# MATHEMATICAL KNOWLEDGE AND INQUIRY[*]

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## 1. What is Mathematical inquiry?

The essence of mathematical inquiry can be summarized as

A. thinking clearly, carefully, and systematically about what is known or assumed, and
B. making inferences from what is known or assumed, using classical deductive reasoning.

Mathematical inquiry, in other words, is a form of ***contemplative inquiry.*** It is quite similar in this sense to the inquiries in both theoretical science and in philosophy, but there are important differences between them in terms of what is taken as known or is assumed, and what kind of reasoning is regarded as legitimate in making inferences.

Why do mathematicians pursue this form of inquiry? Because they have a commitment to discovering deep truths about the universe, and they believe that their inquiry reveals those truths. For Plato, mathematics was the key to the higher realm of reality. Galileo held that the book of nature is written in the language of mathematics. (For an illuminating discussion, see "Why do we Study Geometry?" by Piers Bursill-Hall (2002) at http://www.dpmms.cam.ac.uk/~piers/F-I-G_opening_ppr.pdf.)

Let us take a careful look at the mode of mathematical inquiry, before proceeding to the nature of mathematical truths.

## 2. Self-evident truths and proved truths

Let us begin with the following question to illustrate the nature of mathematical inquiry:

*How many straight lines can you draw through two points?*

Suppose A and B are two points, as given in the following diagram:


*figure 1*

To make the points easy to see, they have been drawn in figure 1 as small solid circles rather than as points, but you should imagine points as occupying no space. Perhaps figure 2 is better for this purpose.


*figure 2*

Think about the question, and come back when you have found an answer. The answer would probably be quite easy and obvious to you. If you want, you could take a piece of paper, make your own points A and B on it, and try drawing straight lines through them before making a decision. But turn to the next page only after you have written down your answer.

[*] I am grateful to Professor Chong Chi Tat for suggestions on improving this write up.

It should be safe to assume that your answer is: *one and only one straight line*. (Right?)

If you were to be asked to provide a justification for this answer (to provide a "proof"), you may not be in a position to do it. You might say something like,

> "Why should I prove it? It is obvious that one and only one straight line can be drawn through two points." Or
>
> "I don't think I can prove it, but it looks so obvious to me, that it doesn't need a proof."

"Obvious truths" of this kind were called "self-evident truths" by the ancient Greeks. Self-evident truths are propositions that can safely be taken to be true without independent justification. Euclid expressed what he viewed as self-evident truths as axioms and postulates.[1] You might be pleasantly surprised to know that the answer you gave to the question about the number of straight lines through two points is the first one in the famous five postulates of Euclid:

***Euclid's postulates***

Postulate 1. Exactly one straight line can be drawn from any point to any point.

Postulate 2. Any straight line can be extended indefinitely in a straight line.

Postulate 3. Given any straight line, a circle can be drawn having the line as radius and one endpoint as center.

Postulate 4. All right angles are equal to one another.

Postulate 5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough

Can you now answer the following question?

*Can two straight lines intersect at two different places?*

When you have arrived at an answer, try to prove it. If you take postulate 1 as given (the grounds), it should not be too difficult to come up with a proof. Turn to the next page only after you have found a proof (or if no inspiration hits you with a proof even after a serious attempt.)

[1] Euclid used the terms *common notions* (which is the same as axioms) and *postulates* with slightly different meanings, but this distinction is not consistently maintained in modern mathematics. Hence we will use *axioms* and *postulates* as synonyms.

If by some chance, you haven't had an *aha!* moment of inspired illumination, here is a clue. Begin by assuming, contrary to what you want to prove, that there exist two lines L-1 and L-2 through points A and B. On paper, we can draw L-1 and L-2 by making them slightly curved, even though they are not supposed to be. (Imagine L-1 and L-2 as straight lines in figure 3.)

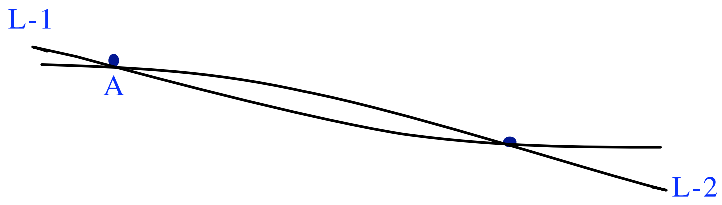


*figure 3*

Can you now prove that L-1 and L-2 are the same line (i.e., that even though look distinct in figure 3, they are not distinct)? Don't peek at the next page without really trying.

Okay, here is the proof. (Don't worry about the format: pay attention to the substantive aspect of the reasoning.)

To prove: Two straight lines cannot intersect at two places.

Proof: Assume, contrary to what we wish to prove, that there exist two distinct straight lines L-1 and L-2 that intersect at two points A and B.

Given Euclidean postulate 1, however, only one straight line can be drawn through two points. Hence, it must be the case that L-1 and L-2 are the same.

But we have assumed that L-1 and L-2 are not the same. This results in a logical contradiction, namely, that L-1 and L-2 are the same and L-1 and L-2 are not the same.

Our starting assumption that there exist two distinct straight lines L-1 and L-2 that intersect at two points A and B is false.

Therefore, we conclude that two straight lines cannot intersect at two places. (QED)[2]

The structure of the proof can be diagrammatically represented as figure 4:

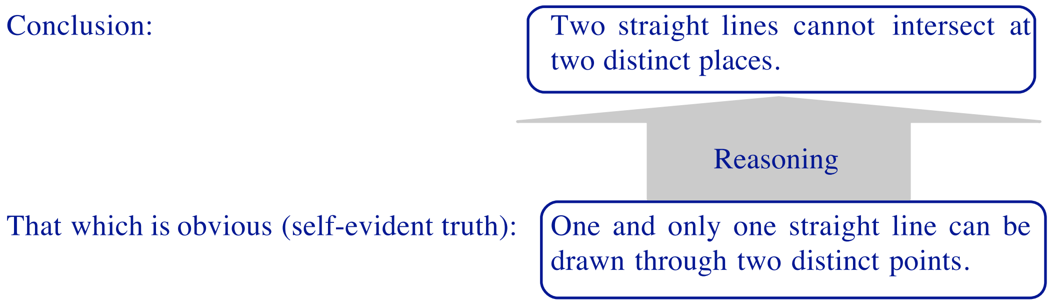


*figure 4*

This is an illustration of how we arrive at a conclusion in mathematics by reasoning from what we take to be self-evident.

If we begin with propositions that we take to be true beyond any doubt, and if the deductive reasoning that we apply to these propositions has the property that if the premises are true, the conclusion from a valid sequence of reasoning cannot be false, we have a way of building totally certain knowledge on totally certain foundations:

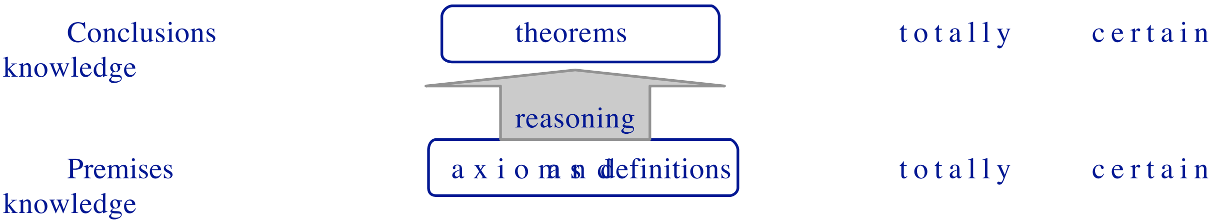


*figure 5*

Let us pause and reflect. We took as "self-evident" the proposition that one and only one straight line can be drawn through two points, because it seemed so intuitively obvious to us. So why can't we adopt the same strategy for the proposition that no two straight lines can intersect at more than one point, responding as follows?

[2] QED is an abbreviation for *Quod erat demonstratum*, a Latin phrase that means 'that which was to be demonstrated/proved."

"Why should I prove it? It is obvious that one and only one straight line can be drawn through two points." Or

"I don't think I can prove it, but it looks so obvious to me, that it doesn't need a proof."

The answer takes us to the very heart of mathematical inquiry. Mathematicians are committed to the project of deducing the largest number of theorems from the smallest number of axioms. *The fewer the axioms of a theory, and the larger the number and range of theorems deduced from them, the better the theory.*

Theorems:
propositions which have been proved

output:
as many as possible

Axioms:
propositions that cannot be proved

input:
as few as possible

The proposition about intersecting lines can be deduced from something else, so we don't take it as an axiom. In contrast, we can't find a way of deducing the number of straight lines through two points, so we are forced to treat it as an axiom. We prove as many knowledge claims as possible, and treat a proposition as an unprovable but necessary assumption only as the last resort.

The idea of pursuing knowledge by deducing as many propositions as possible is central to theoretical science as well, where "axioms" correspond to the statements of the theory (laws, principles, etc) and "theorems" correspond to the predictions deduced from the theory. You have probably heard about it as the famous *Occam's Razor*, the simplicity criterion, or as "parsimony." The counterpart of this principle in practical affairs is *efficiency*, accomplishing the maximum effect with the least amount of work.

Why are mathematicians and scientists committed to the simplicity criterion in their inquiry? Because they believe that Reality is ultimately governed by a very small set of organizational principles. This is one of the central tenets of the secular faith of the community of mathematicians and theoretical scientists.

### 3. Self-evident truths are not so self-evident after all

The straight lines you were thinking of in the proof in the previous section were most likely straight lines on a plane surface, like the paper or computer screen you are looking at now. What about straight lines on a curved surface?

Imagine a transparent hemi-spherical cup, filled with water. The surface of the water where it meets the air would be a flat plane, while the surface where it meets the cup would be curved (hemispherical).

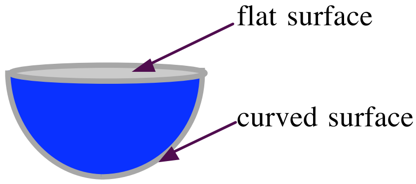


*figure 6*

What would straight lines be like on a curved surface?

We can find a way to answer this question by thinking about the intersection between two flat surfaces, like this:

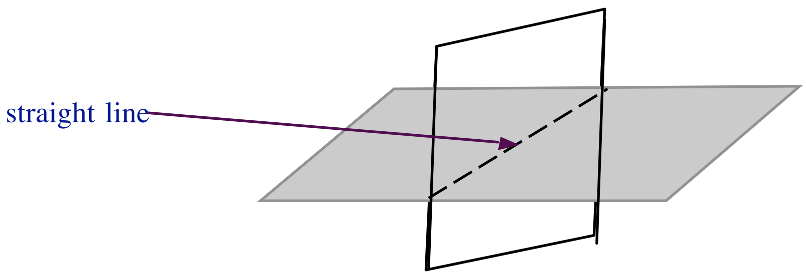


*figure 7*

The intersection of two flat surfaces is a straight line (the dotted line in figure 7), just as the intersection of two straight lines is a point:

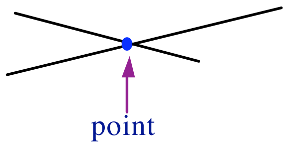


*figure 8*

Suppose we generalize from this result, and define a line, a straight line and a point as follows:

Definition 1: A point is an intersection between two lines.

Definition 2: A line is an intersection between two surfaces.

Definition 3: A straight line on a surface S is an intersection of a flat surface with S. [3]

Let us go back to the water in our hemispherical cup. Imagine now a sphere half filled with water, as in figure 9. For an ant sitting on the surface of the water, the edge of the water would be a circle, not a straight line. For an ant sitting on the surface of the sphere outside, however, the edge of the water would be the result of the intersection of the spherical surface with a flat plane, which means by definition 3 it is a straight line:

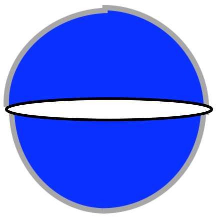

*figure 9*

Think of it this way. If an ant on a sphere were to start from point X and walk on a straight line, it will return to point X?

Going by definitions 1-3, how many straight lines can you draw between two points on a spherical surface?

Do not peek at the next page until you have thought through this carefully and formulated your answer.

[3] Euclid begins with the intuitions of point, line, and straight line, and defines a flat surface as one that lies on two straight lines. Definitions 1-3 reverse this move, and defines (straight) lines in terms of (plane) surfaces.

You must have realized by now that infinitely many straight lines can be drawn across two points on a spherical surface. Imagine a perfectly spherical apple. Select two points X and Y on the surface of the apple such that XY is a line passing through the centre of the apple (i.e, XY is the diameter.) Now imagine cutting the apple with a sharp metal plane through XY, creating crescent shaped slices of the apple.

If you put the slices together to get back the whole sphere, each line on the apple (the edge created by the cut) will be a straight line passing through XY. There will be as many straight lines through X and Y as there are cuts.

What you thought was a self-evident truth now turns out to be a self-evident truth about flat surfaces. The geometry of curved surfaces works differently.

Is the statement that two straight lines cannot intersect at two distinct points true in the geometry of spherical surfaces?

Can you prove that the following statements are true in the geometry of spherical surfaces?

If two straight lines intersect at any point, they also intersect at some other point.

No two straight lines can intersect at more than two points.

Can you think of surfaces on which straight lines intersect at more than two points?

### 4. The nature of mathematical knowledge and inquiry

When developing the foundational axioms of his theory of geometry, Euclid implicitly assumed a (flat) plane surface, resulting in the axiom that exactly one (one and only one) straight line can be drawn through two straight lines. Hence, what was taken as "self-evident" truth for more than two thousand years turns out to be true only in a particular kind of geometry.

<u>*Mathematical axioms*</u>

*In the geometry of flat surfaces, one and only one straight line can be drawn across two points.*

*In the geometry of spherical surfaces, infinitely many straight lines can be drawn across two points.*

The property of "true in theory X but false in theory Y" applies to mathematical theorems as well:

<u>*Mathematical theorems.*</u>

*In the geometry of flat surfaces, no two straight lines can intersect at more than one point.*

*In the geometry of spherical surfaces, two straight lines can interest at two points.*

Euclidean geometry is geometry of plane surfaces. Geometries of curved surfaces were not developed until the nineteenth century by mathematicians who deviated from Euclid in their foundational axioms. The example you have seen above, the geometry of spherical surfaces, called elliptical geometry, was developed by Georg Friedrich Bernhard Riemann in 1854. Riemannian geometry was used by Albert Einstein in developing his theory of relativity. Another form of non-Euclidean geometry, called hyperbolic geometry, was developed independently by

János Bolyai and Nikolai Ivanovich Lobachevsky in the 1830's, prior to the birth of spherical geometry.

Take a question that most school children are prepared to answer without hesitation.

*What is the sum of angles in a triangle?*

Chances are that they would say, "Two right angles." Or "180 degrees." This answer would be correct within Euclidean geometry, but not in non-Euclidean geometries. In Riemannian geometry, the sum of angles in a triangle would be more than two right angles, and in Bolyai-Lobachevsky geometry, it would be less than two right angles. This means that

*Is it true that the sum of angles in a triangle is two right angles?*

is not an answerable question. We should be asking

*Is it true that in Euclidean geometry, the sum of angles in a triangle is two right angles?*
*Is it true that in Riemannian geometry, the sum of angles in a triangle is two right angles?*

The answer to the first question is yes, while the answer to the second question is no.

These statements might sound somewhat unsettling if you haven't heard about non-Euclidean geometries prior to this discussion. Still more unsettling might be the realization that if Einstein's theory of relativity is correct – and there is irrefutable evidence to demonstrate that it is – then the universe we live in is Riemannian, not Euclidian.

The existence of alternative theories is not restricted to geometry. It is also true of numbers. For instance, in ordinary arithmetic – the kind that school children are taught – the sum of 8 and 6 is 14. In what is called modular arithmetic, with base ten, the sum of 8 and 6 is 4. This may sound shocking to you, but if you find out what modular arithmetic is, the shock will wear off quite soon. Likewise, in the kind of algebra that children are taught,

$$a\,(b + c) = a\,(c + b)$$

This is not true in what is called Grassman's algebra, where

$$a\,(b + c) = -\,a\,(c + b)$$

That is,

$$a\,(b + c) + a\,(c + b) = 0$$

If you think that Grassman's algebra is some kind of weird stuff that we don't have to take seriously, think again: this body of mathematics has a central place in the string theory of subatomic physics.

The existence of alternative theories of mathematics, and the idea of mathematical statements being true only relative to a theory, has important consequences for the understanding of the notion of mathematical truth, and the nature of mathematical knowledge and inquiry, which we will not go into in this brief introduction.

## 5. Why should non-mathematicians bother about mathematics?

The teaching of mathematics in primary and secondary schools focus almost exclusively on helping students to

1) understand the body of knowledge resulting from mathematical inquiry, and
2) acquire proficiency in making calculations as part of the application of that body of knowledge in a set of standard situations.

Thus, a primary school student is taught the arithmetical operations of adding, subtracting, dividing and multiplying, and trained in the use of these operations in calculating how many apples each person would get when a basket of 60 apples is divided among 12 children, or how much money a cashier ought to give you back if you give him a ten dollar note as payment for eight apples that cost 75 cents each.

As a result of this mode of teaching, most educated people have the impression that mathematics is largely a matter of making calculations with abstract symbols, and that except for arithmetical operations, mathematics is irrelevant in their lives. What is missing in this tradition are two ingredients that are of value in an educated individual's capacity for thinking and inquiry, namely:

3) the ability to model (physical, biological, mental, and socio-cultural) phenomena in terms of precise and explicit mathematical concepts, and pursue their logical consequences in a rigorous way, going beyond what calculators can do (applied mathematics), and

4) the ability to employ the modes of mathematical inquiry outside mathematics, that is, as
    a) thinking clearly, carefully, and systematically about what is known or assumed, and
    b) making inferences from what is known or assumed, using classical deductive reasoning.

We explore (3) in another context. What I have tried to do in the preceding sections is to provide and experiential feel for the mode of inquiry in (4), crucially involving (intuitively obvious and non-obvious) axioms, definitions, conjectures, theorems and proofs. How this mode of inquiry can be used in domains outside mathematics, we will explore at a later point. While (1) and (2) are indeed useful outside of research and teaching in mathematics, the deeper but unappreciated value of mathematics for the educated non-specialist lies in (3) and (4).