequences match the propositions that we are already committed to; and,
- if there is a mismatch, eliminating it by *modifying* either the definition or the judgment that we are committed to.

This strategy applies in other domains as well: science, morality, law, religious beliefs, and so on. To answer the question, "Does free will exist?" or to evaluate experiments in neuroscience to demonstrate that free will does not exist, we need to first define free will. Likewise, to critically evaluate the claim that IQ tests constitute a reliable measure of intelligence, we need to define intelligence. And to rationally inquire into the existence of soul and God, we need to define soul and God. Any debate on such issues without a definition agreed upon by the debating parties would be a waste of time, without any advancement of understanding.

Part 2 of the video begins by extending the conceptual inquiry strategies learned in mathematics to the question, "What is democracy?" such that we can answer the question, "Is there democracy in India?" The central point demonstrated by this exercise is that we get the answer 'yes' if we adopt definition 1, and 'no' if we adopt definition 2:

Definition 1: Democracy is a system of voting to elect representatives to govern us.

Definition 2: Democracy is a system in which everyone affected by a decision has an opportunity to influence the decision.

Under definition 1, a system of monarchy in which rulers inherit their right to rule from their parent does not have democracy, even if the ruler provides every opportunity for the citizens to engage in discussions and debates on public issues, and decisions are shaped by the consensus arrived at through this process. In contrast, the above system would be democratic under definition 2, though there is no voting and election. By definition 1, a system in which people have the right to vote to elect members of the government body from among the leading mafia families in the country would be a democratic system, even if the government's decisions are in conflict with the people's interests. By definition 2, such a system would be undemocratic, regardless of voting and election. By definition 1, the issue of democracy is irrelevant in a classroom, but by definition 2, a classroom in which the students have a voice in deciding the topics for a course, the kinds of assignments and tests they do, and the deadlines for submitting assignments, and so on, would be a democratic classroom. By definition 1, an elected despot would be democratic, but not by definition 2.

Similar strategies apply to, "Is it morally right for a government to set up a law that requires people who attempt suicide to be put in prison?" To address this question, we need to begin with, "What is a moral action?" Moral inquiry involves two components: arriving at conclusions on the moral status of an action or practice on the basis of a moral theory; and as the basis for that activity, constructing a moral theory based on shared moral judgments. The latter calls for conceptual inquiry.

Two other case studies that illustrate different aspects of conceptual inquiry are:

"What is a salad?" at http://wiki.nus.edu.sg/display/aki/1.2.5.1.+Ex.+E+-+Activity (a tongue-in-cheekdialogue as an audio-podcast)

"What is excellence in teaching?" at http://wiki.nus.edu.sg/display/aki/1.2.5.1.+Ex.+E+-+Activity (a set of interactive tasks as text)

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The Concept of 'Same X': Triggers for further thought

If you move a circle to a different location, is it the same circle? If you rotate a triangle, is that the same triangle? What if you flip it? What if you enlarge it?

Given that circles and triangles are geometric objects, such questions suggest that we need to have clarity on the concept of 'same geometric object'. Consider the following candidates:

Two geometric objects are the same iff they:

1) occupy the same space.
2) occupy the same space or can be shown to occupy the same space after rotation and/or translation.
3) occupy the same space, or can be shown to occupy the same space after rotation, translation, and/or reflection.
4) occupy the same space, or can be shown to occupy the same space after rotation, translation, and/or scaling (enlarged/reduced).
5) occupy the same space, or can be shown to occupy the same space after rotation, translation, reflection, and/or deformation.

(If you are not familiar with the terms 'translation', 'rotation', 'reflection', 'scaling', and 'deformation', do an internet search.)

The statements in (1)-(5) define 'same geometric object' in terms of 'same space', without defining 'same space', but we will sidestep that issue for now, and assume that we know what we mean by 'occupy the same space.'

If you have watched the two-part video, more or less digested the written material above, and watched the video again, you may find it useful to go through the tasks below, based on the definitions in (1)-(5) above.

Task 1
Under which of the above definitions do the concepts of 'same geometric object' and 'congruent geometric object' converge?

Task 2
Which of the definitions would allow us to prove that given a triangle, there is exactly one circle that circumscribes it?

Task 3
Which of the definitions would allow us to prove that for a given triangle, there is exactly one circle that circumscribes it?

Task 4
Which of the definitions would allow us to prove that for given a circle, there is exactly one equilateral triangle that it circumscribes?

Task 5
What definition of circumscription would you need to prove that if A circumscribes B, then A and B cannot both be circles? (Hint: proper subset)

Task 6
Connect the definitions in (1)-(5) to (a) projective geometry, and (b) topology. (Do an internet search if you are not familiar with these terms.)

Tasks 7 to 12 below nudge you to move outside mathematics, but using the same strategies.

Task 7
Bob and Alice are looking at two oranges on the table. Bob points at them and says, "They are the same fruit." Alice shakes her head and says, "No, they are not the same fruit."

Define 'same fruit' such that Bob's claim can be proved. Then define 'same fruit' such that Alice's claim can be proved. Construct both proofs.

Task 8
Bob borrows Ernest Hemingway's novel *Old Man and the Sea* from the library in June 2012, and Alice borrows it in May 2013, and says to Bob, "I've borrowed the same book that you borrowed last June." Bob shakes his head and says, "No, we have never borrowed the same book from any library."

Define 'same book' such that Alice's claim can be proved. Then define 'same book' such that Bob's claim can be proved. Describe the facts of the matter, such that given the same facts, Alice's claim would be true under her definition, and Bob's would be true under his definition.

Task 9
In the course of a conversation, Bob says, "You are not the same person you were when you married me ten years ago." "That's ridiculous," says Alice, "I am indeed the same person."

Describe a state of affairs that Alice and Bob can both agree on. Formulate different definitions of 'same person' such that given the state of affairs, Alice's claim follows logically from her definition, and Bob's follows from his.

Task 10 (This might be a hard question.)
Having discovered why they disagreed on task 9, Alice tells Bob, "Given your definition, Bob, it wouldn't be legitimate for you to say, 'I was born in 1985.' All you can say is 'The person inhabiting this body was born in 1985,' and even that would create problems with the legitimacy of the expression 'this body'."

If Bob was born in 1985, why would his definition of 'same person' make it illegitimate for him to say "I was born in 1985"? (This might be a hard question.)

Task 11 (This too might be hard.)
Define 'same' such that we can get the definitions of 'same geometric object', 'same fruit', 'same book', and 'same person' by simply adding further specifications to the core definition of 'same'.

Task 12 (This is extra-hard.)
Explore the conceptual similarities and differences between the following:

X and Y are the same.
X and Y are identical.
X and Y are equal.
X and Y are equivalent.
X and Y are analogous.

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If you enjoyed working on tasks 9-10 (also 11-12), you will enjoy the course on Death (= end of 'personhood') by Yale Professor Shelley Kagen at http://oyc.yale.edu/philosophy/phil-176

If you thought that tasks 9 and 10 are instances of word play or splitting hairs, do watch the first three lectures of Kagen's course, and take another look at the tasks.

If you are interested in biology, you might try connecting "X and Y are the same" to "X and Y are analogous", and "X and Y are homologous". You might also try doing an internet search for analogy, homology, homoplasy, orthology, and paralogy and apply the strategies of conceptual inquiry to these terms.