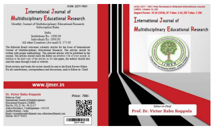
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PROPAGATION OF DILATATION AND DISTROTIONAL WAVES IN MICROMORPHIC MEDIUM

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ABSTRACT:

In this paper, the propagation of dilatation and distortional waves in micro-isotropic. Micro-elastic medium has been discussed. Frequency equations have been deduced in both the cases. It is found that additional waves are propagating in the Micro-morphic medium. The classical results are obtained as some particular cases of these.

Introduction: The theory of micro-morphic materials was proposed by Eringen [1]. This theory was modified by Koh [3] by extending the concept of coincidence of principal directions of stress and strain of classical elasticity to micro-Elastic solid and assuming a particular form of isotropy. He called this modified theory as theory of micro-isotropic, micro-elastic materials. In this paper the problem of propagation of dilatations and distortional waves is discussed in medium as developed by Koh [3].

In Classical elasticity [2] it was shown that the equations of motion correspond to two types of waves which can be propagated through an elastic solid. These two types of waves are called dilatation and distortional. In the present case, in each of the situation some additional waves are found compared to the classical elasticity. These additional waves are exiting due to extra degree of freedom such as micro-rotation and micro-strain of micro-isotropic micro-elastic medium.

BASICEUQATIONS: The micro-displacement in the micro-elastic continuum is denoted by u_k and micro-deformation by ϕ_{mn}. Further the macro-strain $e_{km} = u_{(k,m)}$,the macro-rotation $r_k = \frac{1}{2} \in_{kmn} u_{n,m}$, the micro strain $\phi_{(mn)}$ and the micro-rotation vector $\phi_p = \frac{1}{2} \in_{pkm} \phi_{km}$ where( ) denotes the symmetric part, comma denotes differentiation with respect to the coordinate (x_k). The stress measures are the asymmetric stress t_{km}, the relative stress σ_{km} and the stress moment t_{kmn}. The couple stress tensor m_{kl} is defined by $m_{kl} = \in_{pnm} t_{kmn}$

The constitutive equations for the theory of micro-isotropic, micro-elastic materials are given by Koh and Parameshwaran.[4] They are

$$t_{(km)} = A_1 \tau_{pp} \delta_{Km} + 2A_2 e_{Km} \qquad (1)$$

$$t_{[km]} = \sigma_{[km]} = 2A_3 \in_{pkm} (r_p + \phi_p) \qquad (2)$$

$$\sigma_{[km]} = -A_4 \phi_{pp} \delta_{km} - 2A_5 \phi_{(km)} \qquad (3)$$

$$t_{k(mn)} = B_1 \phi_{pp,k} \delta_{mn} + 2B_2 \phi_{(mn),k} \qquad (4)$$

$$m_{(kl)} = -2\left(B_5 \phi_{l,k} + B_4 \phi_{k,l} + B_5 \phi_{p,p} \delta_{kl}\right) \qquad (5)$$

where [ ] denotes anti-symmetric part,

$$
\begin{aligned}
&A_1 = \lambda + \sigma_1, && B_1 = \tau_3 \\
&A_2 = \mu + \sigma_2, && 2B_2 = \tau_1 + \tau_{10} \\
&A_3 = \sigma_5, && B_3 = 2\tau_4 + 2\tau_9 + \tau_7 - \tau_{10} \\
&A_4 = -\sigma_1, && B_4 = -2\tau_4 \\
&A_5 = -\sigma_2 && B_5 = -2\tau_9
\end{aligned}
\qquad (6)
$$

And λ, μ, σ_1, σ_2, σ_5, τ_3, τ_4, τ_7, τ_9 and τ_{10} are elastic constants.

The displacement equations of motion for micro-elastic body occupying a region R are given by

$$(A_1 + A_2 - A_3)u_{p,pm} + (A_2 + A_3)u_{m,pp} + 2A_3 \in_{pkm} \phi_{p,k} + \rho f_m = \rho \frac{\partial^2 u_m}{\partial t^2} \qquad (7)$$

$$B_1\phi_{pp,kk}\delta_{ij} + 2B_2\phi_{(ij),kk} - A_4\phi_{pp}\delta_{ij} - 2A_5\phi_{(ij)} + \rho f_{(ij)} = \frac{1}{2}\rho j \frac{\partial^2 \phi_{(ij)}}{\partial t^2} \qquad (8)$$

$$2B_3\phi_{p,mm} + 2(B_4 + B_5)\phi_{m,mp} - 4A_3(r_p + \phi_p) - \rho l_p = \rho j \frac{\partial^2 \phi_p}{\partial t^2} \qquad (9)$$

where ρ is the average mass density, f_m is the body force per unit mass, $f_{(ij)}$ is the symmetric body moment per unit mass, j is the micro-inertia, l_p is the body couple per unit mass and do symbol denote differentiation with respect to time t,

subject to the boundary conditions

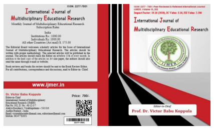
$$t_{km}n_k = T_m \quad t_{kmn}n_k = T_{mn} \qquad \text{on the boundary} \qquad (10)$$

where $\vec{n}$ is the unit outward drawn normal to the boundary T_m and T_{mn} are the surface tractions and surface moments respectively?

Dilatation waves:

The equations of motion (7) under the absence of body forces and body couples are given by

$$(A_1 + A_2 - A_3)\frac{\partial}{\partial x_1}\Delta + (A_2 + A_3)\nabla^2 u_1 + 2A_3\left(\frac{\partial \phi_2}{\partial x_3} - \frac{\partial \phi_3}{\partial x_2}\right) = \rho \frac{\partial^2 u_1}{\partial t^2} \qquad (11)$$



$$(A_1+A_2-A_3)\frac{\partial}{\partial x_2}\Delta+(A_2+A_3)\nabla^2u_2+2A_3\left(\frac{\partial\phi_3}{\partial x_1}-\frac{\partial\phi_1}{\partial x_3}\right)=\rho\frac{\partial^2u_2}{\partial t^2} \qquad (12)$$

$$(A_1+A_2-A_3)\frac{\partial}{\partial x_3}\Delta+(A_2+A_3)\nabla^2u_3+2A_3\left(\frac{\partial\phi_1}{\partial x_2}-\frac{\partial\phi_2}{\partial x_1}\right)=\rho\frac{\partial^2u_3}{\partial t^2} \qquad (13)$$

where $\Delta=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$ (14)

Similarly, under the absence of body forces and the body couples the equation (9) can be expressed as

$$2(B_4+B_5)\frac{\partial}{\partial x_1}\Delta^1+2B_3\nabla^2\phi_1$$

$$-2A_3(u_{3,2}-u_{2,3})-4A_3\phi_1=\rho j\frac{\partial^2\phi_1}{\partial t^2} \qquad (15)$$

$$2(B_4+B_5)\frac{\partial}{\partial x_2}\Delta^1+2B_3\nabla^2\phi_2$$

$$-2A_3(u_{1,3}-u_{3,1})-4A_3\phi_2=\rho j\frac{\partial^2\phi_2}{\partial t^2} \qquad (16)$$

$$2(B_4+B_5)\frac{\partial}{\partial x_3}\Delta^1+2B_3\nabla^2\phi_3$$

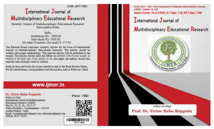
$$-2A_3(u_{2,1}-u_{1,2})-4A_3\phi=\rho j\frac{\partial^2\phi_3}{\partial t^2} \qquad (17)$$

Where $\Delta^1=\frac{\partial\phi_1}{\partial x_1}+\frac{\partial\phi_2}{\partial x_2}+\frac{\partial\phi_3}{\partial x_3}$ (18)

Differentiating (11) with respect to x_1 we get,

$$(A_1+A_2-A_3)\frac{\partial^2}{\partial x_1^2}\Delta+(A_2+A_3)\nabla^2\frac{\partial u_1}{\partial x_1}$$

$$+2A_3(\phi_{2,31}-\phi_{3,12})=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_1}\right) \qquad (19)$$



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Similarly, differentiating (12) with respect to x_2 we get,

$$(A_1+A_2-A_3)\frac{\partial^2}{\partial x_2^2}\Delta+(A_2+A_3)\nabla^2\frac{\partial u_2}{\partial x_2}$$

$$+2A_3(\phi_{3,12}-\phi_{1,32})=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_2}{\partial x_2}\right) \qquad (20)$$

differentiating (13) with respect to x_3 we get

$$(A_1+A_2-A_3)\frac{\partial^2}{\partial x_3^2}\Delta+(A_2+A_3)\nabla^2\frac{\partial u_3}{\partial x_3}$$

$$+2A_3(\phi_{1,23}-\phi_{2,13})=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_3}{\partial x_3}\right) \qquad (21)$$

Adding (19) to (21) we obtain

$$(A_1+A_2-A_3)\nabla^2\Delta+(A_2+A_3)\nabla^2\Delta=\rho\frac{\partial^2\Delta}{\partial t^2}$$

i.e., $(A_1+2A_2)\nabla^2\Delta=\rho\dfrac{\partial^2\Delta}{\partial t^2}$

$$\therefore\nabla^2\Delta=\frac{1}{C_1^2}\frac{\partial^2\Delta}{\partial t^2} \qquad (22)$$

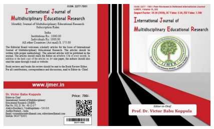
where $C_1^2=\dfrac{A_1+2A_2}{\rho}$ (23)

The equation (22) shows that the volume dilation or compression is transmitted in the form of waves through the micro-morphic medium with a velocity $C_1=\sqrt{\dfrac{A_1+2A_2}{\rho}}$.

This wave is called a wave of dilatation. The equation (22) corresponds to the classical elasticity.

In view of (6) the equation (23) reduces to

$$C_1^2=\frac{A_1+2A_2}{\rho}=\frac{\lambda+2\mu+\sigma_1+2\sigma_2}{\rho}$$



and it reduces to $\frac{\lambda+2\mu}{\rho}$ when $\sigma_1=0$, $\sigma_2=0$, which is the classical result.[2]

Now, differentiating (15), (16), (17) with respect to x_1, x_2 and x_3 respectively, we have

$$2(B_4+B_5)\frac{\partial^2}{\partial x_1^2}\Delta^1+2B_3\nabla^2\frac{\partial\phi_1}{\partial x_1}-2A_3(u_{3,21}-u_{2,31})$$

$$-4A_3\frac{\partial\phi_1}{\partial x_1}=\rho j\frac{\partial^2}{\partial t^2}\left(\frac{\partial\phi_1}{\partial x_1}\right) \qquad (24)$$

$$2(B_4+B_5)\frac{\partial^2}{\partial x_2^2}\Delta^1+2B_3\nabla^2\frac{\partial\phi_2}{\partial x_2}-2A_3(u_{1,32}-u_{3,12})$$

$$-4A_3\frac{\partial\phi_2}{\partial x_2}=\rho j\frac{\partial^2}{\partial t^2}\left(\frac{\partial\phi_2}{\partial x_2}\right) \qquad (25)$$

$$2(B_4+B_5)\frac{\partial^2}{\partial x_3^2}\Delta^1+2B_3\nabla^2\frac{\partial\phi_3}{\partial x_3}-2A_3(u_{2,13}-u_{1,23})$$

$$-4A_3\frac{\partial\phi_3}{\partial x_3}=\rho j\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\phi_3}{\partial x_3}\right) \qquad (26)$$

Adding (24), (25) and (26) we get

$$2(B_4+B_5)\nabla^2\Delta^1+2B_3\nabla^2\Delta^1-4A_3\Delta^1=\rho j\frac{\partial^2\Delta^1}{\partial t^2}$$

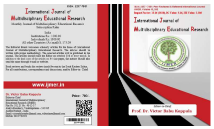
i.e. $$[2(B_4+B_5)+2B_3]\nabla^2\Delta^1-4A_3\Delta^1=\rho j\frac{\partial^2\Delta^1}{\partial t^2} \qquad (27)$$

Plane waves advancing in the positive direction of the unit vector $\vec{n}$ may be expressed as

$$\Delta^1=A\exp\{ik(\vec{n}.\vec{r}-vt)\} \qquad (28)$$

Where A is a constant, k is the wave number, $\vec{r}$ is the position vector, v is the velocity of the wave. Substituting (28) in (27) we get

$$2(B_3+B_4+B_5)k^2+4A_3=\rho j^2k^2v^2$$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$$2\left(B_3+B_4+B_5\right)+\frac{4A_3}{k^2}=\rho j v^2 \tag{29}$$

If we introduce the angular frequency

$$\omega = kv \tag{30}$$

the equation (29) reduces to

$$C_2^2=\frac{2\left(B_3+B_4+B_5\right)}{\rho j\left(1-\frac{\omega_1^2}{\omega^2}\right)} \tag{31}$$

where $\omega_1^2=\frac{4A_3}{\rho j}$ (32)

The speed of these waves depends on the frequency. Hence, they are dispersive. Further these waves depend on purely micro-morphic Elastic constants which are not encountered in classical elasticity.

Now the equations of motion under the absence of force and couples, the equation (8) involving micro-strains can be expressed as

$$B_1\phi_{pp,kk}+2B_2\phi_{11,kk}-A_4\phi_{pp}-2A_5\phi_{11}=\frac{1}{2}\rho j\frac{\partial^2\phi_{11}}{\partial t^2} \tag{33}$$

$$B_1\phi_{pp,kk}+2B_2\phi_{22,kk}-A_4\phi_{pp}-2A_5\phi_{22}=\frac{1}{2}\rho j\frac{\partial^2\phi_{22}}{\partial t^2} \tag{34}$$

$$B_1\phi_{pp,kk}+2B_2\phi_{33,kk}-A_4\phi_{pp}-2A_5\phi_{33}=\frac{1}{2}\rho j\frac{\partial^2\phi_{33}}{\partial t^2} \tag{35}$$

$$2B_2\phi_{(12),kk}-2A_5\phi_{(12)}=\frac{\rho j}{2}\frac{\partial^2\phi_{(12)}}{\partial t^2} \tag{36}$$

$$2B_2\phi_{(13),kk}-2A_5\phi_{(13)}=\frac{\rho j}{2}\frac{\partial^2\phi_{(13)}}{\partial t^2} \tag{37}$$

$$2B_2\phi_{(23),kk}-2A_5\phi_{(23)}=\frac{\rho j}{2}\frac{\partial^2\phi_{(23)}}{\partial t^2} \tag{38}$$

We rewrite these equations for the convenience. Adding (33) to (35) we get

$$(2B_1 + 2B_2)\phi_{pp,kk} - (3A_4 + 2A_5)\phi_{pp} = \frac{\rho j}{2}\frac{\partial^2 \phi_{pp}}{\partial t^2} \qquad (39)$$

Subtracting (34) from (33) it yields

$$2B_2(\phi_{11} - \phi_{22})_{,kk} - 2A_5(\phi_{11} - \phi_{22})_{,kk} = \frac{1}{2}\rho j \frac{\partial^2}{\partial t^2}(\phi_{11} - \phi_{22})_{,kk} \qquad (40)$$

and subtracting (35) from (33) we get

$$2B_2(\phi_{11} - \phi_{33})_{,kk} - 2A_5(\phi_{11} - \phi_{33}) = \frac{1}{2}\rho j \frac{\partial^2}{\partial t^2}(\phi_{11} - \phi_{33})_{,kk} \qquad (41)$$

Now we discuss the waves corresponding to (36) to (41). Plane waves advancing in the positive direction of unit vector $\vec{n}$ may be expressed as

$$\phi_{(ij)} = D_{(ij)} \exp\{ik(\vec{n}.\vec{r} - vt)\} \qquad (42)$$

where D_{ij} are constants and

$$D_{ij} = D_{(ji)} \qquad (43)$$

Substituting (42) in the equation (39) and using (30), we get

$$[(3B_1 + 2B_2)k^2 + (3A_4 + 2A_5)]\phi_{pp} = \frac{\rho j}{2}\omega^2$$

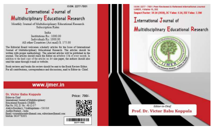
We obtain an expression for the phase velocity v_3 as

$$V_3^2 = \frac{\omega^2}{k^2} = \frac{2(3B_1 + 2B_2)}{\rho j(\omega^2 - \omega_2^2)} \qquad (44)$$

where $$\omega_2^2 = 2\frac{(3A_4 + 2A_5)}{\rho j} \qquad (45)$$

It is longitudinal micro-dilation wave. V_3 is real not finite or imaginary according as $\omega > \omega_2$, $\omega = \omega_2$ and $\omega < \omega_2$ respectively.

Now substituting (42) in the equation (40) we obtain the phase velocity V_4 and it is given by $$V_4^2 = \frac{\omega^2}{k^2} = \frac{4B_2\omega^2}{\rho j(\omega^2 - \omega_3^2)} \qquad (46)$$



where $\omega_3^2 = \dfrac{4A_5}{\rho j}$ (47)

We call this wave as transverse micro-extensional wave.

We can observe that the equation (26) to (28) and (41) are identical to the equation (40). Thus, we have another four waves whose phase velocities are v_5, v_6, v_7, v_8 each of which is equal to.V4

The wave corresponds to the equation (41), we call it as transverse micro-extensional wave. The waves given by (36) to (38) called micro-shear waves. All these waves are dispersive and having some cut-off frequencies. Thus it is possible to have eight waves propagating in micro-morphic medium with four district velocities.

Distortional waves:

The waves of distortion in the classical elasticity are obtained

when $\Delta = 0$. i.e., $\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = 0$

The velocity of the wave of distortion is given by $\sqrt{\mu/\rho}$ [4].

In this case there is neither a volume expansion nor volume compression.

In micro-morphic theory we shall show that these distortional waves can be obtained by assuming $\Delta = 0$, $\Delta^{1} = 0$ $\mathrm{Curl}\vec{u} = 0$, $\mathrm{Curl}\vec{\phi} = 0$ (48)

In view of (48) the equation of motion given by (11) to (16) reduce to

$$(A_2 + A_3)\nabla^2 u_1 = \rho \frac{\partial^2 u_1}{\partial t^2} \tag{49}$$

$$(A_2 + A_3)\nabla^2 u_2 = \rho \frac{\partial^2 u_2}{\partial t^2} \tag{50}$$

$$(A_2 + A_3)\nabla^2 u_3 = \rho \frac{\partial^2 u_3}{\partial t^2} \tag{51}$$

$$2B_3\nabla^2 \phi_1 - 4A_3\phi_1 = \rho j \frac{\partial^2 \phi_1}{\partial t^2} \tag{52}$$

$$2B_3\nabla^2 \phi_2 - 4A_3\phi_2 = \rho j \frac{\partial^2 \phi_2}{\partial t^2} \tag{53}$$

$$2B_3\nabla^2\phi_3 - 4A_3\phi_3 = \rho j\frac{\partial^2\phi_3}{\partial t^2} \qquad (54)$$

The equations (49) to (51) to can be written as

$$\nabla^2 u_1 = \frac{1}{v_1^{1^2}}\frac{\partial^2 u_1}{\partial t^2} \qquad (55)$$

$$\nabla^2 u_2 = \frac{1}{v_1^{1^2}}\frac{\partial^2 u_2}{\partial t^2} \qquad (56)$$

$$\nabla^2 u_3 = \frac{1}{v_1^{1^2}}\frac{\partial^2 u_3}{\partial t^2} \qquad (57)$$

where $v_1^{1^2} = \frac{A_2 + A_3}{\rho}$ (58)

The velocity of distortional wave of the classical case is obtained a particular case of (58) when $A_3 = 0$ and $\sigma_1 = 0$. The equations (52) to (54) frequency equations corresponds micro-rotation.

The equations (52) to (54) represent waves whose velocities are v_2^1, v_3^1, v_4^1 (are all equal) and they are given by

$$v_2^{1\,2} = v_3^{1^2} = v_4^{1^2} = \frac{4B_2\omega^2}{\rho j\left(\omega^2 - \omega_1^{1^2}\right)} \qquad (59)$$

where $\omega_1^{1^2} = \frac{4A_3}{\rho j}$ (60)

The equations of motion involving micro-strains give another six waves propagating with two district velocities and all are dispersive.

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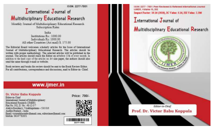

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