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Relavency of Ancient Mathematics To The Current Digital Trends

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FOREWORD

I am happy to say that the Department of Mathematics, National Sanskrit University, Tirupati-A.P, is conducting the XXXI Annual Congress of APTSMS and **Inter National Conference on Relavency of Ancient Mathematics To The Current Digital Trends** during 9^{th} -11^{th}, August, 2022. It provides a platform for researchers and academicians to exchange their latest views and information about Mathematics in various fields and to acquire experience from the existing developments and to explore future evolutions.

I am grateful to Prof.G.S.R.Krishna Murthy, Vice-Chancellor, National Sanskrit University, for his support and encouragement. I am thankful to Commander Chall Venkateswar (Retd), Registrar National Sanskrit University, for his Co-operation. I thank all the Members of organizing Committee and Supporting Staff of the department who have supported in a big way to organize this conference.

Tand Also thanks toY.V.Krisna Rao, D.R &Finance officer (i/c),National Sanskrit University for their financial support to conduct the conference without which it would have been not possible to organize this event.

May this conference be successful in every respect I wish every participant to have a good experience of the conference as well as recreation at our university.

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C O N T E N T S

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Editorial……

It is heartening to note that our journal is able to sustain the enthusiasm and covering various facets of knowledge. It is our hope that IJMER would continue to live up to its fullest expectations savoring the thoughts of the intellectuals associated with its functioning .Our progress is steady and we are in a position now to receive evaluate and publish as many articles as we can. The response from the academicians and scholars is excellent and we are proud to acknowledge this stimulating aspect.

The writers with their rich research experience in the academic fields are contributing excellently and making IJMER march to progress as envisaged. The interdisciplinary topics bring in a spirit of immense participation enabling us to understand the relations in the growing competitive world. Our endeavour will be to keep IJMER as a perfect tool in making all its participants to work to unity with their thoughts and action.

The Editor thanks one and all for their input towards the growth of the **Knowledge Based Society**. All of us together are making continues efforts to make our predictions true in making IJMER, a Journal of Repute

Dr.K.Victor Babu
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Shear Waves in Dissipative Poroelastic Transversely Isotropic Magnetic Inhomogeneous Solid Located Between Two Initially Stressed Half- Spaces

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Abstract: The purpose of this paper is to study shear waves in poroelastic medium consists of transversely isotropic magnetic solid located between two initially stressed inhomogeneous half-spaces. The equations are drawn from that of Biot's incremental deformation framework. Boundary conditions at interfaces are employed to obtain frequency equation. Because of dissipative medium, waves attenuate. The frequency and attenuation coefficient are computed against wavenumber fixing inhomogeneous parameter, magneto poroelastic coupling factor, and initial stress. The numerical results are presented in the form of graphs.

Key words: Shear waves, Magneto poroelastic, Inhomogeneous, Initial stress

1. Introduction

The Earth has a inhomogeneous layer structure, and has effect on the elastic waves. The Earth crust has the great diversity of metamorphic, igneous and sedimentary rocks. These rocks can generate magnetic field, because of minerals iron, nickel, and cobalt in them. Hence, these rocks are the magneto-elastic in nature. The study of seismic waves in inhomogeneous media is important in Earthquake Engineering and Seismology. The investigation of the SH-waves in magneto-elastic solids is quite important for the progress of fundamental work in the mechanics of a magnetic body. The shear waves in visco-elastic inhomogeneous layer placed an initially stressed half-space is studied by Chatterjee et. al. [1]. In the paper [1], the study of propagation of shear waves in an isotropic initially stressed visco-elastic, heterogeneous layer lying over a homogeneous half-space is made. The SH-type wave in an anisotropic layer located between an orthotropic medium and an in-homogeneous half-space is made by Sultana and Gupta [2]. The shear waves in monoclinic irregular layer is investigated by Chattopadhyay et. al. [3]. In the paper [3], the horizontally polarized shear waves in an internal magneto elastic monoclinic stratum with lower irregular interface is investigated. Sindhuja et. al. [4] studied shear waves in poroelastic magneto medium between self-reinforced poroelastic medium and half- space. In the paper [4], the reinforcement, inhomogeneity and initial stress effects on the shear waves is studied. Reddy et. al. [5], studied SH-waves in an initially stressed triclinic layer between transversely heterogeneous isotropic poroelastic half-spaces. In the paper [5], the SH-waves in an initially stressed triclinic layer welded between two half-spaces is studied. Using Biot's theory, the shear waves in a transversely isotropic poroelastic layer between two elastic layers is studied [6]. The paper [7] deals with shear wave propagation in fiber-reinforced poroelastic medium between two dry sandy poroelastic half spaces. Balu et. al. [8] studied shear waves in magneto-poroelastic isotropic dissipative medium between two poroelastic half-spaces. SH-wave propagation in magneto-elastic anisotropic medium between two elastic half-spaces is studied in several papers [9-11]. However, the problem of shear waves in magneto transversely isotropic dissipative poroelastic medium between two poroelastic half-spaces is not yet studied. Hence, in

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this paper, the same is studied.

The remaining part of the paper is organized as follows: In section 2, the problem solution is given. Boundary conditions and secular equation are discussed in section 3. In section 4, numerical results are given. Finally, in section 5, the conclusion is placed.

2. Formulation and Solution of the Problem

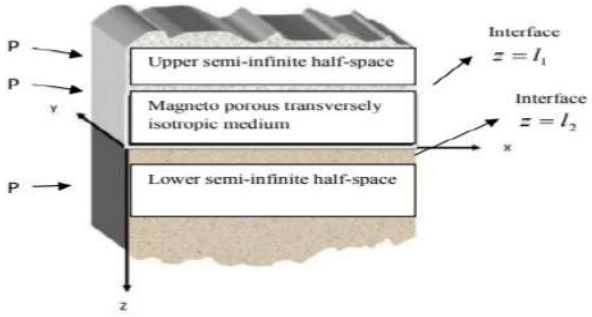

Fig.1. Geomentry of the problem

The magneto poroelastic isotropic medium between two heterogeneous semi-infinite half spaces, all in influence of gravity and initial stress is considered here. The x-axis is considered positively downwards along the shear waves direction. In absence of the viscosity of body forces and fluid, the equations of motion are [12] given below:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(S_{ik}\, e_{kj} + S_{ij}\, e - p\,\omega_{ik}\right) = \frac{\partial^2}{\partial t^2}\left(\rho_{11} u_i + \rho_{12} U_i\right) + b\frac{\partial}{\partial t}\left(u_i - U_i\right),$$

$$s'_{,i} = \frac{\partial^2}{\partial t^2}\left(\rho_{12} u_i + \rho_{22} U_i\right) - b\frac{\partial}{\partial t}\left(u_i - U_i\right). \tag{1}$$

In the above, $\sigma_{ij}\ (i,j=1,2,3)$ is the stress tensor of the solid, S_{ij} and S_{ik} are the components of the initial stress, b is co-efficient of dissipation. ',' denotes differentiation with respect to position, $s' = (-\eta\, p^*)$ is fluid phase stress, p^* is fluid pressure, η is porosity, u_i and U_i are components of displacement for the solid and fluid, respectively, e_{kj} and ω_{ik} are the components of the strain and rotational tensors, respectively, p is initial stress, and ρ_{ij} are mass coefficients. The constitutive relations for transversely poroelastic isotropic solid in the cylindrical system are given by

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{r\theta} \\ \sigma_{\theta z} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 & 0 & 0 \\ M_{12} & M_{11} & M_{13} & 0 & 0 & 0 \\ M_{13} & M_{13} & M_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{55} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ e_{r\theta} \\ e_{\theta z} \\ e_{rz} \end{bmatrix} - \begin{bmatrix} \alpha \\ \alpha \\ \alpha' \\ 0 \\ 0 \\ 0 \end{bmatrix} P, \tag{2}$$

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where, $P = M[\varepsilon' - \alpha(e_{rr} + e_{\theta\theta}) - \alpha' e_{zz}]$.

In the Eq. (2), e_{jj} and e_{jk} are the components of normal and shear strains, P is the pore pressure, α and α' are Biot's effective stress coefficients in isotropic plane and in z -direction, respectively. M is Biot's modulus, ε' is the variation of fluid content per unit reference volume, and M_{jk} are components of the drained elastic moduli, depend on E, E', ν, ν', G and G'. E and ν are drained Young's moduli and Poisson ratio in the isotropic plane, E' and ν' are similar quantities as that of E and ν corresponding to the direction of the axis of symmetry, G and G' are the shear moduli related to the direction of the isotropic plane and axis of the symmetry, respectively. For given anisotropic ratios of $N_E = \frac{E'}{E}$ and $N_\nu = \frac{\nu'}{\nu}$, E' and ν' can be computed. Different N_E and N_ν ratios define different degrees of anisotropy. The strain and rotational components are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{3}$$

We consider three cases namely upper semi-infinite porous half-space, intermediate layer of magneto-poroelastic media and lower semi-infinite porous half-space, separately in the following sections.

2.1. Upper semi-infinite porous half-space

Consider the upper semi-infinite porous heterogeneous half-space M_1 (say). Let (u_1, u_2, u_3) and (U_1, U_2, U_3) be displacement components of solid and fluid part, respectively. For the SH-wave, $u_1 = 0, u_3 = 0$, $u_2 = u_2(x, z, t)$ and $U_1 = 0, U_3 = 0, U_2 = U_2(x, z, t)$.

Then Eq. (1), become

$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{23}}{\partial z} - p\frac{\partial \omega_{12}}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho_{11}u_2 + \rho_{12}U_2) + b\frac{\partial}{\partial t}(u_2 - U_2),$$

$$0 = \frac{\partial^2}{\partial t^2}(\rho_{12}u_2 + \rho_{22}U_2) - b\frac{\partial}{\partial t}(u_2 - U_2). \tag{4}$$

Since half-space M_1 is heterogeneous, the densities and initial stress variations are considered as given below.

$$\rho_{11} = \rho_{110}(1 + \frac{z}{\xi_1}), \quad \rho_{12} = \rho_{120}(1 + \frac{z}{\xi_1}), \quad \rho_{22} = \rho_{220}(1 + \frac{z}{\xi_1}), \quad p = p_{100}(1 + \frac{z}{\xi_1}). \tag{5}$$

In the above, $\rho_{110}, \rho_{120}, \rho_{220}$ are starting point values of mass coefficients, and p_{100} is starting point value of initial stress.

Substitution of Eqs. (2) and (5) in Eq. (4), the following equations are obtained:

$$(\frac{M_{55}}{2} - \frac{p_{100}}{2})\frac{\partial^2 u_2}{\partial x^2} + \frac{M_{55}}{2}\frac{\partial^2 u_2}{\partial z^2} = (1 + \frac{z}{\xi_1})(\rho_{110}\frac{\partial^2 u_2}{\partial t^2} + \rho_{120}\frac{\partial^2 U_2}{\partial t^2}) + b(\frac{\partial u_2}{\partial t} - \frac{\partial U_2}{\partial t}),$$

$$0 = (1 + \frac{z}{\xi_1})(\rho_{120}\frac{\partial^2 u_2}{\partial t^2} + \rho_{220}\frac{\partial^2 U_2}{\partial t^2}) - b(\frac{\partial u_2}{\partial t} - \frac{\partial U_2}{\partial t}). \tag{6}$$

For harmonic waves, the component of displacements are taken as follows:

$$u_2(x, z, t) = f_1(z)\, e^{ik(x - ct)}, \; U_2(x, z, t) = F_1(z)\, e^{ik(x - ct)}. \tag{7}$$

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In Eq. (7), k is wavenumber, i is unity of the complex number system and t is time. Substitution of Eq. (7) in Eq. (6), gives

$$\frac{M_{55}}{2} f_1^{//}(z) + ((\frac{p_{100}}{2} - \frac{M_{55}}{2})k^2 - \rho_{110}k^2c^2(1+\frac{z}{\xi_1}) + bikc) f_1(z) + (\rho_{120}k^2c^2(1+\frac{z}{\xi_1}) + bikc)) F_1(z) = 0,$$

$$(\rho_{120}k^2c^2(1+\frac{z}{\xi_1}) + bikc))\, f_1(z) + (\rho_{220}k^2c^2(1+\frac{z}{\xi_1}) + bikc))\, F_1(z) = 0. \tag{8}$$

Eq. (8) can be re-written as,

$$\frac{d^2 f_1}{dz^2} + q_1^2 f_1(z) = 0. \tag{9}$$

Solution of Eq. (9) is

$$f_1(z) = C_1\, e^{q_1 z}. \tag{10}$$

In Eq. (10), C_1 is arbitrary constant, and

$$q_1 = \frac{2}{(M_{55}\rho_{220}^2\omega^2 + b^2(1+\frac{z}{\xi_1}))^{\frac{1}{2}}}(a_1 + ia_2)^{\frac{1}{2}}, a_1 = s_1((1+\frac{z}{\xi_1})(\rho_{110}\rho_{220}^2\omega^4 + 2b^2\rho_{220}^2(\frac{M_{55}}{2} -$$

$$\frac{p_{100}}{2})\omega^4k^2 + \rho_{220}\rho_{120}^2\omega^4 + +b^2k^2\rho_{110} - 2b^2\rho_{220}^2\omega^4), a_2 = s_1 b\omega(\rho_{220}^2 - 2\rho_{120}\rho_{220} + \frac{2}{M_{55}}).$$

Substitution of Eq. (10) in Eq. (8), gives

$$F_1(z) = -D_1 D_2^{-1} f_1(z), \tag{11}$$

where $D_1 = \rho_{120} - ib(1+\frac{z}{\xi_1})\omega^{-1}$, $\quad G_{22} = \rho_{220} + ib(1+\frac{z}{\xi_1})\omega^{-1}$.

Substitution of Eq. (10) in Eq. (7), and then using displacement-stress relations, the relevant stress components are obtained as follow:

$$(\sigma_{23})_{M_1} = \frac{M_{55}}{2} q_1 C_1\, e^{q_1 z}\, e^{ik(x-ct)}. \tag{12}$$

2.2. Intermediate magneto poroelastic media

Consider the middle layer of magneto transversely isotropic material M_2(say) of thickness 'l_1 which is under initially stressed. Let (u_4, u_5, u_6) and (U_4, U_5, U_6) be the solid and fluid components of the displacement, respectively. In this case, the equation of motion for the propagation of SH-waves are given below.

$$\frac{\partial\sigma_{12}}{\partial x} + \frac{\partial\sigma_{23}}{\partial z} - p\frac{\partial\omega_{12}}{\partial x} + (\vec{J}\times\vec{B})_y = \frac{\partial^2}{\partial t^2}(\rho_{11}u_5 + \rho_{12}U_5) + b\frac{\partial}{\partial t}(u_5 - U_5),$$

$$0 = \frac{\partial^2}{\partial t^2}(\rho_{12}u_5 + \rho_{22}U_5) - b\frac{\partial}{\partial t}(u_5 - U_5), \tag{13}$$

where $(\vec{J}\times\vec{B})_y$ is the y-component of electromagnetic force (emf), $\vec{J}$ is the electric current density and $\vec{B}$ the magnetic induction vector, the Maxwell's equations which given on the electromagnetic field are

$$\vec{\nabla}.\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{H} = \vec{J}, \vec{B} = \mu_e\vec{H}, \quad \vec{J} = \sigma(\vec{E} + \frac{\partial u_5}{\partial t}\times\vec{B}). \tag{14}$$

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In the above, $\vec{E}$ is the induced electric field, $\vec{H}$ is the magnetic intensities consist of both primary and induced magnetic fields, μ_e and σ are the induced permeability and conduction co-efficient, respectively.

The linearized Maxwell stress tensor $(\tau_{ij})^{M_x}$ of magnetic field is given by

$$(\tau_{ij}^0)^{M_x} = \mu_e(H_i\, a_j - H_j\, a_i - H_k\, a_k\, \delta_{ij}). \tag{15}$$

Let $\vec{H} = (H_x, H_y, H_z)$, and let change in magnetic field be (a_1, a_2, a_3). Here it is assumed that the displacement current is absent. With the help of Eq. (14), the below equation is obtained.

$$\nabla^2 H = \mu_e\, \sigma\left(\frac{\partial \vec{H}}{\partial t} + \vec{\nabla} \times \left(\frac{\partial \vec{u}_5}{\partial t} \times \vec{H}\right)\right). \tag{16}$$

The components of Eq. (16) can be written as

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_e \sigma}\nabla^2 H_x, \quad \frac{\partial H_z}{\partial t} = \frac{1}{\mu_e \sigma}\nabla^2 H_z, \tag{17}$$

$$\text{and } \frac{\partial H_y}{\partial t} = \frac{1}{\mu_e \sigma}\nabla^2 H_y + \frac{\partial}{\partial x}\left(H_x \frac{\partial u_5}{\partial t}\right) + \frac{\partial}{\partial z}\left(H_z \frac{\partial u_5}{\partial t}\right). \tag{18}$$

If the medium is fully conducting $(i.e\ \sigma \to \infty)$, the equations (17) and (18) reduce to

$$\frac{\partial H_x}{\partial t} = \frac{\partial H_z}{\partial t} = 0, \quad \frac{\partial H_y}{\partial t} = \frac{\partial}{\partial x}\left(H_x \frac{\partial u_5}{\partial t}\right) + \frac{\partial}{\partial z}\left(H_z \frac{\partial u_5}{\partial t}\right). \tag{19}$$

Here it is assumed that the primary magnetic field is uniform in the entire space. From the Eq. (19) it is clear that there is no perturbation in H_x and H_z, but there is perturbation in H_y. Therefore, considering small perturbation a_2 in H_y. Thus, they can written as $H_x = H_{01}$, $H_y = H_{02} + a_2$ and $H_z = H_{03}$, where (H_{01}, H_{02}, H_{03}) are components of the initial magnetic field $\vec{H}_0$. Since $(H_0 \cos\phi,\ 0,\ H_0 \sin\phi)$, where $H_0 = |\vec{H}_0|$ and ϕ is the angle at which wave crosses the magnetic field. Thus one can write

$$\vec{H} = (H_0 \cos\phi,\ a_2,\ H_0 \sin\phi). \tag{20}$$

Here one can assume that initially a_2 to be zero. Using Eq. (20) in (19), one can obtain

$$\frac{\partial a_2}{\partial t} = \frac{\partial}{\partial x}\left(H_0 \cos\phi \frac{\partial u_5}{\partial t}\right) + \frac{\partial}{\partial z}\left(H_0 \sin\phi \frac{\partial u_5}{\partial t}\right). \tag{21}$$

Now integration of Eq. (21) with respect to t provides

$$a_2 = H_0 \cos\phi \frac{\partial u_5}{\partial x} + H_0 \sin\phi \frac{\partial u_5}{\partial z}. \tag{22}$$

Using Eq. (14) and the relation, $\nabla\left(\frac{H^2}{2}\right) = -(\vec{\nabla} \times \vec{H}) \times \vec{H} + (\vec{H}.\vec{\nabla}).\vec{H}$ provides the electromagnetic force as

$$(\vec{J} \times \vec{B}) = \mu_e\left[-\vec{\nabla}\left(\frac{H^2}{2}\right) + (\vec{H}.\vec{\nabla}).\vec{H}\right]. \tag{23}$$

The component form of $\vec{J} \times \vec{B}$ can written as

$$(\vec{J} \times \vec{B})_x = 0, \quad (\vec{J} \times \vec{B})_z = 0, \text{ and}$$

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$$(\vec{J}\times\vec{B})_y = \mu_e H_0^2(\sin^2\phi\frac{\partial^2 u_5}{\partial z^2}+\sin 2\phi\frac{\partial^2 u_5}{\partial x\,\partial z}+\cos^2\phi\frac{\partial^2 u_5}{\partial x^2}). \tag{24}$$

Substitution of Eq. (24) in Eq. (13), gives

$$(\frac{M_{55}}{2}-\frac{p}{2}+\mu_e H_0^2\cos^2\phi)\frac{\partial^2 u_5}{\partial x^2}+(M_{55}+\mu_e H_0^2\sin^2\phi)\frac{\partial^2 u_5}{\partial z^2}+\mu_e H_0^2\sin 2\phi\frac{\partial^2}{\partial x\,\partial z}=$$

$$\rho_{11}\frac{\partial^2 u_5}{\partial t^2}+\rho_{12}\frac{\partial^2 U_5}{\partial t^2}+b(\frac{\partial u_5}{\partial t}-\frac{\partial U_5}{\partial t}),$$

$$0=\rho_{12}\frac{\partial^2 u_5}{\partial t^2}+\rho_{22}\frac{\partial^2 U_5}{\partial t^2}-b(\frac{\partial u_5}{\partial t}-\frac{\partial U_5}{\partial t}). \tag{25}$$

For harmonic waves, the displacement components of SH-wave in intermediate layer of magneto-poroelastic layer are taken as follows:

$$u_5(x,z,t)=f_2(z)\,e^{ik(x-ct)},\qquad U_5(x,z,t)=F_2(z)\,e^{ik(x-ct)}. \tag{26}$$

The equations (25) and (26) give

$$\frac{d^2 f_2}{dz^2}+\frac{\alpha}{A_1}\frac{df_2}{dz}-\frac{t}{A_1}f_2=0. \tag{27}$$

The solution of Eq. (27) is

$$f_2(z)=e^{-\frac{\alpha z}{2A_1}}(C_2\cos(q_2 z)+C_3\sin(q_2 z)). \tag{28}$$

In Eq. (28), C_2 and C_3 are arbitrary constants,

$$q_2=\frac{\sqrt{\alpha^2 A_1^{-2}+4tA_1^{-1}}}{2},\quad \alpha=ik\mu_e H_0^2\sin 2\phi,\quad A_1=M_{55}+\mu_e H_0^2\sin^2\phi,$$

$$t=(\frac{1}{M_{55}}((\frac{M_{55}}{2}-\frac{p}{2})+\mu_e H_0^2\cos^2\phi)k^2-\rho_{11}k^2c^2-bikc-(\rho_{11}k^2c^2-bikc(\frac{-\rho_{12}kc+ib}{\rho_{22}kc+ib})).$$

Eqs. (25), (26) and (28), yield

$$F_2(z)=-M_{12}M_{22}^{-1}f_2(z), \tag{29}$$

In the above, $M_{12}=\rho_{12}-\frac{ib}{\omega}$, $M_{22}=\rho_{22}-\frac{ib}{\omega}$. Substitution of Eq. (28) in Eq. (26), gives the relevant stress component as follows:

$$(\sigma_{23})_{M_2}=-\frac{M_{55}}{2A_1}\alpha q_2\,e^{-\frac{\alpha z}{2A_1}}(C_2\cos(q_2 z)+C_3\sin(q_2 z))e^{ik(x-ct)}. \tag{30}$$

2.3. Lower semi-infinite poroelastic half-space

The lower heterogeneous poroelastic half-space M_3(say) of thickness 'l_2' under the effect of initial stress is considered. Let (u_7,u_8,u_9) and (U_7,U_8,U_9) be the displacement of components solid and fluid, respectively. Then the equation of motion for SH-wave can be written as

$$\frac{\partial\sigma_{12}}{\partial x}+\frac{\partial\sigma_{23}}{\partial z}-p\frac{\partial\omega_{12}}{\partial x}=\frac{\partial^2}{\partial t^2}(\rho_{11}u_8+\rho_{12}U_8)+b\frac{\partial}{\partial t}(u_8-U_8),$$

$$0=\frac{\partial^2}{\partial t^2}(\rho_{12}u_8+\rho_{22}U_8)-b\frac{\partial}{\partial t}(u_8-U_8), \tag{31}$$

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If half-space is heterogeneous with respect to initial stress and mass coefficients, now one can have

$$\rho_{11} = \rho_{110}(1+nz), \rho_{12} = \rho_{120}(1+nz), \rho_{22} = \rho_{220}(1+nz) \text{ and } p = p_{300}(1+p^* z). \tag{32}$$

In the above, p_{300} is starting values of p and n, p^* are constants. Substitution of Eqs. (2) and (32) in (31), gives

$$(\frac{M_{55}}{2} - \frac{p_{300}}{2}(1+p^* z))\frac{\partial^2 u_8}{\partial x^2} + \frac{\partial^2 u_8}{\partial z^2} = \frac{\rho_{110}}{M_{55}}(1+nz)\frac{\partial^2 u_8}{\partial t^2} + \frac{\rho_{120}}{M_{55}}(1+nz)\frac{\partial^2 U_8}{\partial t^2} + \frac{b}{M_{55}}(\frac{\partial u_8}{\partial t} - \frac{\partial U_8}{\partial t}),$$

$$0 = \rho_{120}(1+nz)\frac{\partial^2 u_8}{\partial t^2} + \rho_{220}(1+nz)\frac{\partial^2 U_8}{\partial t^2} - b(\frac{\partial u_8}{\partial t} - \frac{\partial U_8}{\partial t}). \tag{33}$$

For harmonic waves, the components displacement of SH-wave were are taken as follows:

$$u_8(x,z,t) = f_3(z)\, e^{ik(x-ct)}, \quad U_8(x,z,t) = F_3(z)\, e^{ik(x-ct)}. \tag{34}$$

Substitution of Eq. (34) in Eq. (33), gives

$$\frac{d^2 f_3}{dz^2} + q_3^2 f_3(z) = 0. \tag{35}$$

Solutions of Eq. (35) is

$$f_3(z) = C_4\, e^{q_3 z}. \tag{36}$$

In Eq. (36), C_4 is arbitrary constant, and

$$q_3 = \frac{1}{(\rho_{220}^2 k^2 c^2 (1+nz)^2 + b^2)^{\frac{1}{2}}}(a_5 + ia_6)^{\frac{1}{2}}, a_5 = s_3((\frac{p_{300}}{2M_{55}}(1+p^* z) - 1)(\rho_{220}^2 \omega^2 (1+nz) + b^2) + (1+nz)^3 \omega^2 (\rho_{110}\rho_{220}^2 - \rho_{220}\rho_{120}^2)$$

$$+ b^2(1+nz)(\rho_{110} + \frac{\rho_{220}}{M_{55}} + \rho_{120}(1+\frac{1}{M_{55}}))), a_6 = s_3 b(1+nz)^2 \omega(\rho_{120}^2 + \rho_{120}\rho_{220} + \frac{1}{M_{55}}(\rho_{220}^2 + \rho_{120}\rho_{220})).$$

Substituting this Eq. (34) in (33), we obtain

$$F_3(z) = -D_3\, D_4^{-1}\, f_3(z), \tag{37}$$

where $D_3 = \rho_{120}(1+nz) - \frac{ib}{\omega}$, $D_4 = \rho_{220}(1+nz) + \frac{ib}{\omega}$.

Substitution of Eq. (36) in (33), and using the displacement-stress relations, the relevant stress component is obtained as

$$(\sigma_{23})_{M_3} = M_{55}\, q_3 C_4\, e^{q_3 z}\, e^{ik(x-ct)}. \tag{38}$$

3. Boundary Conditions and Secular Equation

Assume that displacement components and stresses are continuous at the interface of upper half-space (M_1) and middle layer (M_2)

i.e, $(u_2)_{M_1} = (u_5)_{M_2}$ and $(\sigma_{23})_{M_1} = (\sigma_{23})_{M_2}$, at $z = l_1$ (39)

Assume that displacements and stresses are continuous at the interface of lower and middle.

i.e, $(u_5)_{M_2} = (u_8)_{M_3}$ and $(\sigma_{23})_{M_2} = (\sigma_{23})_{M_3}$, at $z = l_2$ (40)

The above boundary conditions result in

$$\left[A_{ij}\right]\ [C_i] = 0, \quad (i,j = 1,2,3,4), \tag{41}$$

where

$$A_{11} = e^{q_1 l_1},\ A_{12} = -\cos(q_2 l_1)\, e^{\frac{-\alpha l_1}{2A_1}},\ A_{13} = -\sin(q_2 l_1)\, e^{\frac{-\alpha l_1}{2A_1}},\ A_{21} = 2A_1 q_1\, e^{q_1 l_1},$$

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$A_{22} = \alpha_2 q_2 \cos(q_2 l_1)\, e^{\frac{-\alpha l_1}{2A_1}}$, $A_{23} = \alpha_2 q_2 \sin(q_2 l_1)\, e^{\frac{-\alpha l_1}{2A_1}}$, $A_{14} = A_{24} = A_{31} = A_{41} = 0$,

A_{32}, A_{33} =similar expressions as A_{12}, A_{13} with l_1 and negative sign replaced by l_2 and positive sign, respectively.

A_{34} =similar expression as A_{11} with l_1 and positive sign replaced by l_2 and negative sign, respectively.

A_{42}, A_{43} =similar expressions as A_{22}, A_{23} with l_1 replaced by l_2, respectively.

A_{44} =similar expression as A_{21} with l_1 and q_1 replaced by l_2 and q_3, respectively.

Eq. (41), results in a system of four homogeneous equations in four arbitrary constants C_1, C_2, C_3 and C_4. For a non zero solution, the determinant of coefficients matrix is zero. Accordingly, the following complex valued frequency equation is obtained.

$$\left|a_{ij}\right| + i\left|a'_{ij}\right| = 0. \qquad (i, j = 1,2,3,4). \qquad (42)$$

In Eq. (42), the elements a_{ij} and a'_{ij} are given in Appendix-A.

4. Computational Results

Employing parameter values in the Eq. (42), the implicit relation between frequency (ω), wavenumber, and the thickness $(h = l_2/l_1)$ is obtained. The frequency and attenuation coefficient (Q^{-1}) against wavenumber are computed at fixed inhomogeneous parameter, magneto coupling factor and initial stress. The attenuation coefficient (Q^{-1}) is $Q^{-1} = \dfrac{2\,\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}$. In this formula, $\mathrm{Im}(\omega)$ is frequency of imaginary part in Eq. (42) and $\mathrm{Re}(\omega)$ is frequency of real part in Eq. (42). Lower and upper semi-infinite porous half-space are assumed to be are Berea sandstone saturated with water (Material-I, say) [13]. The material values of the same are as follows:

$\rho_{11} = 2415.2\ kg/m^3$, $\rho_{12} = -300\ kg/m^3$, $\rho_{22} = 500\ kg/m^3$.

Intermediate layer of magneto-poroelastic medium is assumed to be Water saturated sandstone (Material-II, say) [14]. The material values of the same are given below:

$\rho_{11} = 1.90302 \times 10^3\ kg/m^3$, $\rho_{12} = 0\ kg/m^3$, $\rho_{22} = 0.2268 \times 10^3\ kg/m^3$.

Using the said values in secular equation, the frequency and attenuation coefficient against wavenumber at fixed inhomogeneous parameter, initial stress and magneto coupling factor are calculated when the anisotropic ratio $N_E = 1$ at various N_V values $N_V = 1$, at various N_E values. The material constant M_{55} involves E, E', ν and ν'. The values of Young's modulus (E), Poisson's ratio (ν) for water saturated Berea sandstone and Shale rock are taken to be $14.4\,Gpa, 0.20$ and $1.854\,Gpa, 0.22$ as presented in the papers [14, 15]. The values are calculated by the bisection method implemented in MATLAB environment, and the numerical results are presented in Fig. 2-3. Figs. 2 and 3 depict the frequency and attenuation coefficient against wavenumber at fixed inhomogeneous parameters, initial stress and magneto-poroelastic coupling factor. From Figures 2 and 3, it is clear, in general, that the frequency values are less than that of attenuation coefficient.

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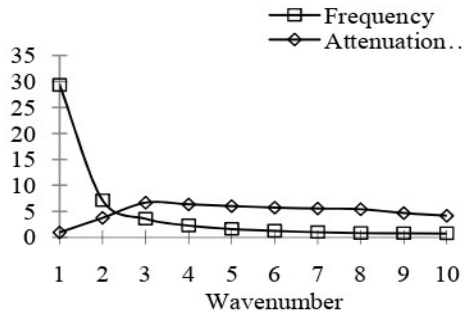

Fig.2. Frequency and attenuation coefficient against wavenumber at heterogeneous parameters $(\frac{n}{k}, \frac{p}{k}, \frac{1}{k\xi_1} = 0.01)$, magneto-poroelastic coupling factor $(\frac{\mu_e H_0^2}{N} = 0.01)$ and $P = 1$.

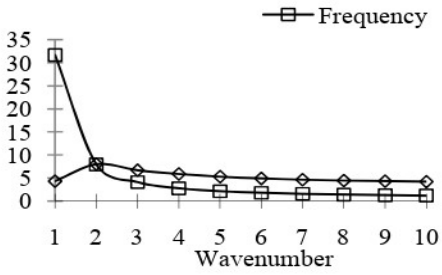

Fig. 3. The frequency and attenuation coefficient against wavenumber at heterogeneous parameters $(\frac{n}{k}, \frac{p}{k}, \frac{1}{k\xi_1} = 0.02)$, magneto-poroelastic coupling factor $(\frac{\mu_e H_0^2}{N} = 0.02)$ and $P = 2$.

5. Concluding Remarks

In the framework of the Biot's incremental Poroelasticity theory, SH-waves in dissipative magneto transversely isotropic poroelastic media located between two poroelastic half-spaces is studied. Frequency and attenuation coefficient values are calculated at varying wavenumber at fixed inhomogeneous parameter, poroelastic magneto coupling factor and initial stress. From the results, it is clear that, in general, frequency values are less than that of attenuation coefficient.

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Appendix-A

$$a_{11} = \cos(l_2)\exp(l_1), \quad a_{12} = \sin(t_1)\sin(l_5)\sinh(l_6) - \cos(t_1)\cos(l_5)\cosh(l_6),$$
$$a_{13} = -\sin(t_1)\cos(l_5)\sinh(l_6) - \cos(t_1)\sin(l_5)\cosh(l_6), \quad a_{22} = -z_1 + z_2 - z_3 + z_4,$$
$$a_{21} = 2L_1R_1(\cos(T_1)\cos(l_2) - \sin(T_1)\sin(l_2))\exp(l_1), \quad a_{23} = -z_9 - z_{10} + z_{11} - z_{12},$$
$$a_{32} = \sin(t_2)\sin(l_7)\sinh(l_8) - \cos(t_2)\cos(l_7)\cosh(l_8), \quad a_{42} = -z_{17} + z_{18} + z_{19} + z_{20},$$
$$a_{33} = \cos(t_2)\sin(l_7)\cosh(l_8) + \sin(t_2)\cos(l_7)\sinh(l_8), \quad a_{43} = -z_{25} - z_{26} + z_{27} - z_{28},$$
$$a'_{12} = \cos(t_1)\sin(l_5)\sinh(l_6) + \sin(t_1)\cos(l_5)\cosh(l_6), \quad a'_{22} = z_5 - z_6 + z_7 + z_8,$$
$$a'_{13} = \sin(t_1)\sin(l_5)\cosh(l_6) - \cos(t_1)\cos(l_5)\sinh(l_6), \quad a'_{23} = z_{13} + z_{14} - z_{15} + z_{16},$$
$$a'_{21} = 2L_1R_1(\cos(T_1)\sin(l_2) + \sin(T_1)\cos(l_2))\exp(l_1), \quad a'_{42} = z_{21} - z_{22} + z_{23} - z_{24},$$

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$$a_{32}^{/} = \sin(t_2)\cos(l_7)\cosh(l_8) - \cos(t_2)\sin(l_7)\sinh(l_8), \qquad a_{43}^{/} = z_{29} + z_{30} - z_{31} + z_{32},$$

$$a_{33}^{/} = \cos(t_2)\cos(l_7)\sinh(l_8) - \sin(t_2)\sin(l_7)\cosh(l_8), \qquad a_{14} = a_{24} = a_{31} = a_{41} = 0,$$

$$a_{14}^{/} = a_{24}^{/} = a_{31}^{/} = a_{41}^{/} = 0,\ t_1 = \frac{kaM}{2A_1},\quad l_1 = ka\,R_1\,s_1\cos(T_1),\ l_5 = ka\,R_2\,\cos(T_2),$$

$$z_1 = R_2 M\cos(T_2)\cos(t_1)\sin(l_5)\sinh(l_6), \qquad z_2 = R_2 M\cos(T_2)\sin(t_1)\cos(l_5)\cosh(l_6),$$

$$z_3 = R_2 M\sin(T_2)\cos(t_1)\cos(l_5)\cosh(l_6), \qquad z_4 = R_2 M\sin(T_2)\sin(t_1)\sin(l_5)\sinh(l_6),$$

$$z_9 = R_2 M\sin(T_2)\sin(t_1)\cos(l_5)\sinh(l_6), \qquad z_{10} = R_2 M\cos(T_2)\cos(t_1)\cos(l_5)\sinh(l_6),$$

$$z_{11} = R_2 M\cos(T_2)\sin(t_1)\sin(l_5)\cosh(l_6), \qquad z_{12} = R_2 M\cos(T_2)\sin(t_1)\cos(l_5)\sinh(l_6),$$

$$z_5 = R_2 M\cos(T_2)\cos(t_1)\cos(l_5)\cosh(l_6), \qquad z_6 = R_2 M\cos(T_2)\sin(t_1)\sin(l_5)\sinh(l_6),$$

$$z_7 = R_2 M\sin(T_2)\cos(t_1)\sin(l_5)\sinh(l_6), \qquad z_8 = R_2 M\sin(T_2)\sin(t_1)\cos(l_5)\cosh(l_6),$$

$$z_{13} = R_2 M\cos(T_2)\cos(t_1)\sin(l_5)\cosh(l_6), \qquad z_{14} = R_2 M\cos(T_2)\sin(t_1)\cos(l_5)\sinh(l_6),$$

$$z_{15} = R_2 M\sin(T_2)\cos(t_1)\cos(l_5)\sinh(l_6), \qquad z_{16} = R_2 M\sin(T_2)\sin(t_1)\sin(l_5)\cosh(l_6),$$

$$s_1 = \frac{1}{(\rho_{22}^2 k^2 c^2 + b^2(1+\frac{z}{\xi}))^{\frac{1}{2}}}, \qquad s_2 = \frac{1}{\left(M_{55}(\rho_{22}^2 k^2 c^2 + b^2)\right)^{\frac{1}{2}}},\ A_1 = 1 + \frac{\mu_e H_0^2}{M_{55}}\sin^2\phi,$$

$$s_3 = \frac{1}{\left(\rho_{22}^2(1+nz) + b^2\right)^{\frac{1}{2}}},\quad R_1 = \left(a_1^2 + a_2^2\right)^{\frac{1}{4}},\quad R_2 = \left(a_3^2 + a_4^2\right)^{\frac{1}{4}},\quad R_3 = \left(a_5^2 + a_6^2\right)^{\frac{1}{4}},$$

$$T_1 = \frac{1}{2}Tan^{-1}\left(\frac{a_2}{a_1}\right), \qquad T_2 = \frac{1}{2}Tan^{-1}\left(\frac{a_4}{a_3}\right), \qquad T_3 = \frac{1}{2}Tan^{-1}\left(\frac{a_6}{a_5}\right),\ M = \frac{\mu_e H_0^2}{M_{55}}\sin 2\phi,$$

$$a_1 = s_1((1+\frac{z}{\xi_1})\rho_{110}\rho_{220}^2\omega^4 + b^2\rho_{220}^2(1+\frac{z}{\xi_1})^2(\frac{M_{55}}{2} - \frac{p_{100}}{2})\omega^4 k^2 + (1+\frac{z}{\xi_1})^2\rho_{220}\rho_{120}^2\omega^4$$

$$- b^2(\frac{M_{55}}{2} - \frac{p_{100}}{2})k^2 + b^2k^2\rho_{110}(1+\frac{z}{\xi_1}) - 2b^2\rho_{220}^2\omega^4(1+\frac{z}{\xi_1}))$$

$$a_2 = s_1 b\omega(\rho_{220}^2 - 2\rho_{120}\rho_{220} + \frac{2}{M_{55}}).$$

$$a_3 = 2s_2\omega^2((\rho_{220}^2(1+\frac{p}{2M_{55}}) + \frac{\mu_e H_0^2}{M_{55}}\cos^2\phi) + b^2(1 - \frac{p}{2M_{55}} + \frac{\mu_e H_0^2}{M_{55}}\cos^2\phi) + b^2(2\rho_{110} +$$

$$\rho_{220} - \rho_{120})) - \frac{M^2}{2A_1},\ a_4 = -2s_2 b_1\omega\left(\rho_{220}^2 + 2\rho_{110}\rho_{220} + \rho_{120}\rho_{220}\right),$$

$$a_5 = s_3((\frac{P_{300}}{2M_{55}}(1+p^*z) - 1)(\rho_{220}^2\omega^2(1+nz) + b^2) + (1+nz)^3\omega^2(\rho_{110}\rho_{220}^2 - \rho_{220}\rho_{120}^2)$$

$$+ b^2(1+nz)(\rho_{110} + \frac{\rho_{220}}{M_{55}} + \rho_{120}(1+\frac{1}{M_{55}}))),$$

$$a_6 = s_3 b(1+nz)^2\omega(\rho_{120}^2 + \rho_{120}\rho_{220} + \frac{1}{M_{55}}(\rho_{220}^2 + \rho_{120}\rho_{220})).$$

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a_{34}=similar expression as a_{11} with l_1, l_2 and positive sign replaced by l_{11}, l_{12} and negative sign, respectively.

a_{44}=similar expression as a_{21} with R_1, T_1 and l_2 replaced by R_3, T_3 and l_{12}, respectively.

$a_{34}^{/}$ = similar expression as $a_{11}^{/}$ with l_1, l_2 and positive sign replaced by l_{11}, l_{12} and negative sign, respectively.

$a_{11}^{/}$ = similar expression as a_{11} with $\cos(l_2)$ replaced by $\sin(l_2)$.

$a_{44}^{/}$=similar expression as $a_{21}^{/}$ with R_1, T_1 and l_2 replaced by R_3, T_3 and l_{12}, respectively.

t_2 = similar expression as t_1 with l_1 replaced by l_2.

l_2 = similar expression as l_1 with $\cos(T_1)$ replaced by $\sin(T_1)$.

l_3, l_4 = similar expressions as l_1, l_2 with l_2 replaced by l_2, respectively.

l_6 = similar expression as l_5 with $\cos(T_1)$ replaced by $\sin(T_1)$.

l_7, l_8 = similar expressions as l_5, l_6 with l_1 replaced by l_2, respectively.

l_9 = similar expression as l_1 with R_1, s_1 and T_1 replaced by R_3, s_3 and T_3, respectively.

l_{10} = similar expression as l_2 with R_1, s_1 and T_1 replaced by R_3, s_3 and T_3, respectively.

l_{11}, l_{12} = similar expressions as l_9, l_{10} l_1 replaced by l_2, respectively.

$z_{17}, z_{18}, z_{19}, z_{20}, z_{25}, z_{26}, z_{27}, z_{28}$ = similar expressions as $z_1, z_2, z_3, z_4, z_9, z_{10}, z_{11}, z_{12}$ with t_1, l_5 and l_6 replaced by t_2, l_7 and l_8, respectively.

$z_{21}, z_{22}, z_{23}, z_{24}, z_{29}, z_{30}, z_{31}, z_{32}$ = similar expressions as $z_5, z_6, z_7, z_8, z_{13}, z_{14}, z_{15}, z_{16}$ with t_1, l_5 and l_6 replaced by t_2, l_7 and l_8, respectively.


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THE CHARACTERIZED CONCEPT OF L-VAGUE RINGS

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Abstract: In this paper we introduce L-Vague Ring, L-Vague ideal and studied some properties on L-Vague rings and L-Vague ideals also define L-Vague coset on L-vague rings, L-Vague ideals and studied some of the fundamental operations on L-Vague ring, L-Vague ideal and L-Vague quotient ring respectively. These concepts are used in the development of some of their important properties have been investigated.

Keywords: Vague set, L-Vague cut-set, L-Vague ring, L-Vague ideal, L-vague Quotient ring

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Introduction: The theory suggested by Zadeh.L.A.[13] to explain a fuzzy subset A of a given universe X describing the truth city of an element of x of X belonging to A by means of a truth city function μ_A defined from X into [0 1] has remodelled the approach of Mathematical modelling,managing etc., in conducting the real-life situations mathematically. Now, several branches of fuzzy mathematics like fuzzy algebra, fuzzy analysis, fuzzy fication, defuzzificaton, fuzzy intervals etc., have emerged. But in the decision making, the fuzzy theory takes care of membership of an element x only, that is the evidence against x belonging to A. It is felt by lot of decision makers and researchers that in proper decision making, the demonstrate belongs to A and evidence not belongs to A are both necessary and how much X belongs to A or how much x does not belong to A are necessary.

Several generalisations of Zadeh's fuzzy set theory have been proposed, such as L-fuzzy sets[12]. Interval valued fuzzy sets, Gau.W.L. and Bueher D.J.[4] have initiated the study of vague sets with the hope that they forma better tool to understand, interpret and solve real life problems which are in general vague than the theory of vague sets. Ranjit Biswas [10] initiated the study of vague groups and Ramakrishna.N [6], [7],[8], Ramakrishna N.T. Eswarlal.T [12],[6],[8] are grate extended the study of vague algebra. The study L-vague sets of a vague set, as a pair of function. In this paper we introduced some operations on L-vague sets and studied some properties on L-vague rings and define L-vague coset, also studied some of their operations on L-vague coset. These concepts are used in the development of some of their important properties have been inspected.

1. Preliminaries

In this section we briefly present the necessary material on lattices, Boolean lattices, Brouwerian lattices and illustrate with examples.

Definition 1.1 [12]:An L-vague set μ in X is a pair $\mu = (m_\mu, n_\mu)$ where $m_\mu : X \to L$, $n_\mu : X \to L$ are mappings such that $m_\mu \leq 1_L - n_\mu$ and $n_\mu \leq 1_L - m_\mu$

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the mapping $m_\mu : X \to L$ is defined the degree of membership function and $n_\mu : X \to L$ defined the degree of membership function and $n_\mu : X \to L$ defined by the degree of non-membership function of the element x$\in$ X *to* $\mu \subset X$ respectively. The function m_μ , n_μ satisfy the condition $m_\mu \leq 1_L - n_\mu$ and $n_\mu \leq 1_L - m_\mu$ i.e., $m_\mu(x) \leq 1_L - n_\mu(x)$ and $n_\mu(x) \leq 1_L - m_\mu(x)$ for x in X, where $1_L - m_\mu(x)$ and $1_L - n_\mu(x)$ are elements of lattice L.

Definition 1.2 [12]: Let (G,$*$) be a group. An L-vague set A= (m_μ , n_μ) of G is said to be L-vague group of G if it satisfies the following conditions

(I) $V_A(xy) \geq \Lambda\{V_A(x), V_A(y)\}\ for\ x, y \in G$

(II)$V_A(x^{-1}) \geq \{V_A(x)\}$

i.e., (1) $m_A(xy) \geq \Lambda\{m_A(x), m_A(y)\}$ (2) $n_A(xy) \leq \vee\{n_A(x), n_A(y)\}$

(3) $m_A(x^{-1}) \geq m_A(x)$(4) $n_A(x^{-1}) \leq n_A(x)$ for all x ,y$\in$ G

Definition 1.3 [5] : Let μ: $R \to L$ then μ is called an L-fuzzy ring of R, if for all x,y in R

(i) $\mu(x - y) \geq \mu(x) \Lambda\, \mu(y)$
(ii) $\mu(xy) \geq \mu(x)$.

Definition 1.4 [5] : A L-fuzzy set μ of R is called an L-fuzzy ideal of R , if for x,y in R

(i) $\mu(x - y) \geq \mu(x) \Lambda\, \mu(y)$
(ii) $\mu(xy) \geq \mu(x) \Lambda\, \mu(y)$

2. L-vague Rings

In this section we define the following

Definition 2.1 : Let (L,$\leq$) be a lattice with involutive order reversing operation N:L$\to$ L. Let (R,+,.) be a commutative ring. An L-vague set A in R is defined as an object of the form A= (m_A,n_A). Where m_A: $R \to L$ and n_A: $R \to L$ defines the degree of membership function and degree of non-membership for x$\in$ R

Definition 2.2 : Let L be a lattice. An L-vague set A= (m_A,n_A) of L is said to be L-vague ring if it satisfies the following properties

(1) $m_A(x - y) \geq \Lambda\{m_A(x), m_A(y)\}$ and $n_A(x - y) \geq \vee\{ n_A(x), n_A(y)\}$
(2) $m_A(xy) \geq \Lambda\{m_A(x), m_A(y)\}$ and $n_A(xy) \leq \vee\{n_A(x), n_A(y)\}$
(3) $m_A(x \Lambda y) \geq \Lambda\{m_A(x), m_A(y)\}$
(4) $n_A(x \vee y) \geq \vee\{n_A(x), n_A(y)\}$

Definition 2.3 : Let A= (m_A,n_A) be an L-vague ring of a ring R then the true $\alpha - cut\ set\ m_{A\alpha}$ can be defined by $m_{A\alpha} = \{x \in R: m_A(x) \geq \alpha\}$

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Definition 2.4: Let A= (m_A,n_A) be an L-vague ring of a ring R then the false $\alpha - cut\ set\ n_{A\beta}$ can be defined by $n_{A\beta} = \{x \in R: n_A(x) \leq \beta\}$.

Definition 2.5: Let A= (m_A,n_A) be an L-vague ring of a ring R then the L-vague cut set of A is $A_{(\alpha,\beta)}$ can be defined by

$$A_{(\alpha,\beta)}(x) = \{x \in R: m_A(x) \geq \alpha\ and\ n_A(x) \leq \beta\} \quad for\ all\ \alpha, \beta\ \in [0\ 1]$$

Theorem 2.6: Let A= (m_A,n_A) be an L-vague set of a ring R. then

$m_A(0) \geq m_A(x)$ and $n_A(0) \leq n_A(x)$ for all 0, x$\in R$

Proof: Let x,0 be any two elements in R then

(i) $m_A(0) = m_A(x - x) \geq \Lambda\{m_A(x), m_A(x)\} = m_A(x)$
$\Rightarrow m_A(0) \geq m_A(x)$

(ii) $n_A(0) = n_A(x - x) \leq \vee\{n_A(x), n_A(x)\} = n_A(x)$
$\Rightarrow n_A(0) \leq n_A(x)$
for all x,y $\in$ [0 1]

Theorem 2.7: Let A= (m_A,n_A) be an L-vague set of a ring R. then

$m_A(x) = m_A(-x)$ and $n_A(x) = n_A(-x)$ for all x$\in R$

Proof: Let x,0 be any two elements in R then

(i) $m_A(x) = m_A(-(-x)) \geq \Lambda\{m_A(-x)\} \geq m_A(x)$

$\Rightarrow m_A(x) = m_A(-x)$.

(i) $n_A(x) = n_A(-(-x)) \leq V\{n_A(-x)\} \leq n_A(x)$

$\Rightarrow n_A(x) = n_A(-x)$.

Theorem 2.8: Let A= (m_A,n_A) and B= (m_B,n_B) are L-vague rings of ring R then A$\cap B$ is also L-vague ring of a ring R for all x$\in R$

Proof: Let x,y be any two elements in R then

(i) $m_{(A\cap B)}(x - y) = \Lambda\{m_A(x - y), m_B(x - y)\}$

$\geq \Lambda\{m_A(x), m_A(y), m_B(x), m_B(y)\}$

$\geq \Lambda\{m_{A\cap B}(x), m_{A\cap B}(y)\}$

$\therefore m_{(A\cap B)}(x - y) \geq \Lambda\{m_{A\cap B}(x), m_{A\cap B}(y)\}$

(ii) $n_{(A\cap B)}(x - y) = \vee\{n_A(x - y), n_B(x - y)\}$

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$$\leq \vee\{n_A(x), n_A(y), n_B(x), n_B(y)\}$$
$$\leq \vee\{n_{A\cap B}(x), n_{A\cap B}(y)\}$$
$$\therefore n_{(A\cap B)}(x-y) \leq \vee\{n_{A\cap B}(x),\ n_{A\cap B}(y)\}$$

(iii) $m_{(A\cap B)}(xy) = \Lambda\{m_A(xy), m_B(xy)\}$
$$\geq \Lambda\{m_A(x), m_A(y), m_B(x), m_B(y)\}$$
$$\geq \Lambda\{m_{A\cap B}(x), m_{A\cap B}(y)\}$$

$$\therefore m_{(A\cap B)}(xy) \geq \Lambda\{m_{A\cap B}(x), m_{A\cap B}(y)\}$$

(iv) $n_{(A\cap B)}(xy) = \vee\{n_A(xy), n_B(xy)\}$

$$\leq \vee\{n_A(x), n_A(y), n_B(x), n_B(y)\}$$
$$\leq \vee\{n_{A\cap B}(x), n_{A\cap B}(y)\}$$
$$\therefore n_{(A\cap B)}(xy) \leq \vee\{n_{A\cap B}(x), n_{A\cap B}(y)\}$$

$(A \cap B)$ is also L-vague ring of rings A,B for x,y$\in R$

Theorem 2.9: Let A= (m_A, n_A) and B= (m_B, n_B) are L-vague rings of ring R then AB is also L-vague ring of a ring R for all x$\in R$

Proof: Let x,y$\in R$

(i) $(m_A.m_B)(x-y) = (m_A.m_B)(x+(-y))$
$\geq \Lambda\{((m_A.m_B)(x)), ((m_A.m_B)(-y))\}$
$= \Lambda\{((m_A.m_B)(x)), ((m_A.m_B)(y))\}$
$(m_A.m_B)(x-y) \geq \Lambda\{((m_A.m_B)(x)), ((m_A.m_B)(y))\}$

(ii) $(n_A.n_B)(x-y) = (n_A.n_B)(x+(-y))$
$\leq \vee\{((n_A.n_B)(x)), ((n_A.n_B)(-y))\}$
$= \vee\{((n_A.n_B)(x)), ((n_A.n_B)(y))\}$
$(n_A.n_B)(x-y) \leq \vee\{((n_A.n_B)(x)), ((n_A.n_B)(y))\}$

(iii) $(m_A.m_B)(xy) = max\{\Lambda_{i=1}^{n}\Lambda_{j=1}^{m} m_A(x_iy_j)m_B(x_iy_j), / x_i, x_i^1, y_j, y_j^1 \in R \ \sum_{i=1}^{n}\sum_{j=1}^{m} / x_ix_i^1y_j, y_j^1 = xy\}$
$\geq \min\{\sum_{i=1}^{n} x_ix_i^1, \sum_{j=1}^{m} y_jy_j^1 = xy\ for\ i,j \in N\}$

Therefore $(m_A.m_B)(xy) \geq min((m_A.m_B)(x)\ \Lambda(m_A.m_B)(y)$

$$(iv)\ \ (n_A.n_B)(xy) = min\{\bigvee_{i=1}^{n}\bigvee_{j=1}^{m}(n_A(x_iy_j)\bigvee n_B(x_i^1y_j^1)\ /x_i, y_j, , x_i^1, y_j^1 \in R\ , \sum_{i=1}^{n}\sum_{j=1}^{m} / x_ix_i^1y_j, y_j^1 = xy\}$$

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$$\leq min\{\bigvee_{i=1}^{n}\bigvee_{j=1}^{m}(n_A(x_i)\vee n_A(y_j)\bigvee(n_B(x_i^1)\vee n_B(y_j^1)/x_i,y_{j,},x_i^1,y_j^1\in R,\sum_{i=1}^{n}\sum_{j=1}^{m}x_ix_i^1y_{j,}y_j^1=xy\}$$

$$=[\min\{\bigvee_{i=1}^{n}(n_A(x_i)\vee n_B(x_i^1))/\sum x_ix_i^1=x,\}]\bigvee[\{min\bigvee_{j=1}^{m}n_B(y_j)\vee n_B(y_j^1)/\sum y_jy_j^1=y\}$$

$$=((n_A.n_B)(x)\vee(n_A.n_B)(y)$$

Therefore $(n_A.n_B)(xy)=((n_A.n_B)(x)\vee(n_A.n_B)(y)$

3. Operations on L-vague ideals of a ring.

In this section we define the following

Definition 3.1 : Let L be a lattice . An L-vague set A=(m_A,n_A) of L is said to be L-vague ideal of ring R if it satisfies the following properties

1. $m_A(x-y)\geq\Lambda\{m_A(x),m_A(y)\}$
2. $n_A(x-y)\leq\mathrm{V}\{n_A(x),n_A(y)\}$
3. $m_A(xy)\geq\Lambda\{m_A(x),m_A(y)\}$
4. $n_A(xy)\leq\mathrm{V}\{n_A(x),n_A(y)\}$
5. $m_A(x\wedge y)\geq\Lambda\{m_A(x),m_A(y)\}$
6. $n_A(x\vee y)\leq\mathrm{V}\{n_A(x),n_A(y)\}$

Definition 3.2: Let A=(x,m_A,n_A) *be* an L-vague ideal of ring R (if and only if)

1. $(m_A)(*)=\{x\in R:m_A(x)=m_A(0)\}$
2. $(n_A)(*)=\{x\in R:n_A(x)\geq n_A(0)\}$

Theorem 3.3: Let A$\subseteq R$then A is an ideal of R if and only if$A^1=(x,m_A,n_A)$ is also an vague ideal of R.

Proof: Let x,y in A then A is an ideal of R

if and only if

(i) $m_A(0)(x-y)=m_A(0)$

$=\Lambda\{\mathrm{m_A(0)(x),m_A(0)(y)}\}$

$$m_A(0)(x-y)=\bigwedge\{\mathrm{m_A(0)(x),m_A(0)(y)}\}$$

(ii) $m_A(0)(xy)=m_A(0)$

$=\Lambda\{\mathrm{m_A(0)(x),m_A(0)(y)}\}$

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$$m_A(0)(xy) = \bigwedge\{m_A(0)(x), m_A(0)(y)\}$$

(iii) $n_A(0)(x-y) = n_A(0)$

$=V\{n_A(0)(x), n_A(0)(y)\}$

$$n_A(0)(x-y) = \bigvee\{n_A(0)(x), n_A(0)(y)\}$$

(iv) $n_A(0)(xy) = n_A(0)$

$=V\{n_A(0)(x), n_A(0)(y)\}$

$$n_A(0)(xy) = \bigvee\{n_A(0)(x), n_A(0)(y)\}$$

Theorem 3.4: Let $A = (x, m_A, n_A)$ be a L-vague ideal of ring R and R be a ring with identity then

(i) $m_A(1) \leq m_A(x)$
(ii) $n_A(1) \geq n_A(x)$ for all x$\in R$

Proof:

(i) For x$\in R$, $m_A(x) = m_A(x.1) \geq \Lambda\{m_A(x), m_A(1)\} \geq m_A(1)$

$\Rightarrow m_A(x) \geq m_A(1)$

(ii) For x$\in R$, $n_A(x) = n_A(x.1) \leq V\{n_A(x), n_A(1)\} \leq n_A(1)$

$\Rightarrow n_A(x) \leq n_A(1)$

Theorem 3.5: Let $A = (x, m_A, n_A)$ be a L-vague ideal of ring R.

(i) If $m_A(x-y) = m_A(0)$ then $m_A(x) = m_A(y)$
(ii) If $n_A(x-y) = n_A(0)$ then $n_A(x) = n_A(y)$ for x,y $\in R$

Proof:

For x,y$\in R$

(i) $m_A(x) = m_A(x-y+y) \geq \Lambda\{m_A(x-y), m_A(y)\} = m_A(y)$

$\Rightarrow m_A(x) \geq m_A(y) \rightarrow (1)$

$and\ m_A(y) = m_A(y-x+x) \geq \bigwedge\{m_A(x-y), m_A(x)\} = m_A(x)$

$\Rightarrow m_A(y) \geq m_A(x) \rightarrow (2)$

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From (1) and (2)

$$\therefore m_A(x) = m_A(y)$$

$(ii)\ n_A(x) = n_A(x - y + y) \leq \vee\{n_A(x - y), n_A(y)\} = n_A(y)$

$$\Rightarrow n_A(x) \leq n_A(y) \rightarrow (1)$$

$$and\ n_A(y) = n_A(y - x + x) \geq \bigvee\{n_A(x - y), n_A(y)\} = n_A(x)$$

$\Rightarrow n_A(y) \geq n_A(x) \rightarrow (2)$

From (1) and (2)

$$\therefore n_A(x) = n_A(y)$$

Theorem 3.6: Let $A = (x, m_A, n_A)$ be a L-vague ideal of ring R then

(i) $m_A(*)$ is an ideal of R
(ii) $n_A(*)$ is need not be an ideal of R

Proof: (i)For x, y $\in m_A(*)$ and r $\in R$

$$m_A(x - y) \geq \Lambda\{m_A(x), m_A(y)\} = m_A(0)$$

$$\therefore m_A(x - y) \geq m_A(0)$$

$$\text{And } m_A(xr) \geq m_A(x) = m_A(0)$$

$$\therefore m_A(xr) \geq m_A(0)$$

Therefore $m_A(*)$ is an ideal of R

(ii) Let x,y $\in (n_A)_*$and r $\in R$.

If n_A(x-y)=n_A(x) then n_A(x-y)> n_A(0) implies (x-y) $\in (n_A)_*$.

But if n_A(x-y)< $max\{n_A(x), n_A(x)\}$ then (x-y) does not belongs to $(n_A)_*$.

Similarly, if n_A(xr)≥ n_A(x) then x$\in (n_A)_*$.

If n_A(xr)< n_A(x) the xr does not belongs to $(n_A)_*$.

Theorem 3.7: Let R be a ring with identity and $A = (x, m_A, n_A)$ be an L-vague ideal of a ring R then

(i) $m_A(x) = m_A(x^{-1})$ is an ideal of R and $n_A(x) = n_A(x^{-1})$ is an ideal of R
Where x is the unit element.

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(ii) Let x,y be associates then $m_A(x) = m_A(y)$ and $n_A(x) = n_A(y)$

Proof: (i) For x∈ R such that $xx^{-1} = x^{-1}x = 1$

Let$m_A(x^{-1}) = m_A(x^{-1}) = m_A(x^{-1}xx^{-1}) \geq m_A(x^{-1}x) \geq m_A(x)$

$$m_A(x^{-1}) \geq m_A(x) \rightarrow (1)$$

And $m_A(x) = m_A(x.1) = m_A(xx^{-1}x) \geq m_A(xx^{-1}) \geq m_A(x^{-1})$

$$m_A(x) \geq m_A(x^{-1}) \rightarrow (2)$$

From eq'ns (1) and (2)

$$m_A(x) = m_A(x^{-1})$$

Also $m_A(xx^{-1}) = m_A(1) \geq m_A(x)$ and $m_A(x^{-1}x) = m_A(1) \geq m_A(x^{-1})$

Therefore $m_A(x) \leq m_A(1)$ $and m_A(x^{-1}) \leq m_A(1)$

$$\Rightarrow m_A(x) = m_A(x^{-1}) = m_A(1)$$

Also $n_A(xx^{-1}) = n_A(1) \leq n_A(x) and\ n_A(x^{-1}x) = n_A(1) \leq n_A(x^{-1})$

$$n_A(x) \leq n_A(1) \text{ and } n_A(x^{-1}) \leq n_A(1)$$

$$n_A(x) = n_A(x^{-1}) = n_A(1)$$

(iii) Let x=uy where u is the unit element then

$$m_A(x) = m_A(ux) \geq m_A(y) and\ m_A(y) = m_A(xu^{-1}) \geq m_A(x)$$

$$n_A(x) = n_A(ux) \leq n_A(y) and\ n_A(y) = n_A(xu^{-1}) \leq n_A(x)$$

Therefore $m_A(x) = m_A(y)$ and $n_A(x) = n_A(y)$

Theorem 3.8: Let R be a ring with identity and$A_j = (x, m_{Aj}, n_{Aj}), B = (x, m_{Bj}, n_{Bj})$ are two L-vague ideal of a ring R then $(Aj \cap Bj)$ is an vague ideal of R. More over for any j∈ I,indexed set, $A_j = (x, m_{Aj}(x), n_{Aj}(x): x \in R)$ is an L-vague ideal of R.

Proof: For x,y∈ R then

(i) For x, y∈ R then

$$m_{(A_j \cap B_j)}(x) = m_{(A_j \cap B_j)}(x - y + y)$$

$$\geq \Lambda\{m_{(A_j \cap B_j)}(\text{x-y}), m_{(A_j \cap B_j)}(y)\}$$

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$$=m_{(A_j\cap B_j)}(y)$$

$$m_{(A_j\cap B_j)}(y) \quad =m_{(A_j\cap B_j)}(y-x+x)$$

$$\geq \Lambda\{m_{(A_j\cap B_j)}(x-y), m_{(A_j\cap B_j)}(x)\}$$

$$=m_{(A_j\cap B_j)}(x)$$

(ii) For x, y$\in R$ then

$$n_{(A_j\cap B_j)}(x) = n_{(A_j\cap B_j)}(x-y+y)$$

$$\leq \mathrm{V}\{n_{(A_j\cap B_j)}(x-y), n_{(A_j\cap B_j)}(y)\}$$

$$=n_{(A_j\cap B_j)}(y)$$

$$n_{(A_j\cap B_j)}(y) \quad =n_{(A_j\cap B_j)}(y-x+x)$$

$$\leq \mathrm{V}\{n_{(A_j\cap B_j)}(x-y), n_{(A_j\cap B_j)}(x)\}$$

$$=n_{(A_j\cap B_j)}(x)$$

(iii) $m_{(A_j\cap B_j)}(xy) = \mathrm{V}\{m_{Aj}(xy), m_{Bj}(xy)\}$

$$\geq \bigvee\{m_{Aj}(x), m_{Aj}(y), m_{Bj}(x), m_{Bj}(y)\}$$

$$\geq \{m_{Aj\cap Bj}(x), m_{Aj\cap Bj}(y)\}$$

Therefore $m_{(A_j\cap B_j)}(xy) \geq \{m_{Aj\cap Bj}(x), m_{Aj\cap Bj}(y)\}$

(iv) $n_{(A_j\cap B_j)}(xy) = \mathrm{V}\{n_{Aj}(xy), n_{Bj}(xy)\}$

$$\leq \bigvee\{n_{Aj}(x), n_{Aj}(y), n_{Bj}(x), n_{Bj}(y)\}$$

$$\leq \{n_{Aj\cap Bj}(x), n_{Aj\cap Bj}(y)\}$$

Therefore $n_{(A_j\cap B_j)}(xy) \leq \{n_{Aj\cap Bj}(x), n_{Aj\cap Bj}(y)\}$

Therefore $(Aj \cap Bj)$ is also L-vague ideal , where Aj,Bj$\in R$ *and for all* $x, y \in R$

4. L-vague cosets.

In this section we explain the L-vague coset and study quotient ring of R by the truth ship function and false ship function of an L-vague ring and L-vague ideal respectively.

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Definition 4.1: Let $A = (x.m_A(x), n_A(x))$ be an L-vague set of an L-vague ring R then $D = \{(x, m_A(0)_{\{x\}}+m_A(x), n_A(0)_{\{x\}}+n_A(x))\}$ is called an L-vague coset (LVC)of A and it is denoted as

$C= \{(x,\ (x+m_{(A)})(x), (x + n_{(A)})(x) : x \in R\}$

Theorem 4.2: Let x ,y$\in R$ and let $D = \{(x, m_A(0)_{\{0\}}+m_A(x), n_A(0)_{\{0\}}+n_A(x))\}$ be an L-vague coset of L-vague ring of A then

(i) $(m_A(0)_{\{x\}}+m_A)(y) = m_A(x-y)$

(ii) $(n_A(0)_{\{x\}}+n_A)(y) = n_A(x-y)$

Proof: For any x,y$\in R$

(i) $(m_A(0)_{\{x\}}+m_A)(y) = \max\{\min\{m_A(0)_{\{x\}}(u), m_A)(v): u+v=y\}\}$
$= \min\{m_A(0)_{\{x\}}(x), m_A(y-x)\}$
$= m_A(y-x)\}$

(ii) $(n_A(0)_{\{x\}}+n_A)(y) = \min\{\max\{n_A(0)_{\{x\}}(u), n_A)(v): u+v=y\}\}$
$= \max\{n_A(0)_{\{x\}}(x), n_A(y-x)\}$
$= n_A(y-x)\}$

Remark: 4.3 The above result becomes

$(x+m_A)(y) = m_A(x-y)$ and $(x+n_A)(y) = n_A(x-y)$

Theorem 4.4 : Let $A = (x, m_A(x), n_A(x))$be an L-vague ideal of R then

(i) $x + m_A = y + m_A$ if and only if $m_A(x-y) = m_A(0)$

(ii) $x + n_A = y + n_A$ if and only if $n_A(x-y) = n_A(0)$

Proof:

(i) Suppose $x + m_A = y + m_A$then $(x+m_A(x)) = y + m_A(x)$

$implies \quad m_A(0) = m_A(x-y).$

Conversely , if $m_A(x-y) = m_A(0)$, then for z$\in R$,

$(x+m_A)(z) = m_A(x-z) = m_A(x-y+y-z)$

$\geq \min\{m_A(x-y), m_A(y-z)\}$

$\geq \min\{m_A(0), m_A(y-z)\}$

$= m_A(y-z).$

Similarly $m_A(y-z) = (y+m_A)(z) \geq (x + m_A(z))$.

Therefore $m_A(x-z) = m_A(y-z)$

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Hence $x + m_A = y + m_A$

(ii) Suppose $x + n_A = y + n$ then $(x+n_A(x)) = y + n_A(x)$

$$implies \quad n_A(0) = n_A(x - y).$$

Conversely , if $n_A(x - y) = n_A(0)$, then for z∈ R,

$$(x+n_A)(z) = n_A(x - z) = n_A(x - y + y - z)$$

$$\leq max\{n_A(x - y), n_A(y - z)\}$$

$$\geq \max\{n_A(0), n_A(y - z)\}$$

$$= n_A(y - z).$$

Similarly $n_A(y - z) = (y + n_A)(z) \leq \big(x + n_A(z)\big)$.

Therefore $n_A(x - z) = n_A(y - z)$

Hence $x + n_A = y + n_A$

Definition 4.5: Let A= $(x, t_A(x), f_A(x))$ bean L-vague set of an l-vague ring R, then $\frac{R}{A}$ is called L-vague quotient ring R by t_A andf_A if it satisfies the following properties.

(i) $0 + t_A = t_A$, $0+f_A = f_A$ is the zero element of $\frac{R}{A}$

(ii) For all $x \in R - (x + t_A) = (-x) + t_A \; and \; -(x + f_A) = (-x) + f_A$ additive inverse of

(iii) For all x,y,z ∈ $\frac{R}{A}$, (x+y)+z=(x+y)+z

(iv) For all x,y,z ∈ $\frac{R}{A}$, (x+y) = (y+x)

(v) For all x,y,z ∈ $\frac{R}{A}$, x.(y.z)=(x.y).z

(vi) For all x,y,z ∈ $\frac{R}{A}$, x.(y+z) = x.y+x.z and (x+y).z=x.z+y.z

Conclusion: In this paper we introduced some operations on L-vague sets and studied some properties on L-vague rings and define L-vague ideals, also studied some of their operations on L-vague ring and L-vague ideals. These concepts are used in the development of some of their important properties have been inspected.

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DEFINING RELATIONS FOR GENERALIZED ACCESSIBLE RINGS

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ABSTRACT

In this paper we give a simple system of defining relations for the generalized accessible rings established in [2]. We showthat ((wσ,xσ,yσ),z) = sgnσ ((w,x,y),z) for all w,x,y,z∈ R and (vσ,xσ,yσ) = sgnσ (v,x,y) for all v,x,y∈ R, where σ is any permutation of the three elements. Also it is shown that f(w,x,y,z) is a skew symmetric function of w,x,y,z. Using these results, we prove that a flexible ring R is generalized accessible if and only if it satisfies the following identities (i) (x,(z,y,y) = 0 (ii) (x,y,(w,z)) + (x,(w,z),y) = 0 for all w,x,y,z∈ R and a 2- and 3- divisible generalized standard ring R is a generalized accessible ring. It is also proved that a non-commutative generalized accessible Jordan ring R is generalized standard.

INTRODUCTION

Schafer [3] defined Generalized standard rings and determined the structure of 2- and 3- divisible simple, finite dimensional generalized standard algebras by showing that they must be either commutative, Jordan, or alternative. Kleinfeld et. al. [1] introduced the generalized accessible rings. Shestakov [4] obtained the defining relations for the class of generalized accessible rings.

PRELIMINARIES

Throughout this section R represents a generalized accessible ring.

A ring is flexible if the following identity holds (x,y,x) = 0.

A flexible ring is non commutative Jordan if the following identity holds $(x^2,y,x) = 0$.

A ring is a generalized standard ring if it satisfies the following identities :

(x,y,x) = 0, ... 1.1

(x,y,z)x + (y,z,x)x + (z,x,y)x = (x,y,zx) + (y,xz,x) + (xz,x,y), ... 1.2

(x,y,wz) + (w,y,xz) + (z,y,xw) = (x,(w,z,y)) + (x,w,(y,z)). 1.3

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It is observed that these identities imply (y,y,(x,z)) = 0 ..... 1.4

and if 2- and 3- divisible, $(x,y,x^2) = 0$. 1.5

A Ring R is a generalized accessible ring if it satisfies the identities

(x,y,x) = 0,

(x,(z,y,y)) = 0. ..... 1.6

3 (x,y,(w,z)) = - (w,(x,y,z)) - 2(x,(y,z,w)) + 2(y,(z,w,x)) + (z,(w,x,y)). 1.7

In any arbitrary ring, we have the following identities:

S(x,y,z) = (xy,z) – x(y,z) – (x,z)y, ... 1.8

(x,yoz,w) = (xoy),z,w) + (x,y,zow) – xo(y,z,w) – (x,y,z)ow, ... 1.9

(w,x,(y,z)) – (w,(x,y),z) + ((w,x),y,z) = (w,(x,y,z)) – (z,(w,x,y), ... 1.10

G(x,y,w,z) = 3(x,y,(w,z)) + (w,(x,y,z)) + 2(x,(y,z,w)) – 2(y,(z,w,x)) - (z, (w,x,y)),
..... 1.11

Lemma 1.1:((wσ,xσ,yσ),z) = sgnσ ((w,x,y),z), where w,x,y and z are arbitrary in R and σ is any permutation of the three letters w,x and y.

Proof: Let us take three elements w,x and y to form an alternative triple. That is, ((w,y,y),z) = 0, ((y,y,w),z) = 0 and ((y,w,y),z) = 0.

By linearizing 1.6 and 1.1, we obtain

((w,x,y),z) = -((w,y,x),z)

= ((x,y,w),z)

= -((y,x,w),z)

= ((y,w,x),z) .

Therefore ((w,x,y),z) = ((x,y,w),z) = ((y,w,x),z).

Hence ((wσ,xσ,yσ),z) = sgnσ ((w,x,y),z). □

Lemma 1.2: Let V = Σ(R,R), Let v∈V and x,y∈R. Then (vσ,xσ,yσ) = sgnσ (v,x,y), where σ is any permutation of the three elements v,x and y.

Proof: Since from G(y,y,w,z) = 0, we have

3(y,y, (w,z)) = -(w,(y,y,z)) - 2(y,(y,z,w)) + 2(y,(z,w,y)) + (z,(w,y,y)). 1.12

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From lemma 1.1, the right hand side of equation 1.12 is zero.

Thus we have (y,y,(w,z)) = 0. ..... 1.13

By linearizing 1.13, we have (x,y,(w,z)) = -(y,x, (w,z)).

Now apply flexible law to 1.13, we have ((w,z),y,y) = 0.

By linearizing this last equation, we get ((w,z), x,y) = -((w,z),y,x).

Thus we have (x,y, (w,z)) = -(y,x,(w,z)) = ((w,z),x,y) = -((w,z),y,x)).

We also know from the flexible law that (x,(w,z),y) = -(y,(w,z),x).

Now we have need to prove that (x,(w,z),y) = -(x,y,(w,z)).

To prove this we define

K(w,x,y,z) = (w,x,(y,z)) – (w,(x,y),z) + ((w,x), y,z) – (w,(x,y,z)) + (z,(w,x,y)).

Now from 1.10 we states precisely that K(w,x,y,z) = 0

From 3K (x,z,w,y) + G(y,w,x,z) – G(x,z,w,y) = 0, we obtain

3(x, (z,w),y) = 3((x,z),w,y) + 3(x,z,(w,y)) – 3(x,(z,w,y))

-3((x,z,w),y) + 3(y,w,(x,z)) + (x,(y,w,z))

+2(y,(w,z,x)) - 2 (w,(z,x,y)) – (z,(x,y,w))

-3(x,z,(w,y) - (w,(x,z,y)) - 2(x,(z,y,w))

+2(z,(y,w,x)) + (y,(w,x,z). 1.14

By using lemma 1.1 and collecting all terms on the right hand side of 1.14 yields.

-3(x,(w,z),y) = -(w,(x,y,z) - 2(x,(y,z,w)) + 2(y,(z,w,x)) + (z,(w,x,y)). 1.15

But from G(x,y,w,z) = 0 we have

3(x,y,(w,z)) = -(w,(x,y,z)) - 2 (x,(y,z,w)) + 2(y,(z,w,x)) + (z,(w,x,y)). ... 1.16

Comparing 1.15 and 1.16 we have

(x,y,(w,z)) = -(x,(w,z),y). 1.17

This completes the proof of the lemma. □

Lemma 1.3: f(w,x,y,z) is a skew symmetric function of w,x,y,z.

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Proof: From lemma 1.2, we have ((x,y),z,w) = -(z,(x,y),w).

Now we substitute w=z in the above equation,

we get ((x,y),z, z) = - (z,(x,y),z).

By flexible law, ((x,y),z,z) = 0. ..... 1.18

Now we have to show that f(w,x,y,z) = -f(z,w,x,y).

To prove this we define function

f(w,x,y,z) = ((w,x),y,z) - (x,(w,y,z)) - ((x,y,z),w) which is satisfied by the identities 1.6 and 1.17

Now substitute z=y in the function f(w,x,y,z), we have

f(w,x,y,y) = ((w,x),y,y) – (x,(w,y,y)) - ((x,y,y),w).

By using 1.18 and 1.6 in the above equation,

we obtain f(w,x,y,y) = 0. ..... 1.19

By linearizing 1.1, 1.6, 1.18 and using 1.10, 1.17, we get

-f(z,w,x,y) = -((z,w),x,y) + (w,(z,x,y)) + ((w,x,y),z)

= -((z,w),x,y) + (w,(z,x,y)) - (w,(x,y,z)) - (w,(x,y),z) + ((w,x),y,z) + (w,x, (y,z))

= ((w,x),y,z) – ((y,z),x,w) - ((z,w),x,y) + (w,(z,x,y)) + (w,(z,y,x) + (z,(x,y),w)

= ((w,x),y,z) + ((y,z),w,x) - ((z,w),x,y) + (w,(z,x,y)) – (w,(z,x,y)) - ((x,y),z,w)

= ((w,x),y,z) + ((y,z),w,x) - ((x,y),z,w) – ((z,w),x,y).

Consequently, f(w,x,y,z) = ((w,x),y,z) + ((y,z), w,x) - ((x,y),z,w) - ((z,w),x,y),

That is, f(w,x,y,z) = -f(z,w,x,y).

When, in view of 1.19, it follows that f(w,x,y,z) is skew–symmetric.

This completes the proof of the lemma. □

Theorem 1.1: A flexible ring R is generalized accessible if and only if the identities (i) (x, (z,y,y)) = 0 (ii) (x,y, (w,z)) + (x, (w,z),y) = 0.

Proof: First we assume that a flexible ring R is generalized accessible if it satisfies the identities (ii).

The above identity proved in the lemma 1.2. Conversely, if R satisfies the identities (i) and (ii) then R is generalized accessible.

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To prove this we take definition,

f(w,x,y,z) = ((w,x),y,z) - (x,(w,y,z) - ((x,y,z),w).

From this definition, we have

((w,x),y,z) = f(w,x,y,z) + (x,(w,y,z)) + ((x,y,z),w).

((y,z),w,x) = f(y,z,w,x) + (z,(y,w,x)) + ((z,w,x),y).

((x,y),z,w) = f(x,y,z,w) + (y,(x,z,w)) + ((y,z,w),x).

Now, by virtue of 1.10 and 1.17, we have

f(w,x,y,z) + (x,(w,y,z)) + ((x,y,z),w) + f (y,z,w,x) + (z, (y, w,x)) + ((z,w,x),y) – f(x,y,z,w) - (y,(x,z,w)) – ((y,z,w),x).

= ((w,x),y,z) + ((y,z),w,x) – ((x,y), z,w).

= (w,(x,y),z) - (w,x,(y,z) + (w,(x,y,z)) – (z,(w,x,y)) + ((y,z),w,x) – ((x,y),z,w).

= -((x,y),w,z) – (w,x,(y,z)) + (w,(x,y,z)) + ((w,x,y),z) + ((y,z),w,x) – ((x,y),z,w).

Now by linearizing of 1.1 and 1.18, we have

f(w,x,y,z) + (x,(w,y,z)) + ((x,y,z),w) + f (y,z,w,x) + (z,(y,w,x)) + ((z,w,x),y) – f(x,y,z,w) - (y,(x,z,w)) - ((y,z,w),x) = (w,(x,y,z)) + ((w,x,y,),z).

When, in view of the linearized 1.1, 1.6 and lemma 1.3

3f(w,z,x,y) = 2[(w,(x,y,z)) – (z,(w,x,y)) + (y,(z,w,x)) - (x,(y,z,w))]. …. ….1.20

3[((w,z), x,y) - (z,(w,x,y)) - ((z,x,y),w)]

= 2(w,(x,y,z)) - 2(z,(w,x,y)) + 2(y,(z,w,x)) - 2(x,(y,z,w))

3((w,z),x,y) = 2(y,(z,w,x)) - 2(x,(y,z,w)) + (z,(w,x,y)) – (w,(x,y,z)) which is equivalent to 1.7.

Thus, 1.7 can be replaced by the simpler 1.20 in the definition of a generalized accessible ring.

Theorem 1.2*:* A 2- and 3- divisible generalized standard ring R is a generalized accessible ring.

Proof*:* If we substitute w=y in 1.3, then we get

(x,y,yz) + (y,y,xz) + (z,y,xy) = (x,y,(y,z)) …….. 1.21

because of 1.1. By cancellation and subtraction, this last equation becomes

(x,y,zy) + (y,y,xz) + (zy,x,y) = 0.

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By interchanging x and z in 1.21 and subtract with this last equation, we get

$(y,y,(x,z)) = 0.$ ……. 1.22

By linearizing 1.22 and using 1.1, we obtain

$(w,y,(x,z)) = -(y,w,(x,z)) = ((x,z),w,y) = -((x,z),y,w).$ ……. 1.23

Now substitute w=z=x in 1.3 and using 1.22 we get

$3(x,y,x^2) = (x,(x,x,y)).$ ……. 1.24

Now we take z=y=x in 1.10 we obtain $((w,x),x,x) = -(x,(w,x,x))$.

Using 1.22 and the linearization of 1.1 we have $((w,x),x,x) = 0$.

Therefore we have $(x,(w,x,x)) = 0$.

If we put w=y in this last equation and using the linearization of 1.1 then we get $(x, (x,x,y)) = 0$.

Substitute this in 1.24 we get $3(x,y,x^2) = 0$, and since R is a 2- and 3- divisible, it follows that $(x,y,x^2) = 0$.
……. 1.25

Now put w=x in 1.3 and using 1.22, then we obtain

$2(x,y,xz) + (z,y,x^2) = (x, (x,z,y))$.

But by the linearization of 1.25 yields $(x,y,xz + zx) + (z,y,x^2) = 0$.

By comparing this last two equations we get

$(x,y,(x,z)) = (x,(x,z,y))$. ……. 1.26

Now interchange w and z in 1.3 and subtract the resulting equation from 1.3, we get

$(x,y,(w,z)) = (x,(w,z,y)) – (z,w,y)) + (x,w,(y,z)) - (x,z,(y,w))$. ……. 1.27

Substitute w=y in 1.27 and using 1.11, then we obtain

$(x,y,(y,z)) = -(x,(z,y,y)) + (x,y,(y,z))$ which implies $(x,(z,y,y)) = 0$. …..…..1.28

Now by linearization of 1.28 and use of 1.1 we get

$(x,(w\sigma,y\sigma,z\sigma)) = sgn\sigma\ (x,(w,y,z))$. ……. 1.29

Where σ is any permutation of the three letters w,y and z. By using 1.29, we rewrite 1.27 as $(x,y,(w,z)) + (x,w,(z,y)) + (x,z,(y,w)) = 2(x,(w,z,y))$.

……. 1.30

By linearizing 1.26, we get

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(w,y,(x,z)) + (x,y,(w,z)) = (w,(x,z,y)) + (x,(w,z,y)).

Interchange x and y in this last equation, we obtain

(w,x,(y,z)) + (y,x,(w,z)) = (w,(y,z,x)) + (y,(w,z,x)). ……. 1.31

Now apply permutation (wyz) in 1.31 we get

(y,x,(z,w)) + (z,x,(y,w)) = (y,(z,w,x)) + (z,(y,w,x)). ……. 1.32

Now subtract 1.31 from 1.32 and use 1.29 we have

2(x,y,(w,z)) + (x,w,(y,z)) + (z,x,(y,w)) = -(w,(x,y,z)) + 2(y,(z,w,x)) +

(z,(w,x,y)). ……. 1.33

By adding 1.30 and 1.33 and using 1.29 and 1.23, it follows that

3(x,y,(w,z)) = -(w,(x,y,z)) – 2(x,(y,z,w)) + 2(y,(z,w,x)) + (z,(w,x,y)). ….. 1.34

Therefore in other words G(x,y,w,z) = 0.

Since 1.1, 1.28 and 1.34 constitute a generalized accessible ring.

In any arbitrary accessible ring R, it is shown that R is flexible, that all associators satisfy ((x,y,z),w) = 0. From this it follows that R is a generalized accessible rings. □

Theorem1.3*:* A non commutative generalized accessible Jordan ring R is generalized standard.

Proof: By linearizing the Jordan identity $(x^2,y,x) = 0$, we obtain

(x,y,zow) + (z,y,xow) + (w,y, xoz) = 0.

Now we have hence

2(x,y,zw) + 2(w,y,xz) + 2(z,y,xw)

= (x,y,(z,w)) + (w,y,(x,z)) + (z,y,(x,w))

= (x,y,(z,w)) + (w,y,(x,z)) + (z,y,(w,x)) + 2(z,y,(x,w))

= H(x,z,w,y) + 2(z,y,(x,w)).

Where H(x,z,w,y) = (x,y,(z,w)) + (w,y,(x,z)) + (z,y,(w,x)).

Now using function f(w,x,y,z) = ((w,x),y,z) - (x,(w,y,z)) - ((x,y,z),w) in H (x,z,w,y), we have

H(x,z,w,y) = ((z,w),x,y) + ((x,z),w,y) + ((w,x),z,y)

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= f(z,w,x,y) + (w,(z,x,y)) + ((w,x,y),z) + f(x,z,w,y) + (z,(x,w,y)) +

((z,w,y),x) + f(w,x,z,y) + (x,(w,z,y)) + ((x,z,y),w).

By using properties of generalized accessible ring, we obtain

H(x,z,w,y) = -3f(w,z,x,y) + 2(w,(x,y,z)) - 2(z,(w,x,y)) - 2(x,(y,z,w)).

Now using 1.20 in the above equation, we have

H(x,z,w,y) = -2(w,(x,y,z)) + 2(z,(w,x,y)) – 2(y,(z,w,x)) + 2(x,(y,z,w)) +

2(w,(x,y,z) - 2(z,(w,x,y)) – 2(x,(y,z,w)).

= -2(y,(z,w,x)).

Now using the identity of corollary of the lemma 2.1 from "Generalized standard rings".

I.P. Shestakov, Algebra, logika, 13, No.1, 88 – 103 (1974), we have

((y,x),y,z) = ((x,y,z),y).

Using the linearization of above identity, we have

2(x,y,zw) + 2(w,y,xz) + 2(z,y,xw) = H(x,z,w,y) + 2 (z,y,(x,w))

= -2(y,(z,w,x)) + 2(z,y,(x,w))

= -2((z,x,w),y) + 2(z,y,(x,w))

= 2((z,y,w),x) - 2(z,x,(y,w))

= 2(x,(z,w,y)) + 2(x,z,(y,w))

Consequently

(x,y,zw) + (w,y,xz) + (z,y,xw) = (x,(z,w,y) + (x,z,(y,w)). (or)

(x,y,zw) + (z,y,xw) + (w,y,xz) = (x,(z,w,y)) + (x,z,(y,w)).

Therefore R is generalized standard ring.

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CLASSES ON HARMONIC STARLIKE FUNCTIONS

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Abstract. The harmonic function subclasses described by an operator in a unit disk aare discussed in this paper. Obtain the necessary and sufficient conditions for above classes.

INTRODUCTION

Let $f=p+iq$ be harmonic in the domain Ω of complex plane $\square$. We shall express $f=\psi+i\zeta$, with ψ and ζ are respectively analytic part and co-analytic part of f.

Let S_H represents the functions of kind $f=\psi+i\zeta$ which are harmonic one-one and orientation- preserving in $\Delta=\{z\in\square:|z|<1\}$ with $f(0)=\psi(0)=f_z(0)-1=0$, also ψ and ζ as

$$\psi(z)=z+\sum_{n=2}^{\infty}a_n z^n,\ \zeta(z)=\sum_{n=1}^{\infty}b_n z^n,\ |b_1|<1. \quad (1)$$

In 1984, "the class S_H was introduced by Clunie and Sheil-Small [4] and their study gave insight of few coefficient bounds. Since then, many authors ([2], [3], [5], [6], [7], [8], [9], [10] and [11]) have studied the subclasses of harmonic univalent functions.

According to Makinde [6], ξ^m defined by

$$\xi^m f(z)=z+\sum_{n=2}^{\infty}\lambda_{nm}a_n z^n$$

where $\lambda_{nm}=\dfrac{n!}{|n-m|!}$ and

$$\xi^m f(z)=z^m\left[z^{-(m-1)}+\sum_{n=2}^{\infty}\lambda_{nm}a_n z^{n-m}\right],\ m\in\square_0=\square\cup\{0\}$$

and $\xi^0 f(z)=f(z),\ \xi' f(z)=z+\sum_{n=2}^{\infty}\lambda_{n1}a_n z^n=z+\sum_{n=2}^{\infty}na_n z^n.$

We consider the differential operator, proposed by Makinde [6], for the functions f defined in (1) by

$$\xi^m f(z)=\xi^m\psi(z)+(-1)^m\overline{\xi^m f(z)},\ m\in\square_0=\square\cup\{0\},\ \forall z\in\Delta \quad (2)$$

where $\xi^m\psi(z)=z+\sum_{n=2}^{\infty}\lambda_{nm}a_n z^n,\ \xi^m\zeta(z)=\sum_{n=1}^{\infty}\lambda_{nm}b_n z^n$ and $\lambda_{nm}=\dfrac{n!}{|n-m|!}$.

Let $A_H(m,\mu,\beta,\phi)$ be the class of functions f of the form (1) satisfying the condition

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$$\mathrm{Re}\left[(1+\beta e^{i\phi})\frac{\xi^{m+1}f(z)}{\xi^{m}f(z)}-\beta e^{i\phi}\right]>\mu,\ 0\le\mu<1,\ \beta\ge 0,\ \phi\in\square,\quad(3)$$

where $\xi^m f$ defined in (2).

Let $\overline{A_H}(m,\mu,\beta,\phi)$ be the subclass of $A_H(m,\mu,\beta,\phi)$ consisting of functions f such that ψ and ζ are of the form"

$$\psi(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\ \zeta(z)=\sum_{n=1}^{\infty}|b_n|z^n,\ |b_1|<1.\quad(4)$$

Lemma

According to [1]**:** Let $\delta\ge 0$. Then $\mathrm{Re}\{w\}>\delta$ if and only if $|w-(1+\delta)|<|w+(1-\delta)|$, where w is any complex number.

MAIN RESULTS

Theorem 1."Let f be of the form (1). If f holds

$$\sum_{n=2}^{\infty}\left[(1+\beta)|n-m|-(\mu+\beta)\right]\lambda_{nm}|a_n|+\sum_{n=1}^{\infty}\left[(1+\beta)|n-m|+(\mu+\beta)\right]\lambda_{nm}|b_n|\le(1-\mu),\qquad(5)$$

then f is sense-preserving, harmonic, univalent in Δ and $f\in A_H(m,\mu,\beta,\phi)$.

Proof. For proving $f\in A_H(m,\mu,\beta,\phi)$ we must show that f holds the condition (3).

By Lemma [1], it is enough to prove

$|A(z)-(1+\mu)B(z)|-|A(z)+(1-\mu)B(z)|\le 0$, where

$A(z)=(1+\beta e^{i\phi})\xi^{m+1}f(z)-\beta e^{i\phi}\xi^{m}f(z)$ and $B(z)=\xi^{m}f(z)$.

Consider $|\,A(z)-(1+\mu)B(z)\,|$

$$\le\mu|z|+\sum_{n=2}^{\infty}\left|(1+\mu)|n-m|-(\mu+\beta)\right|\lambda_{nm}|a_n||z|^n+\beta\sum_{n=2}^{\infty}\left|1-|n-m|\right|\lambda_{nm}|a_n||z|^n$$
$$+\sum_{n=1}^{\infty}\left||n-m|+(1+\mu)\right|\lambda_{nm}|b_n|\left|\overline{z}\right|^n+\beta\sum_{n=1}^{\infty}\left||n-m|+1\right|\lambda_{nm}|b_n|\left|\overline{z}\right|^n.$$

(6)

Consider $|\,A(z)+(1-\mu)B(z)\,|$

$$\ge(2-\mu)|z|-\sum_{n=2}^{\infty}\left|(\mu-1)-|n-m|\right|\lambda_{nm}|a_n||z|^n-\beta\sum_{n=1}^{\infty}\left|1-|n-m|\right|\lambda_{nm}|a_n||z|^n$$
$$-\sum_{n=1}^{\infty}\left||n-m|-(1-\mu)\right|\lambda_{nm}|b_n|\left|\overline{z}\right|^n-\beta\sum_{n=1}^{\infty}\left||n-m|+1\right|\lambda_{nm}|b_n|\left|\overline{z}\right|^n.$$

(7)

From (6) and (7), we will get, $|A(z)-(1+\mu)B(z)|-|A(z)+(1-\mu)B(z)|\le 0$. Hence the result.

Theorem 2. Let f be given by (4). Then $f\in\overline{A_H}(m,\mu,\beta,\phi)$ if and only if f holds the condition (5)".

Proof: Since $\overline{A_H}(m,\mu,\beta,\phi)\subset A_H(m,\mu,\beta,\phi)$, we shall prove necessary part of the theorem only, for which one can refer Sharma and Ravindar ([3], proof of Theorem 2.2).

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TWO-LAYER JEFFREY FLUID TRANSPORT IN NARROW CHANNELS WITH MILD STENOSIS

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Abstract

This paper examines the movement properties of a double-layer Jeffrey fluid within a mild stenosed narrow channel. The configuration comprises a central core containing Jeffrey fluid and an encircling peripheral section of Newtonian fluid. With the flow treated as steady and the stenosis mild, the governing equations are solved analytically to derive expressions for velocity profiles, effective viscosity, core haematocrit, average haematocrit. The findings demonstrate that effective viscosity diminishes with a rise in the Jeffrey parameter, whereas greater stenosis height and channel width result in elevated viscosity values. The calculated viscosity stays within physiologically acceptable parameters. The research further validates that the 'Fahraeus–Lindqvist effect' accounts for the rise in viscosity with channel radius. The mean haematocrit declines with the Jeffrey parameter and stenosis height, but grows with channel width, while the core haematocrit declines with all three parameters: Jeffrey parameter, stenosis height, and channel width.

1. INTRODUCTION

Microcirculation is the movement of blood with slight vessels like capillaries, venules, and arterioles. There are blood vessels in the human circulatory system that range in size from 20 to 500 micrometers. Furthermore, anomalous effects arise when blood passes vessels with reduced diameters. The 'Fahraeus-Lindqvist impact' [1] exemplifies a phenomenon in which the effective viscosity of blood decreases as channel width increases (Martini et al. [2]). A number of researchers (Azel-vandre and Oiknine [3]) have examined blood flow in small channels, presuming it to display Newtonian fluid characteristics and have validated the Fahraeus-Lindqvist effect due to its increasing relevance. Additionally, Sharan and Popel [4] employed a dual-component model to analyse blood flow in constricted channels. Haynes [5] investigated a double-layered fluid model comprising two Newtonian fluids with differing viscosities. Blood, regarded as a suspension of hemoglobin in plasma, displays non-Newtonian fluid behaviour under less shear rates (Haynes and Burton [6]). Chaturani and Upadhya [7] investigated double-layered blood flow models utilizing a Newtonian fluid in the peripheral section. Shukla et al. [8] scrutinized the effect of the periphery on the peristaltic transport of a biofluid. Srivastava and Saxena [9] have proposed the use of double-layered blood stream models, wherein the central area comprises plasma and Casson fluid, while the peripheral section is characterized by a Newtonian fluid. Aerosty and Gross [10] examined pulsatile flow in small vessels under the assumption that blood behaves as a Casson fluid. Stenosis refers to the constriction of an artery resulting from arteriosclerotic plaques. This occurs due to an excess of fatty deposits and abnormal angiogenesis within the arterial lumen. Stenosis obstructs blood flow, resulting in irregular hemodynamics. This is a primary factor contributing to blood clots, strokes, and heart failure. Stenosis can occur in various forms, such as when multiple shapes intersect or when the shapes lack regularity. Numerous experimental and theoretical investigations (Sankar and Hemalatha [11]) have sought to elucidate the impact of stenoses in the arterial lumen of a plasma channel on blood flow dynamics. Umadevi et al. [12] demonstrate that copper nanoparticles are efficacious in reducing the hemodynamics associated with stenoses. This indicates potential biomedical applications. Dhange et al. [13] Casson concentrated on the continuous, incompressible movement of fluid in a channel with several stenoses. Ponalagusamy [14] examined the influence of non-zero couple pressure at the interface, stenosis height, taper angle, axial distance, and several sloshes on flow parameters and flow line patterns. The Jeffrey fluid illustrates a non-Newtonian liquid characterized by viscoelastic and shear-thinning properties. The Jeffrey fluid model has been utilized to examine various fluid stream phenomena, including flow in porous media, non-Newtonian flow in conduits, and the dynamics of polymer solutions. The model is also utilized in biomechanics to examine the dynamics of blood stream

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and other biological fluids within blood vessels. The peristaltic motion of Jeffrey fluid in a circular vessel is observed by Hayat et al. [15]. Vajravelu et al. [16] investigate the peristaltic movement of Jeffrey fluid with thermal effects on a vertical porous surface under the assumptions of 'long wavelength approximation and less Reynolds number.' Investigations are underway regarding the diffusion of solute substances in the movement of Jeffrey fluid with a porous material in a peristaltic tube, as discussed by Ravi Kiran and Radhakrishnamacharya [17]. Santosh and Rajeshwar Rao [18] examined a double-fluid problem regarding the motion of Jeffrey fluid influenced by slip through a narrow channel. Santosh et al. [19] explored the effect of a porous material on a dual-layered mathematical model for the movement of Jeffrey fluid in channels of varying diameters.

2. MATHEMATICAL MODEL OF THE PROBLEM

This paper presents a two-layered model for the analysis of the two-dimensional movement of a Jeffrey blood in a narrow channel through stenosis. The variable h_0 is used to denote the channel width in the non-stenotic zone. It is composed that the movement area contains of a core with a height h_1 and a periphery with a thickness $\boldsymbol{\epsilon = h_0 - \beta h_0}$. Further, Fig. 1 depicts that the periphery composed that the Newtonian fluid and central core contains Jeffrey fluid. The cartesian coordinates (x, y) are utilized so that the x-axis is equivalent to the centerline of the vessel and the y-axis is vertical to the x-axis. For an incompressible Jeffrey fluid, the constitutive equations given by Hayat et al. [21] are $\overline{\boldsymbol{T}} = -P\overline{\boldsymbol{I}} + \overline{\boldsymbol{S}}$ and $\overline{\boldsymbol{S}} = \frac{\boldsymbol{\mu_c}}{\boldsymbol{1+\lambda_1}}(\dot{\boldsymbol{\gamma}} + \boldsymbol{\lambda_2}\ddot{\boldsymbol{\gamma}})$ here, P indicates pressure, The parameter denoted as λ_1 represents the ratio between relaxation time and retardation time, $\overline{\boldsymbol{S}}$ represents identity tensor, $\boldsymbol{\lambda_2}$ symbolises the retardation time, the symbol $\dot{\boldsymbol{\gamma}}$ signifies the shear rate, and dots are placed above the quantities to denote their changes over time. In this case, the formulae that describe the movement of an incompressible Jeffrey fluid are:

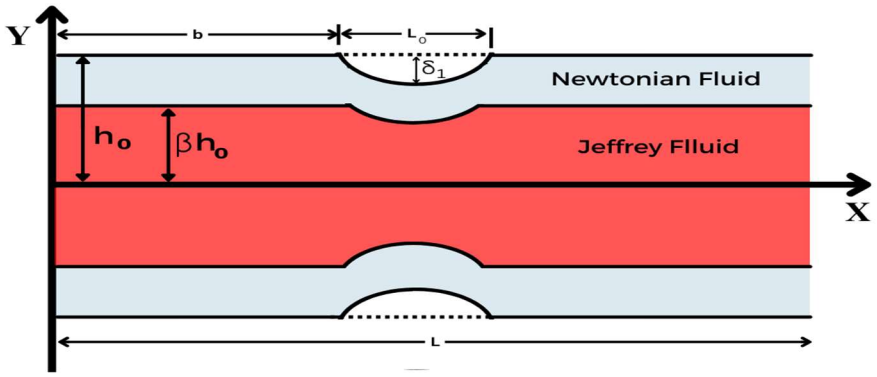

Fig. 1 Geometrical Structure of the Stenosed Channel

Given by the geometry of the wall

$$h = \begin{cases} h_0 - \frac{\delta_1}{2}\left(1 + \cos\frac{2\pi}{L_0}\left(x - b - \frac{L_0}{2}\right)\right), & b \le x \le b + L_0 \\ h_0, & \text{otherwise} \end{cases} \quad (1)$$

$$h_1 = \begin{cases} \beta h_0 - \frac{\delta}{2}\left(1 + \cos\frac{2\pi}{L_0}\left(x - b - \frac{L_0}{2}\right)\right), & b \le x \le b + L_0 \\ \beta h_0, & \text{otherwise} \end{cases} \quad (2)$$

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where L is the channel's length, L_0 is its length, δ_1 is the stenosis's highest point, δ is the interface's highest bulging point because of the stenosis, the parameter β represents the ratio among the average half-width of the middle core and the channel width in the unobstructed area, additionally h_0 denotes the width of the channel in the clear area.

3. ANALYSIS

In this paper, the two-dimensional movement characteristics of concentrated Jeffrey blood are modelled using the governing equations described by Santhosh et al. [20].

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

$$\rho\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]u=-\frac{\partial p}{\partial x}+\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y} \tag{4}$$

$$\rho\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]v=-\frac{\partial p}{\partial y}+\frac{\partial S_{yy}}{\partial y}+\frac{\partial S_{yx}}{\partial x} \tag{5}$$

In this context, v and u signify the velocity components along the x, y co-ordinate axes, p denotes pressure, ρ indicates density, Syy, Syx, Sxy, and Sxx represent various stress quantities, and μ_1 refers to the viscosity in the core section. The velocity in the middle portion is denoted as u_1 (y), whereas that in the peripheral area will be u_2 (y).

Considering the movement to be uniform and considering the stenosis as slight, the equations representing the fluid flow (Jeffrey fluid) in the core area ($0 \le y \le h_1$) lessen to

$$\frac{\mu_1}{(1+\lambda_1)}\frac{\partial^2 u_1}{\partial y^2}=\frac{\partial p}{\partial x},\quad 0 \le y \le h_1 \tag{6}$$

In the periphery area ($h_1 \le y \le h$) the equations representing the fluid flow (Newtonian fluid) for the current problem lessen to

$$\mu_2\frac{\partial^2 u_2}{\partial y^2}=\frac{\partial p}{\partial x},\quad h_1 \le y \le h \tag{7}$$

The boundary conditions for this current paper are

$$u_1=u_2, \tau_1=\tau_2 \text{ at } y=\pm h_1(x) \tag{8}$$

The velocity adheres to without slip boundary conditions as dictated by the governing equations.

$$u_2=0 \text{ at } y=\pm h(x) \tag{9}$$

calculating equations (6) and (7) subject to the conditions (8) and (9), we get

$$u_1=\frac{1}{2}\frac{1}{\mu_2}\frac{\partial p}{\partial x}[\mu'(1+\lambda_1)(y^2-h_1^2)-(h^2-h_1^2)] \tag{10}$$

and

$$u_2=\frac{1}{2\mu_2}\frac{\partial p}{\partial x}[y^2-h^2] \tag{11}$$

The core flow flux and that of the peripheral area are respectively represented by

$$Q_1=2\int_0^{h_1}u_1(y)dy \quad \text{and} \quad Q_2=2\int_{h_1}^{h}u_2(y)\,dy \tag{12}$$

Adding for $u_1(y)$ and $u_2(y)$ from (10) & (11) in (12)

$$Q_1=-\frac{1}{\mu_2}\frac{\partial p}{\partial x}\left[\frac{2h_1^3}{3}(1+\lambda_1)\mu' +h^2h_1-h_1^3\right] \tag{13}$$

$$Q_2=-\frac{1}{\mu_2}\frac{\partial p}{\partial x}\left[\frac{h_1^3}{3}-h^2h_1+\frac{2h^3}{3}\right] \tag{14}$$

Accordingly, the fluid flux in the narrowed stenotic passage is, $Q=Q_1+Q_2$ (15)

By substituting equations (13) and (14) into equation (15), we obtain

$$Q=-\frac{2}{3}\frac{1}{\mu_2}\frac{\partial p}{\partial x}[h_1^3(1+\lambda_1)\mu'+h^3-h_1^3] \tag{16}$$

The effective viscosity is derived through comparison of (16) with the flux relation for movement between double parallel plates.

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$$\mu_{eff} = \left(\frac{\mu' h_0^3}{3[h_1^3(1+\lambda_1)\mu' + h^3 - h_1^3]}\right), \text{ where, } \mu' = \frac{\mu_2}{\mu_1} \quad (17)$$

Mean Haematocrit Values for Cell-Free Layers of Walls

Hematocrit is the volume of RBCs in percentage and human adults contain 40–45% approximately. The blood entering or leaving the core hematocrit H_c is in relation with hematocrit H_0, $H_0 Q = H_c Q_1$ (18)

After adding Q_1 and Q from (13) and (16) in (18),

The following is the result (after simplification), $\overline{H_c} = \frac{H_C}{H_0} = \left[\frac{h_1^3(1+\lambda_1)\mu' + h^3 - h_1^3}{h_1^3(1+\lambda_1)\mu' + \frac{3h^2 h_1}{2} - \frac{3h_1^3}{2}}\right]$ (19)

The variable $\overline{H_c}$ represents the normalized core haematocrit. The core haematocrit (Hc) and the mean haematocrit (H_m) within the channel are connected by $H_m = \beta H_c$

After simplification, the result is

$$\overline{H_m} = \frac{H_m}{H_0} = \beta\left[\frac{h_1^3(1+\lambda_1)\mu' + h^3 - h_1^3}{h_1^3(1+\lambda_1)\mu' + \frac{3h^2 h_1}{2} - \frac{3h_1^3}{2}}\right] \quad (20)$$

$\overline{H_m}$ represents the normalized mean haematocrit.

4. NUMERICAL RESULTS AND DISCUSSION

Equations (10), (11), (17), (19), and (20) present the closed-form solutions for the fluid motion velocities in the Periphery, Core, Effective viscosity (μ_{eff}), Normalized core hematocrit ($\overline{H_C}$), Normalized mean hematocrit ($\overline{H_m}$). The numerical evaluations of the impact of various physical parameters, consisting of Jeffrey parameter λ_1, Hematocrit H_0 and the stenosis height δ_1, on the previously mentioned flow characteristics has been performed using MATHEMATICA. As depicted in Figures 2–13, the results of these computations are presented. Additionally, the noted values are selected for the pertinent study of the present paper. Channel width, $h_0 = 20\,\mu$ *to* $200\,\mu$, $\epsilon = 3.12\,\mu$ for 40%, $3.60\,\mu$ for 30 % haematocrit. Moreover, the value of β is derived from the relation $\beta = 1 - {}^{\epsilon}/_{h_0}$.(Haynes [5], Chaturani and Upadhya [7], Srivastava [9]).

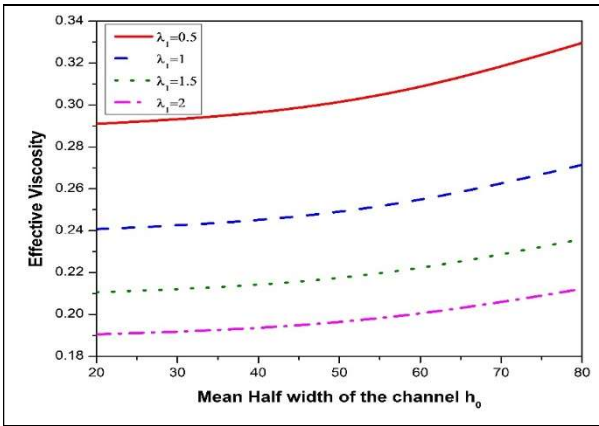

Fig 2: Impact of λ_1 upon μ_{eff} ($\delta_1 = 0.2, H_0 = 40\%$)

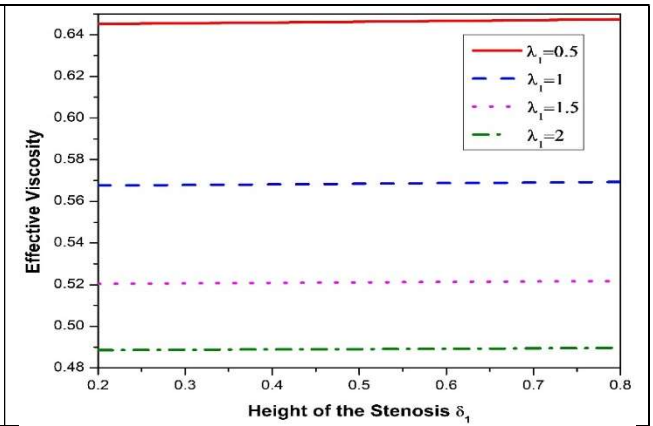

Fig 3: Impact of λ_1 upon μ_{eff} ($H_0 = 40\%$, $h_0 = 20$)

As the channel width increases, the flow becomes more laminar, and the interactions between blood cells and the channel walls diminish in significance. As a result, the effective viscosity of blood flow rises (Caro and Pedley [22]). This outcome is evident from Figures 2 and 4. Additionally, it shows the Fahraeus-Lindqvist effect. An abrupt decrease in the fluid's cross-sectional area occurs in a stenotic zone, which causes the fluid's velocity to rise. The fluid's shear rate, or velocity gradient,

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rises as it moves faster through the constriction. In Jeffrey Fluid, a more prominent shear-thinning behavior may result from this increased shear rate, which would raise the effective viscosity. This outcome is seen in figures 3 and 5. Figures 6 to 9 illustrate that the mean haematocrit decreases with the Jeffrey parameter. The interaction between RBCs and the Jeffrey fluid can disrupt cell dispersion during blood flow. The elastic characteristics of the Jeffrey fluid may affect the deformation and orientation of blood cells, potentially resulting in alterations to the apparent haematocrit. The reduction in average haematocrit within a Jeffrey fluid flowing through a narrow conduit is likely due to the intricate interaction between the fluid's rheological properties and the behaviour of blood molecules in that particular flow context. From the figs 7 and 9 it is perceived that the mean hematocrit declines with the height of the stenosis. In areas of stenosis, RBCs may aggregate or clump together. This aggregation can cause a decrease in the effective volume occupied by RBCs, leading to a lower hematocrit. It is noticed through figures 10-11 that the core hematocrit diminutions with Jeffrey fluid. Figs. 10 and 11 depicted that the core hematocrit declines with the channel width. In a stenotic narrow channel, where the blood experiences higher shear rates, the viscosity of blood may decrease, potentially affecting the distribution of cellular elements like RBCs. This is the reason which causes the decrement in the core hematocrit. The 'Fahraeus-Lindqvist effect' defines the lessening in effective viscosity and hematocrit in the core of rapidly flowing blood. In simpler terms, as blood flows more rapidly through a narrowed vessel, the deposition of RBC in the central flow stream (core) may appear to decrease. This result is observed in figures 10 and 11.

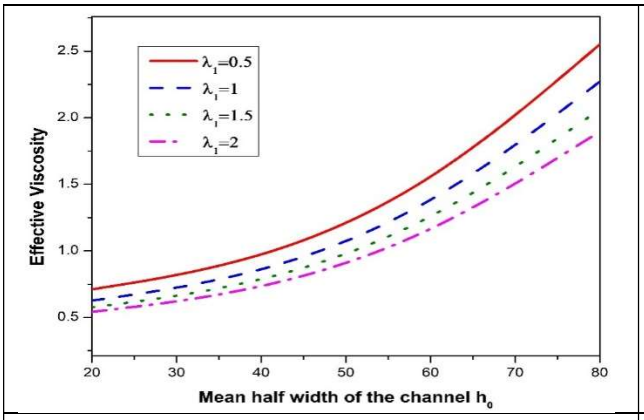

Fig 4: Impact of λ_1 upon μ_{eff} ($\delta_1 = 0.2, H_0 = 30\%$)

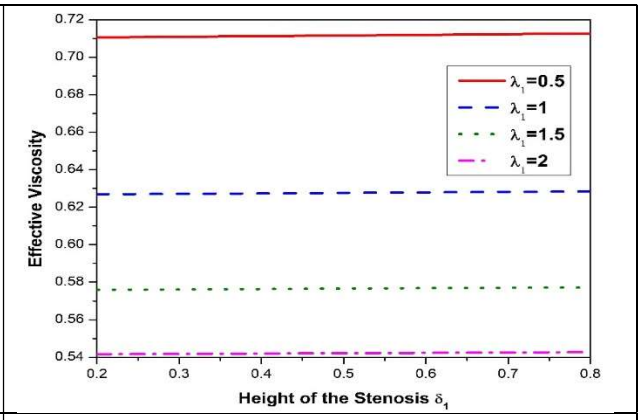

Fig 5: Impact of λ_1 upon μ_{eff} ($H_0 = 30\%$, $h_0 = 20$)

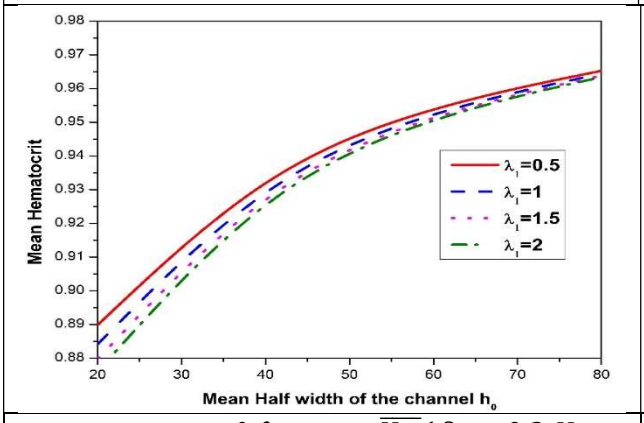

Fig 6: Impact of λ_1 upon $\overline{H_m}$($\delta_1 = 0.2, H_0 = 40\%$)

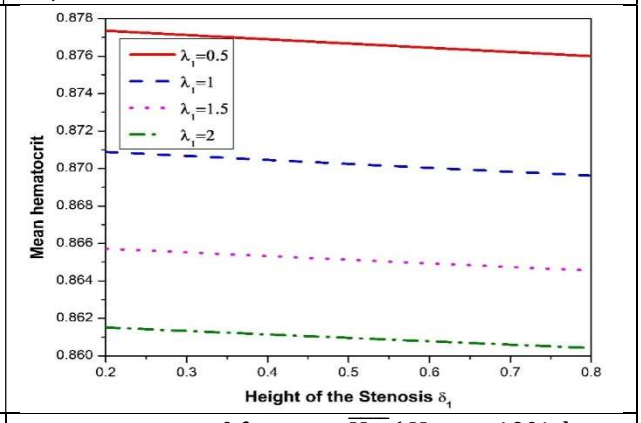

Fig 7: Impact of λ_1 upon $\overline{H_m}$($H_0 = 40\%, h_0 = 20$)

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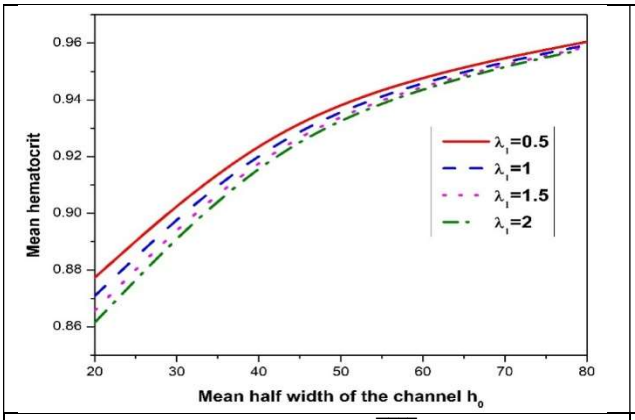

Fig 8: Impact of λ_1 upon $\overline{H_m}$ $(\delta_1 = 0.2, H_0 = 30\%)$

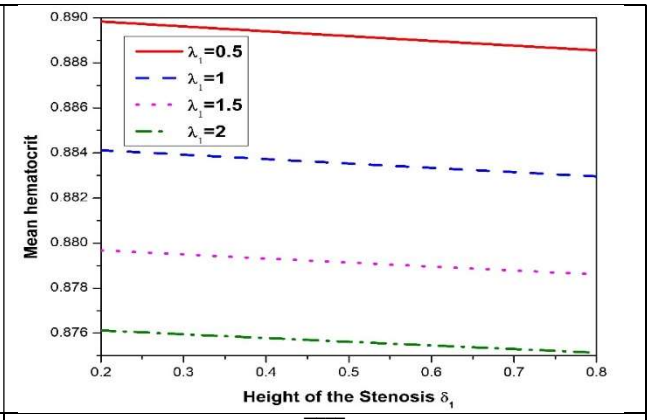

Fig 9: Impact of λ_1 upon $\overline{H_m}$ $(h_0 = 20, H_0 = 30\%)$

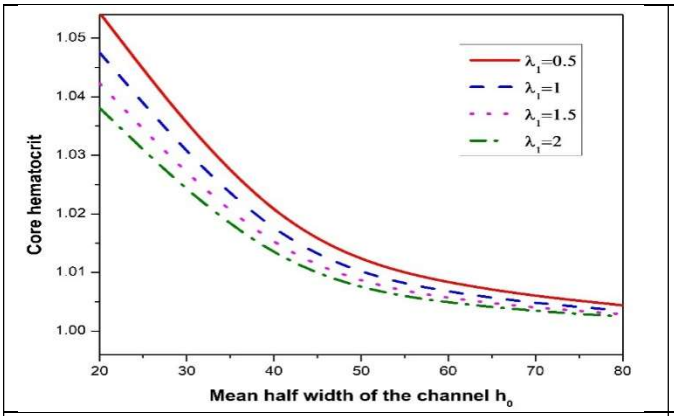

Fig 10: Impact of λ_1 upon $\overline{H_c}(\delta_1 = 0.2, H_0 = 40\%)$

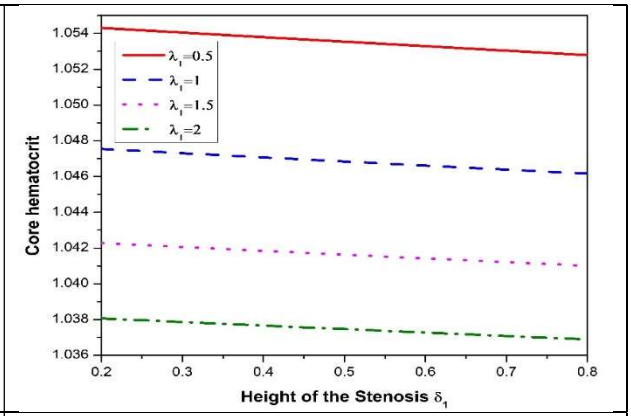

Fig 11: Impact of λ_1 upon $\overline{H_c}(h_0 = 20, H_0 = 40\%)$

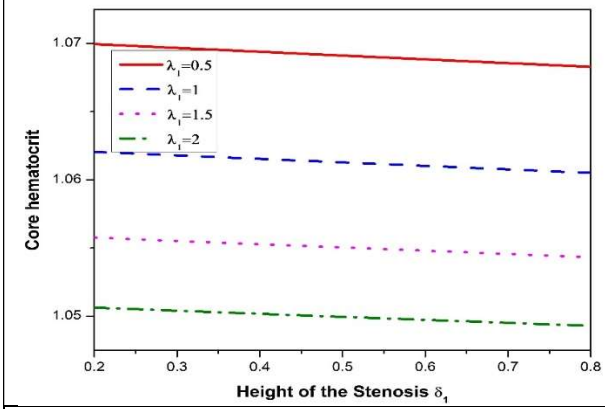

Fig 12: Impact of λ_1 upon $\overline{H_c}(h_0 = 20, H_0 = 30\%)$

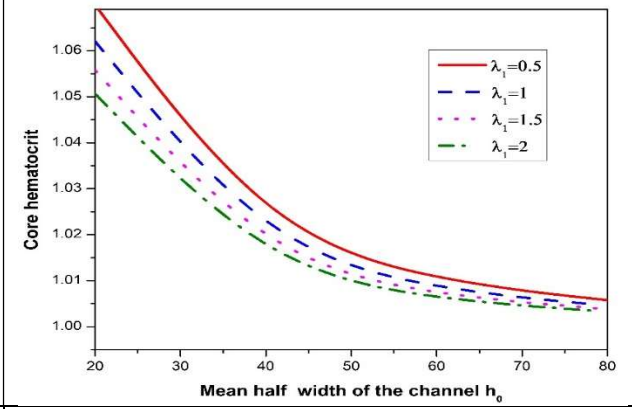

Fig 13: Impact of λ_1 upon $\overline{H_c}(\delta_1 = 0.2, H_0 = 30\%)$

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5. CONCLUSION

This paper examines two-layer Jeffrey fluid transport in narrow channels with mild stenosis. Both fluid models are classified as Jeffrey fluids in the core section, whereas a Newtonian fluid is designated for the periphery portion. Upon simplifying the equations of motion, analytical results were derived for effective viscosity, flow velocities, mean hematocrit, and core hematocrit. The Mathematica software package was employed to execute a numerical computation on each of these flow-related variables. The effective viscosity increases with channel radius and stenosis height, while it decreases with the Jeffrey parameter. A larger channel radius and increased stenosis height correlate with a reduced core hematocrit, which diminishes as the Jeffrey parameter increases. A rise in the Jeffrey parameter corresponded with a decline in mean hematocrit. An increase in channel radius correlates with a greater mean hematocrit associated with stenosis height.

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On axially symmetric vibrations in a poroelastic two layered cylindrical shell

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Abstract

Using Biot's theory on propagation of waves in poroelastic media, plane-strain vibrations of an indefinitely long two layered cylindrical shell is resolved. A two layered poroelastic cylindrical shell is made up of two concentric shells each built of a distinct poroelastic material and each of which is homogeneous and isotropic. In addition to obtaining the secular equation for the vibrations of the poroelastic two-layered cylindrical shell, some specific cases, including the poroelastic two layered bore and poroelastic bore are also discussed. The non-dimensional frequency is calculated against the function of the inner layer's thickness to its inner radius. The outcomes are analyzed after being graphically displayed for two different types of poroelastic cylindrical shells. It has been found that the dilatational and shear vibrations in a poroelastic two layered cylindrical shell are not dependent upon one another.

1. Introduction

Gazis (1958) studied vibrations in thickly walled hollow elastic cylinders in plane-strain form. Whitter & Jones (1982) discussed axisymmetric vibrations in a two layered cylinder. Cui et al. (1997) presented solutions in an inclined borehole made of poroelastic material. Abousleiman and Cui (1998) analyzed vibrations in poroelastic transversely isotropic solid in wellbore cylinders.

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Berryman and Pride (2005) discussed torsional vibrations in poro-elastic cylinder. Radial vibrations in a poroelastic hollow cylinder were studied by Tajuddin and Ahmed Shah (2006). Malla Reddy and Tajuddin (2010) examined radially symmetric vibrations in a two-layered poroelastic cylinder. Radial vibrations in a poroelastic two layered cylinder were established by Shanker et al. (2012). Sandhya Rani et al. (2022) discussed flexural vibrations in a two layered finite cylinders.

The present study investigates the vibrations in a indefinitely long poroelastic two layered cylindrical shell using Biot's (1956). The structure of Biot's model is an elastic matrix that is permeable system of connecting gaps that are all filled with liquid. Two concentric poroelastic cylindrical shells, each made of a dissimilar materials with each one is poroelastic, homogeneous and isotropic, combine to form a poroelastic two layered cylindrical shell. The bounds of the two layered cylindrical shell are free from stress, and the two shells are perfectly bonded together.

2. Problem formulation and solution

In the absence of dissipation, the equations that govern a homogeneous, isotropic poroelastic media (Biot, 1956) are:

$$N\nabla^2\vec{\mathbf{u}} + (A+N)\nabla e + Q\nabla\varepsilon = \frac{\partial^2}{\partial t^2}(\rho_{11}\vec{\mathbf{u}} + \rho_{12}\vec{\mathbf{U}})$$

$$Q\nabla e + R\nabla\varepsilon = \frac{\partial^2}{\partial t^2}(\rho_{12}\vec{\mathbf{u}} + \rho_{22}\vec{\mathbf{U}}) \qquad (1)$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$. $\vec{\mathbf{u}}(u_r, u_\theta, u_z)$ is the displacement in solid and $\vec{\mathbf{U}}(U_r, U_\theta, U_z)$ is the displacement in fluid. A, N, Q, R are poroelastic parameters, e, ε represent the solid and fluid dilatations and $\rho_{11}, \rho_{12}, \rho_{22}$ represent the coefficients of mass.

The relation between stress and strain components and liquid pressure s in terms of dilatations in a poroelastic media can be given by

$$\sigma_{mn} = 2Ne_{mn} + (Ae + Q\varepsilon)\delta_{mn}, \qquad (m,n = r,\theta,z)$$

$$s = Qe + R\varepsilon \qquad (2)$$

where e_{kl} are components of strain in a poroelastic solid.

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Consider a system of cylindrical co-ordinates r, θ, z. A two-layered, poroelastic cylinder with cylindrical outer and inner shells is considered. Assuming that r_1 and r_2 are the inner and outer radii of the inner and outer shells, respectively, and that 'b' is the radius of the interface. The cylinder's axis is used as the z-axis. Inner and outer cylindrical shells are indicated by the superscripts $l = 1, 2$.

Let us consider the vibrations in poroelastic cylinder with the solid displacements $\vec{\mathbf{u}}^{(l)}(u_r^{(l)}, u_\theta^{(l)}, 0)$ and fluid displacements $\vec{\mathbf{U}}^{(l)}(U_r^{(l)}, U_\theta^{(l)}, 0)$ given by

$$u_r^{(l)} = \frac{\partial f_1^{(l)}}{\partial r} + \frac{1}{r}\frac{\partial^2 g_1^{(l)}}{\partial \theta}, \quad u_\theta^{(l)} = \frac{1}{r}\frac{\partial f_1^{(l)}}{\partial \theta} - \frac{\partial g_1^{(l)}}{\partial r},$$
$$U_r^{(l)} = \frac{\partial f_2^{(l)}}{\partial r} + \frac{1}{r}\frac{\partial g_2^{(l)}}{\partial \theta}, \quad U_\theta^{(l)} = \frac{1}{r}\frac{\partial f_2^{(l)}}{\partial \theta} - \frac{\partial g_2^{(l)}}{\partial r}, \tag{3}$$

where $l = 1, 2$ and $f_1^{(l)}, f_2^{(l)} g_1^{(l)}, g_2^{(l)}$ are the functions of the variables r, θ and t.

From equation (1), (2) and (3) the relevant stresses can be obtained as

$$(\sigma_{rr} + s)^{(l)} = \left(A_1^{(l)}T_{11}^{(l)} + A_2^{(l)}T_{12}^{(l)} + A_3^{(l)}T_{13}^{(l)} + A_4^{(l)}T_{14}^{(l)} + B_1^{(l)}T_{15}^{(l)} + B_2^{(l)}T_{16}^{(l)}\right)\sin n\theta \exp(i\omega t),$$
$$(\sigma_{r\theta})^{(l)} = \left(A_1^{(l)}T_{21}^{(l)} + A_2^{(l)}T_{22}^{(l)} + A_3^{(l)}T_{23}^{(l)} + A_4^{(l)}T_{24}^{(l)} + B_1^{(l)}T_{25}^{(l)} + B_2^{(l)}T_{26}^{(l)}\right)\sin n\theta \exp(i\omega t),$$
$$s^{(l)} = \left(A_1^{(l)}T_{31}^{(l)} + A_2^{(l)}T_{32}^{(l)} + A_3^{(l)}T_{33}^{(l)} + A_4^{(l)}T_{34}^{(l)}\right)\cos n\theta \exp(i\omega t), \tag{4}$$

where

$$T_{11}^{(l)}(r) = \left(\xi_1^{(l)}\right)^2\left[2N^{(l)} J_n''(\xi_1^{(l)}r) + \left[(Q^{(l)} + R^{(l)})\left(\delta_1^{(l)}\right)^2 - (A^{(l)} + Q^{(l)})\right] J_n(\xi_1^{(l)}r)\right]$$
$$T_{13}^{(l)}(r) = \left(\xi_2^{(l)}\right)^2\left[2N^{(l)} J_n''(\xi_2^{(l)}r) + \left[(Q^{(l)} + R^{(l)})\left(\delta_2^{(l)}\right)^2 - (A^{(l)} + Q^{(l)})\right] J_n(\xi_1^{(l)}r)\right]$$
$$T_{15}^{(l)}(r) = \frac{n}{r}\left[\frac{-1}{r}J_n(\xi_3^{(l)}r) + \xi_3^{(l)} J_n'(\xi_3^{(l)}r)\right], T_{21}^{(l)}(r) = \frac{N^{(l)}n}{r^2}J_n(\xi_1^{(l)}r) - \frac{2N^{(l)}n}{r}\xi_1^{(l)}J_n'(\xi_1^{(l)}r)$$
$$T_{23}^{(l)}(r) = \frac{N^{(l)}n}{r^2}J_n(\xi_2^{(l)}r) - \frac{2N^{(l)}n}{r}\xi_2^{(l)}J_n'(\xi_2^{(l)}r)$$
$$T_{25}^{(l)}(r) = \frac{-1}{r^2}\left[\left(\xi_3^{(l)}\right)^2 r^2 J_n''(\xi_3^{(l)}r) - \xi_3^{(l)}rJ_n'(\xi_3^{(l)}r) + n^2 J_n(\xi_3^{(l)}r)\right]$$
$$T_{31}^{(l)}(r) = \left(\xi_1^{(l)}\right)^2\left(R^{(l)}\left(\delta_1^{(l)}\right)^2 - Q^{(l)}\right)J_n(\xi_1^{(l)}r), T_{33}^{(l)}(r) = \left(\xi_2^{(l)}\right)^2\left(R^{(l)}\left(\delta_2^{(l)}\right)^2 - Q^{(l)}\right)J_n(\xi_2^{(l)}r)$$

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$$T_{41}^{(l)}(r)=\xi_1^{(l)}J_n'(\xi_1^{(l)}r),T_{43}^{(l)}(r)=\xi_2^{(l)}J_n'(\xi_2^{(l)}r),T_{45}^{(l)}(r)=\frac{n}{r}J_n(\xi_3^{(l)}r)$$

$$T_{51}^{(l)}(r)=\frac{-n}{r}J_n(\xi_1^{(l)}r),T_{53}^{(l)}(r)=\frac{-n}{r}J_n(\xi_2^{(l)}r),T_{55}^{(l)}(r)=-\xi_3^{(l)}J_n'(\xi_3^{(l)}r)$$

$$\left.\begin{aligned}T_{i2}^{(l)}(r)&=T_{i1}^{(l)}(r)\\T_{i4}^{(l)}(r)&=T_{i3}^{(l)}(r)\\T_{i6}^{(l)}(r)&=T_{i5}^{(l)}(r)\end{aligned}\right\}$$

with $J_n=Y_n, J_n'=Y_n'$ and $J_n''=Y_n''$ for $i=1,2,3,4,5.$

$A_1^{(l)},A_2^{(l)},A_3^{(l)},A_4^{(l)},B_1^{(l)}$ and $B_2^{(l)}$ are constants, ω is frequency, $\left(\xi_k^{(l)}\right)^2=\dfrac{\omega^2}{\left(V_k^{(l)}\right)^2}$, $k=1,2$ and 3,

$$\left(\delta_k^{(l)}\right)^2=\frac{\left(P^{(l)}R^{(l)}-\left(Q^{(l)}\right)^2\right)-\left(V_1^{(l)}\right)^2\left(R^{(l)}\rho_{11}^{(l)}-Q^{(l)}\rho_{12}^{(l)}\right)}{\left(V_k^{(l)}\right)^2\left(R^{(l)}\rho_{12}^{(l)}-Q^{(l)}\rho_{22}^{(l)}\right)} \text{ for } k=1,2 \text{ and } 3 \qquad (5)$$

3. Secular equation:

The boundary conditions for the vibrations in a two layered cylindrical shell are

$$\begin{aligned}&at\ r=r_1,\ \ (\sigma_{rr}+s)^{(1)}=0,\ (\sigma_{r\theta})^{(1)}=0\\&at\ r=r_2,\ (\sigma_{rr}+s)^{(2)}=0,\ (\sigma_{r\theta})^{(2)}=0\\&at\ r=b,\ \ \left(\sigma_{rr}+s\right)^{(1)}=\left(\sigma_{rr}+s\right)^{(2)},\ \left(\sigma_{r\theta}\right)^{(1)}=\left(\sigma_{r\theta}\right)^{(2)}\\&\qquad u_r^{(1)}=u_r^{(2)},\ u_\theta^{(1)}=u_\theta^{(2)};\\&at\ r=r_1,r_2 \text{ and } a\ ;\quad s^{(1)}=s^{(2)}=0,\end{aligned} \qquad (6)$$

Eqs. (6) results in a homogeneous system with twelve equations.

The secular equation is derived on eliminating the constants as

$$\left|C_{mn}\right|=0 \text{ for } m,n=1,2,....,12 \qquad (7)$$

with

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$$n=1,2,\ldots 6,\quad C_{1n}=T^{(1)}_{1n}(r_1), C_{4n}=T^{(1)}_{1n}(b),\ C_{2n}=T^{(1)}_{2n}(r_1), C_{5n}=T^{(1)}_{2n}(b), C_{8n}=T^{(1)}_{4n}(b), C_{9n}=M^{(1)}_{5n}(b),$$
$$C_{10,n}=0,\ C_{11,n}=0,\ C_{12,n}=0,$$
$$n=7,8,\ldots 12; C_{1n}=0, C_{2n}=0,\ C_{9n}=T^{(2)}_{5,n-6}(b),\ C_{4n}=T^{(2)}_{1,n-6}(b), C_{8n}=T^{(2)}_{4,n-6}(b),\ C_{10,n}=T^{(2)}_{1,n-6}(r_2),$$
$$C_{5n}=T^{(2)}_{2,n-6}(b),\ C_{11,n}=T^{(2)}_{2,n-6}(r_2), C_{12,n}=T^{(2)}_{3,n-6}(r_2),$$
$$n=1,2,3,4;\ C_{3n}=T^{(1)}_{3n}(r_1),\ C_{6n}=T^{(1)}_{3n}(b),$$
$$n=5,6,\ldots 12; C_{3n}=0,\ C_{6n}=0,$$
$$n=1,2,3,4,5,6,11,12; C_{7n}=0,$$
$$n=7,8,9,10; C_{7n}=T^{(2)}_{3,n-6}(b), \tag{8}$$

We obtain axial symmetric vibrations when $n=0$ thus the secular equation of two layered poroelastic cylindrical shell (7) reduces to

$$\begin{vmatrix} T^{(1)}_{25}(r_1) & T^{(1)}_{26}(r_1) & 0 & 0 \\ T^{(1)}_{25}(b) & T^{(1)}_{26}(b) & T^{(2)}_{25}(b) & T^{(2)}_{26}(b) \\ T^{(1)}_{55}(b) & T^{(1)}_{56}(b) & T^{(2)}_{55}(b) & T^{(2)}_{56}(b) \\ 0 & 0 & T^{(2)}_{25}(r_2) & T^{(2)}_{26}(r_2) \end{vmatrix}$$

$$\begin{vmatrix} T^{(1)}_{11}(r_1) & T^{(1)}_{12}(r_1) & T^{(1)}_{13}(r_1) & T^{(1)}_{14}(r_1) & 0 & 0 & 0 & 0 \\ T^{(1)}_{31}(r_1) & T^{(1)}_{32}(r_1) & T^{(1)}_{33}(r_1) & T^{(1)}_{34}(r_1) & 0 & 0 & 0 & 0 \\ T^{(1)}_{11}(b) & T^{(1)}_{12}(b) & T^{(1)}_{13}(b) & T^{(1)}_{14}(b) & T^{(2)}_{11}(b) & T^{(2)}_{12}(b) & T^{(2)}_{13}(b) & T^{(2)}_{14}(b) \\ T^{(1)}_{31}(b) & T^{(1)}_{32}(b) & T^{(1)}_{33}(b) & T^{(1)}_{34}(b) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T^{(2)}_{31}(b) & T^{(2)}_{32}(b) & T^{(2)}_{33}(b) & T^{(2)}_{34}(b) \\ T^{(1)}_{41}(b) & T^{(1)}_{42}(b) & T^{(1)}_{43}(b) & T^{(1)}_{44}(b) & T^{(2)}_{41}(b) & T^{(2)}_{42}(b) & T^{(2)}_{43}(b) & T^{(2)}_{44}(b) \\ 0 & 0 & 0 & 0 & T^{(2)}_{11}(r_2) & T^{(2)}_{12}(r_2) & T^{(2)}_{13}(r_2) & T^{(2)}_{14}(r_2) \\ 0 & 0 & 0 & 0 & T^{(2)}_{31}(r_2) & T^{(2)}_{32}(r_2) & T^{(2)}_{33}(r_2) & T^{(2)}_{34}(r_2) \end{vmatrix} = 0 \tag{9}$$

where $T^{(l)}_{mn}(r)$ are given in equation. (5) and can be evaluated for $n=0$.

In the equation (9), the first and second equations are the secular equations of axially symmetric shear and dilatational vibrations respectively in a poroelastic two layered cylindrical shell. Also, it is clear from the equation that these two vibrations are uncoupled.

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Poroelastic composite bore

When the outer radius r_2 of outer layer approaches to infinity, the secular equation (7) transforms into

$$\begin{vmatrix}
T_{11}^{(1)}(r_1) & T_{12}^{(1)}(r_1) & T_{13}^{(1)}(r_1) & T_{14}^{(1)}(r_1) & T_{15}^{(1)}(r_1) & T_{16}^{(1)}(r_1) & 0 & 0 & 0 \\
T_{21}^{(1)}(r_1) & T_{22}^{(1)}(r_1) & T_{23}^{(1)}(r_1) & T_{24}^{(1)}(r_1) & T_{25}^{(1)}(r_1) & T_{26}^{(1)}(r_1) & 0 & 0 & 0 \\
T_{31}^{(1)}(r_1) & T_{32}^{(1)}(r_1) & T_{33}^{(1)}(r_1) & T_{34}^{(1)}(r_1) & 0 & 0 & 0 & 0 & 0 \\
T_{11}^{(1)}(b) & T_{12}^{(1)}(b) & T_{13}^{(1)}(b) & T_{14}^{(1)}(b) & T_{15}^{(1)}(b) & T_{16}^{(1)}(b) & T_{12}^{(2)}(b) & T_{14}^{(2)}(b) & T_{16}^{(2)}(b) \\
T_{21}^{(1)}(b) & T_{22}^{(1)}(b) & T_{23}^{(1)}(b) & T_{24}^{(1)}(b) & T_{25}^{(1)}(b) & T_{26}^{(1)}(b) & T_{22}^{(2)}(b) & T_{24}^{(2)}(b) & T_{26}^{(2)}(b) \\
T_{31}^{(1)}(b) & T_{32}^{(1)}(b) & T_{33}^{(1)}(b) & T_{34}^{(1)}(b) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & T_{32}^{(2)}(b) & T_{34}^{(2)}(b) & T_{36}^{(2)}(b) \\
T_{41}^{(1)}(b) & T_{42}^{(1)}(b) & T_{43}^{(1)}(b) & T_{44}^{(1)}(b) & T_{45}^{(1)}(a) & T_{46}^{(1)}(b) & T_{42}^{(2)}(b) & T_{44}^{(2)}(b) & T_{46}^{(2)}(b) \\
T_{51}^{(1)}(b) & T_{52}^{(1)}(b) & T_{53}^{(1)}(b) & T_{54}^{(1)}(b) & T_{55}^{(1)}(a) & T_{56}^{(1)}(b) & T_{52}^{(2)}(b) & T_{54}^{(2)}(b) & T_{56}^{(2)}(b)
\end{vmatrix} = 0 \qquad (10)$$

Eq. (10) is the secular equation of poroelastic composite bore in plane-strain form.

For $n = 0$, the secular equation (10) further reduces to

$$\begin{vmatrix}
T_{25}^{(1)}(r_1) & T_{26}^{(1)}(r_1) & 0 \\
T_{25}^{(1)}(b) & T_{26}^{(1)}(b) & T_{26}^{(2)}(b) \\
T_{55}^{(1)}(b) & T_{56}^{(1)}(b) & T_{56}^{(2)}(b)
\end{vmatrix}$$

$$\begin{vmatrix}
T_{11}^{(1)}(r_1) & T_{12}^{(1)}(r_1) & T_{13}^{(1)}(r_1) & T_{14}^{(1)}(r_1) & 0 & 0 \\
T_{31}^{(1)}(r_1) & T_{32}^{(1)}(r_1) & T_{33}^{(1)}(r_1) & T_{34}^{(1)}(r_1) & 0 & 0 \\
T_{11}^{(1)}(b) & T_{12}^{(1)}(b) & T_{13}^{(1)}(b) & T_{14}^{(1)}(b) & T_{12}^{(2)}(b) & T_{14}^{(2)}(b) \\
T_{31}^{(1)}(b) & T_{32}^{(1)}(b) & T_{33}^{(1)}(b) & T_{34}^{(1)}(b) & 0 & 0 \\
0 & 0 & 0 & 0 & T_{32}^{(2)}(b) & T_{34}^{(2)}(b) \\
T_{51}^{(1)}(b) & T_{52}^{(1)}(b) & T_{53}^{(1)}(b) & T_{54}^{(1)}(b) & T_{52}^{(2)}(b) & T_{54}^{(2)}(b)
\end{vmatrix} = 0 \qquad (11)$$

The first and second equations represent the secular equations of shear and dilatational vibrations in a composite bore. Also, it is clear that these two vibrations are independent in nature.

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Poroelastic bore

Considering the poroelastic parameters of inner and outer cylinders are same then the composite bore transforms into a poroelastic bore with radius r_1 and the secular equation (10) transforms into

$$\begin{vmatrix} T_{12}^{(1)}(r_1) & T_{14}^{(1)}(r_1) & T_{16}^{(1)}(r_1) \\ T_{22}^{(1)}(r_1) & T_{24}^{(1)}(r_1) & T_{26}^{(1)}(r_1) \\ T_{32}^{(1)}(r_1) & T_{34}^{(1)}(r_1) & 0 \end{vmatrix} = 0, \qquad (12)$$

4. Numerical work, Results and Discussion

In order to evaluate non-dimensional frequency, two different types of combinations of cylinders - composite cylinder I and composite cylinder II made of kerosene saturated sandstone (Yew, 1976) and water saturated sandstone are considered. The physical parameters of Water saturated sandstone (Fatt, 1959) are given by

$A = 0.306 \text{ x } 10^{11}\, dyne/cm^2, Q = 0.013 \text{ x } 10^{11}\, dyne/cm^2, R = 0.0637 \text{ x } 10^{11}\, dyne/cm^2,$
$N = 0.922 \text{ x } 10^{11}\, dyne/cm^2, \rho_{11} = 1.9032\ \ gm/cm^3, \rho_{12} = 0\ gm/cm^3, \rho_{22} = 0.268\ gm/cm^3.$

whereas the physical parameters of kerosene saturated sandstone are

$A = 0.4436 \text{ x } 10^{11}\, dyne/cm^2, Q = 0.07635 \text{ x } 10^{11}\, dyne/cm^2, R = 0.0326 \text{ x } 10^{11}\, dyne/cm^2,$
$N = 0.2765 \text{ x } 10^{11}\, dyne/cm^2, \rho_{11} = 1.926137\, gm/cm^3 \rho_{12} = -0.002137\ gm/cm^3,$
$\rho_{22} = 0.215337 gm/cm^3.$

Considering the above values of poroelastic constants, the non-dimensionalized secular equations produces a relation between frequency Ω and ratio h/r_1 of thickness to inner radius of the inner layer. Three values for r_2/a, viz., 1.05, 2 & 5 are considered for poroelastic outer shell which represent respectively, thin outer shell, moderately thick outer shell and thick outer shell.

Figures 1 and 2 presents frequency of shear vibrations in a two layered poroelastic cylindrical shells I and II respectively, each consisting of thin, moderately thick and thick casings. It is clear that the frequency is high for smaller values of thickness ratio r_2/a of outer shell for values $h/r_1 > 0.4$, also the frequency is steady. Whereas for $h/r_1 < 0.4$ the frequency is less for lower

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values of thickness ratio r_2 / a of outer layer and frequency is not steady.

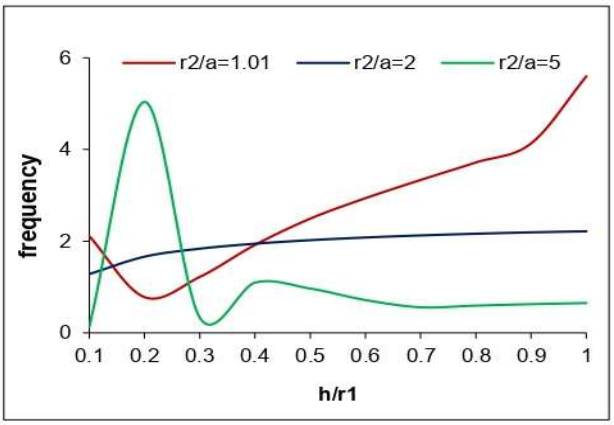

Fig. 1 Axially symmetric Shear vibrations - poroelastic two layered cylindrical shell-I

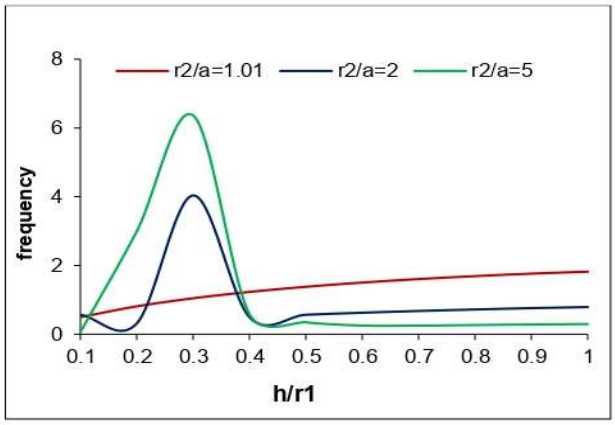

Fig. 2 Axially symmetric Shear vibrations - poroelastic two layered cylindrical shell-II

The frequency for dilatational vibrations is incorporated in Figures 3 and 4 for cylindrical shells I and II, respectively. It is interesting to observe that behavior is different for dilatational vibrations in both cylindrical shells I and II. The frequency is steady in cylindrical shell II when compared with composite cylinder I. The frequency is steady only for thin outer shell and frequency for thick outer shell lies between the frequencies of the other two in the case of cylindrical shell I. Whereas in the case of cylindrical shell II the frequency is more for thick outer shell when comparing with other two when $h / r_1 > 0.4$ also it is observed that the frequency is steady in the three cases when $h / r_1 > 0.4$.

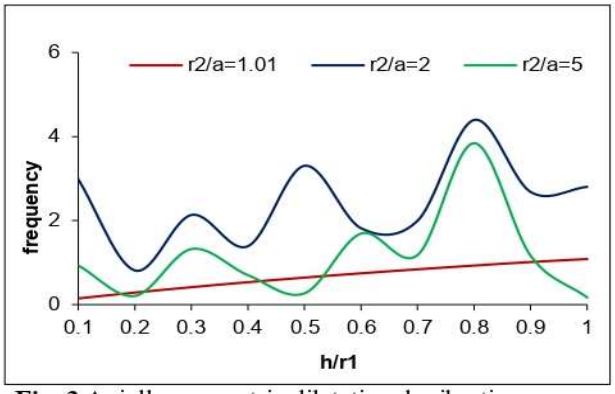

Fig. 3 Axially symmetric dilatational vibrations – poroelastic two layered cylindrical shell -I

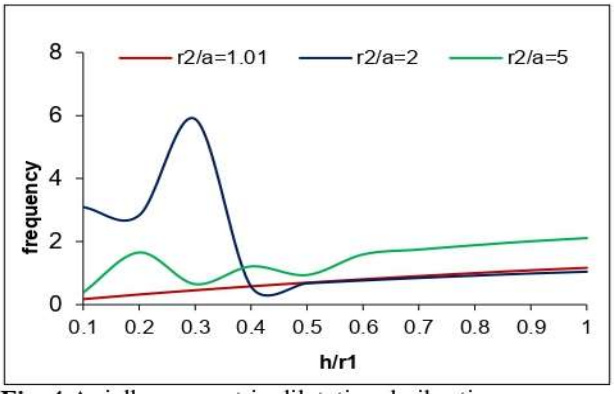

Fig. 4 Axially symmetric dilatational vibrations - poroelastic two layered cylindrical shell -II

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Shear vibrations of composite bore I are depicted in Figure 5. The frequency for composite bore II is stable and less when compared with the frequency of bore I for $h/r_1>0.4$. Dilatational vibrations as presented in Figure 6 the frequency of composite bore II is steady and is less than that of composite bore I for $h/r_1 \geq 0.5$.

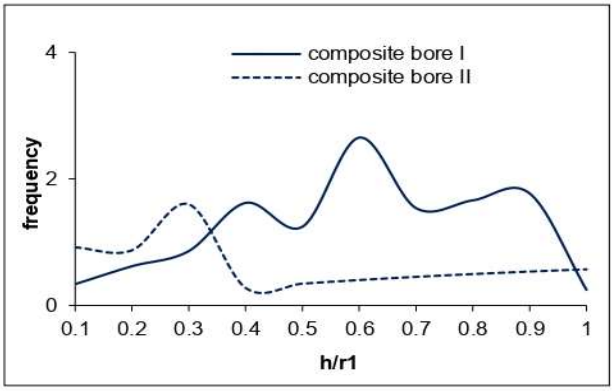

Fig. 5 Axially symmetric shear vibrations - composite bore

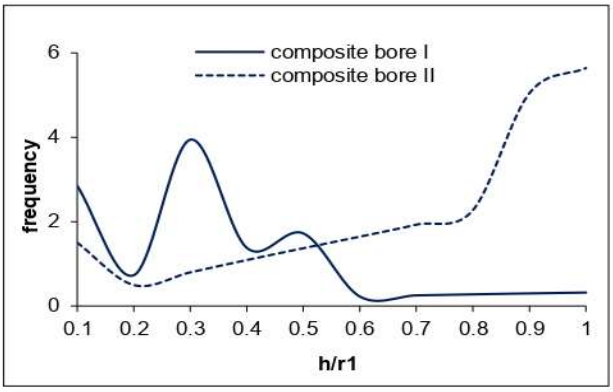

Fig. 6 Axially symmetric dilatational vibrations - composite bore

5. Conclusion:

The investigation of axial symmetric vibrations in two-layered cylindrical shells and composite bore has produced the findings:

(i) The poroelastic composite hollow cylinder's axially symmetric vibrations produced uncoupled dilatational and shear vibrations. Similar findings have been made regarding poroelastic composite bores.

(ii) The frequency of shear vibrations is steady for both cylindrical shells I and II.

(iii) The frequency of shear vibrations is more for both cylindrical shells I and II for higher values of the ratio h / r_1.

(iv) Only for poroelastic cylindrical shell II the frequency of dilatational vibrations is steady.

(v) In poroelastic composite bore II, frequencies of shear and dilatational vibrations are steady.

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HEMODYNAMIC EFFECTS OF STENOSIS ON JEFFREY FLUID TRANSPORT THROUGH NARROW CHANNEL

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Abstract

Hemodynamic impacts of stenosis on Jeffrey fluid transport through narrow channel is considered in this paper under the boundary conditions. The proposed model comprises of a core containing Jeffrey fluid and a periphery portion consisting of Newtonian fluid. Movement variables such as velocity, flow flux and core hematocrit, effective viscosity, average hematocrit has been given analytical expressions, and the influence of several pertinent variables on the above-mentioned variables have been studied. The effective viscosity is noticed to decline with Jeffrey parameter and rise with stenosis height, the channel width. The observed effective viscosity of the model appears to be within the acceptable range of values. Also, it is seen that the 'Fahraeus-Lindqvist effect' is responsible for the rise in effective viscosity as the channel radius grows. Further, mean hematocrit decreased with Jeffrey parameter and stenotic height, increased with channel width. Moreover, core hematocrit decreased with Jeffrey parameter, height of the stenosis and channel width.

1. INTRODUCTION

The flow of blood through the narrow vessels, including capillaries, venules, and arterioles, is referred to as microcirculation. The human circulatory system comprises a network of blood vessels that vary in diameter from 20 to 500 micrometers. Furthermore, anomalous effects arise when blood passes vessels with reduced diameters. The 'Fahraeus-Lindqvist effect' [1] exemplifies a phenomenon in which the apparent viscosity of blood decreases as channel width increases (Martini et al. [2]). A number of researchers (Azel-vandre and Oiknine [3]) have examined blood flow in small channels, presuming it to display Newtonian fluid characteristics and have validated the Fahraeus-Lindqvist effect due to its increasing relevance. Additionally, Sharan and Popel [4] employed a dual-component model to analyze blood flow in constricted channels. Haynes [5] investigated a double-layered fluid model comprising two Newtonian fluids with differing viscosities. Blood, regarded as a suspension of hemoglobin in plasma, exhibits non-Newtonian fluid behavior under low shear rates (Haynes and Burton [6]). Chaturani and Upadhya [7] investigated double-layered blood flow models utilizing a Newtonian fluid in the periphery. Shukla et al. [8] examined the influence of the peripheral layer on the peristaltic transport of a biofluid. Srivastava and Saxena [9] have proposed the use of two-layered blood flow models, wherein the central region comprises plasma and Casson fluid, while the peripheral region is characterized by a Newtonian fluid. Aerosty and Gross [10] examined pulsatile flow in small vessels under the assumption that blood behaves as a Casson fluid. Stenosis refers to the constriction of an artery resulting from arteriosclerotic plaques. This occurs due to an excess of fatty deposits and abnormal angiogenesis within the arterial lumen. Stenosis obstructs blood flow, resulting in irregular hemodynamics. This is a primary factor contributing to blood clots, strokes, and heart failure. Stenosis can occur in various forms, such as when multiple shapes intersect or when the shapes lack regularity. Numerous experimental and theoretical investigations (Sankar and Hemalatha [11]) have sought to elucidate the impact of stenoses in the arterial lumen of a plasma channel on blood flow dynamics. Umadevi et al. [12] demonstrate that copper nanoparticles are efficacious in reducing the hemodynamics associated with stenoses. This indicates potential biomedical applications. Dhange et al. [13] Casson concentrated on the continuous, incompressible flow of fluid in a non-channel with multiple stenoses. Ponalagusamy [14] examined the influence of non-zero couple pressure at the interface, taper angle, stenosis height, axial distance, and various sloshes on flow parameters and flow line patterns.The Jeffrey fluid illustrates a non-Newtonian liquid characterized by viscoelastic and shear-thinning properties. The Jeffrey fluid model has been utilized to examine various fluid stream phenomena, including flow in porous media, non-Newtonian flow in conduits, and the dynamics of polymer solutions. The model is also

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utilized in biomechanics to examine the dynamics of blood flow and other biological fluids within blood vessels. The peristaltic motion of Jeffrey fluid in a circular vessel is examined by Hayat et al. [15]. Vajravelu et al. [16] investigate the peristaltic motion of Jeffrey fluid with thermal effects on a vertical porous surface under the assumptions of 'long wavelength approximation and low Reynolds number.' Investigations are underway regarding the diffusion of solute substances in the flow of Jeffrey fluid through a porous medium in a peristaltic channel, as discussed by Ravi Kiran and Radhakrishnamacharya [17]. Santosh and Rajeshwar Rao [18] examined a two-fluid problem regarding the motion of Jeffrey fluid influenced by slip through a narrow channel. Santosh et al. [19] investigated the influence of a porous layer on a dual-layered mathematical model for the flow of Jeffrey fluid in channels of varying diameters.

2. MATHEMATICAL MODEL OF THE PROBLEM

This paper presents a two-layered model for the analysis of the two-dimensional movement of a Jeffrey blood in a narrow channel through stenosis. The variable h_0 is used to denote the channel width in the non-stenotic zone. It is composed that the movement area contains of a core with a height h_1 and a periphery with a thickness $\boldsymbol{\epsilon = h_0 - \beta h_0}$. Further, Fig. 1 depicts that the periphery composed that the Newtonian fluid and central core contains Jeffrey fluid. The cartesian coordinates (x, y) are utilized so that the x-axis is equivalent to the centerline of the vessel and the y-axis is vertical to the x-axis. For an incompressible Jeffrey fluid, the constitutive equations given by Hayat et al. [21] are $\overline{\boldsymbol{T}} = -P\overline{\boldsymbol{I}} + \overline{\boldsymbol{S}}$ and $\overline{\boldsymbol{S}} = \frac{\mu_c}{1+\lambda_1}(\dot{\gamma} + \lambda_2\ddot{\gamma})$ here, P indicates pressure, The parameter denoted as λ_1 represents the ratio between relaxation time and retardation time, $\overline{\boldsymbol{S}}$ represents identity tensor, $\boldsymbol{\lambda_2}$ symbolises the retardation time, the symbol $\dot{\boldsymbol{\gamma}}$ signifies the shear rate, and dots are placed above the quantities to denote their changes over time. In this case, the formulae that describe the movement of an incompressible Jeffrey fluid are:

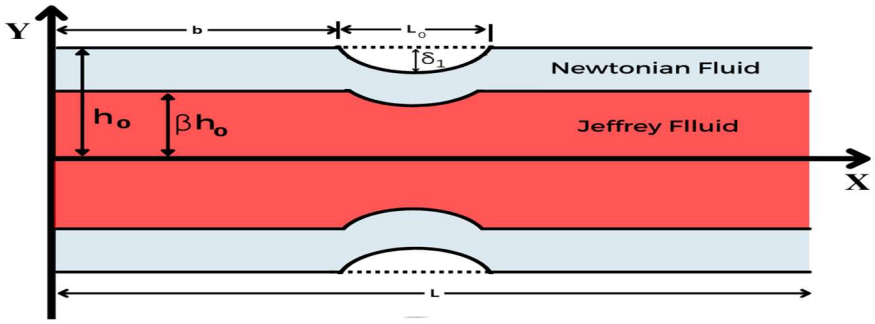

Fig. 1 Geometrical Structure of the Stenosed Channel

Given by the geometry of the wall

$$h = \begin{cases} h_0 - \frac{\delta_1}{2}\left(1 + \cos\frac{2\pi}{L_0}\left(x - b - \frac{L_0}{2}\right)\right) & , b \le x \le b + L_0 \\ h_0 , & \text{otherwise} \end{cases} \quad (1)$$

$$h_1 = \begin{cases} \beta h_0 - \frac{\delta}{2}\left(1 + \cos\frac{2\pi}{L_0}\left(x - b - \frac{L_0}{2}\right)\right) & , b \le x \le b + L_0 \\ \beta h_0 , & \text{otherwise} \end{cases} \quad (2)$$

where L is the channel's length, L_0 is its length, δ_1 is the stenosis's highest point, δ is the interface's highest bulging point because of the stenosis, the parameter β represents the ratio among the average half-width of the central core and the channel width in the unobstructed region, additionally h_0 denotes the width of the channel in the clear area.

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3. ANALYSIS

In this paper, the two-dimensional movement characteristics of concentrated Jeffrey blood are modelled using the governing equations described by Santhosh et al. [20].

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (3)$$

$$\rho\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]u=-\frac{\partial p}{\partial x}+\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y} \quad (4)$$

$$\rho\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]v=-\frac{\partial p}{\partial y}+\frac{\partial S_{yy}}{\partial y}+\frac{\partial S_{yx}}{\partial x} \quad (5)$$

In this context, v and u signify the velocity components along the x, y co-ordinate axes, respectively, p denotes pressure, ρ indicates density, Syy, Syx, Sxy, and Sxx represent various stress quantities, and μ_1 refers to the viscosity in the core section. The velocity in the middle portion is denoted as u_1 (y), whereas that in the peripheral area will be u_2 (y).

Considering the movement to be uniform and considering the stenosis as slight, the equations representing the fluid flow (Jeffrey fluid) in the core area ($0 \le y \le h_1$) lessen to

$$\frac{\mu_1}{(1+\lambda_1)}\frac{\partial^2 u_1}{\partial y^2}=\frac{\partial p}{\partial x}, \quad 0 \le y \le h_1 \quad (6)$$

In the periphery area ($h_1 \le y \le h$) the equations representing the fluid flow (Newtonian fluid) for the current problem lessen to

$$\mu_2\frac{\partial^2 u_2}{\partial y^2}=\frac{\partial p}{\partial x}, \quad h_1 \le y \le h \quad (7)$$

The boundary conditions for this current paper are

$$u_1=u_2, \tau_1=\tau_2 \text{ at } y=\pm h_1(x) \quad (8)$$

The velocity adheres to without slip boundary conditions as dictated by the governing equations.

$$u_2=0 \text{ at } y=\pm h(x) \quad (9)$$

calculating equations (6) and (7) subject to the conditions (8) and (9), we get

$$u_1=\frac{1}{2}\frac{1}{\mu_2}\frac{\partial p}{\partial x}[\mu'(1+\lambda_1)(y^2-h_1^2)-(h^2-h_1^2)] \quad (10)$$

and

$$u_2=\frac{1}{2\mu_2}\frac{\partial p}{\partial x}[y^2-h^2] \quad (11)$$

The core flow flux and that of the peripheral area are respectively represented by

$$Q_1=2\int_0^{h_1}u_1(y)dy \quad \text{and} \quad Q_2=2\int_{h_1}^{h}u_2(y)\,dy \quad (12)$$

Adding for $u_1(y)$ and $u_2(y)$ from (10) & (11) in (12)

$$Q_1=-\frac{1}{\mu_2}\frac{\partial p}{\partial x}\left[\frac{2h_1^3}{3}(1+\lambda_1)\mu'+h^2h_1-h_1^3\right] \quad (13)$$

$$Q_2=-\frac{1}{\mu_2}\frac{\partial p}{\partial x}\left[\frac{h_1^3}{3}-h^2h_1+\frac{2h^3}{3}\right] \quad (14)$$

Accordingly, the fluid flux in the narrowed stenotic passage is, $Q=Q_1+Q_2$ (15)

By substituting equations (13) and (14) into equation (15), we obtain

$$Q=-\frac{2}{3}\frac{1}{\mu_2}\frac{\partial p}{\partial x}[h_1^3(1+\lambda_1)\mu'+h^3-h_1^3] \quad (16)$$

The effective viscosity is derived through comparison of (16) with the flux relation for movement between two parallel plates.

$$\mu_{eff}=\left(\frac{\mu' h_0^3}{3[h_1^3(1+\lambda_1)\mu'+h^3-h_1^3]}\right), \text{ where, } \mu'=\frac{\mu_2}{\mu_1} \quad (17)$$

Mean Haematocrit

Values for Cell-Free Layers of Walls

Hematocrit is the volume of RBCs in percentage and human adults contain 40–45% approximately. The blood entering or leaving the core hematocrit H_c is in relation with hematocrit H_0, $H_0Q=H_cQ_1$ (18)

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After adding Q_1 and Q from (13) and (16) in (18),

The following is the result (after simplification), $\overline{H_c} = \frac{H_C}{H_0} = \left[\frac{h_1^3(1+\lambda_1)\mu' + h^3 - h_1^3}{h_1^3(1+\lambda_1)\mu' + \frac{3h^2h_1}{2} - \frac{3h_1^3}{2}}\right]$ (19)

The variable $\overline{H_c}$ represents the normalized core haematocrit. The core haematocrit (Hc) and the mean haematocrit (H_m) within the channel are connected by $H_m = \beta H_c$

After simplification, the result is

$$\overline{H_m} = \frac{H_m}{H_0} = \beta\left[\frac{h_1^3(1+\lambda_1)\mu' + h^3 - h_1^3}{h_1^3(1+\lambda_1)\mu' + \frac{3h^2h_1}{2} - \frac{3h_1^3}{2}}\right] \quad (20)$$

$\overline{H_m}$ represents the normalized mean haematocrit.

4. NUMERICAL RESULTS AND DISCUSSION

Equations (10), (11), (17), (19), and (20) present the closed-form solutions for the fluid motion velocities in the Periphery, Core, Effective viscosity (μ_{eff}), Normalized core hematocrit ($\overline{H_C}$), Normalized mean hematocrit ($\overline{H_m}$). The numerical evaluations of the impact of various physical parameters, consisting of Jeffrey parameter λ_1, Hematocrit H_0 and the stenosis height δ_1, on the previously mentioned flow characteristics has been performed using MATHEMATICA. As depicted in Figures 2–13, the results of these computations are presented. Additionally, the noted values are selected for the pertinent study of the present paper. Channel width, $h_0 = 20\,\mu\ to\ 200\,\mu$, $\epsilon = 3.12\,\mu$ for 40%, $3.60\,\mu$ for 30 % haematocrit. Moreover, the value of β is derived from the relation $\beta = 1 - {}^{\epsilon}/_{h_0}$.(Haynes [5], Chaturani and Upadhya [7], Srivastava [9]).

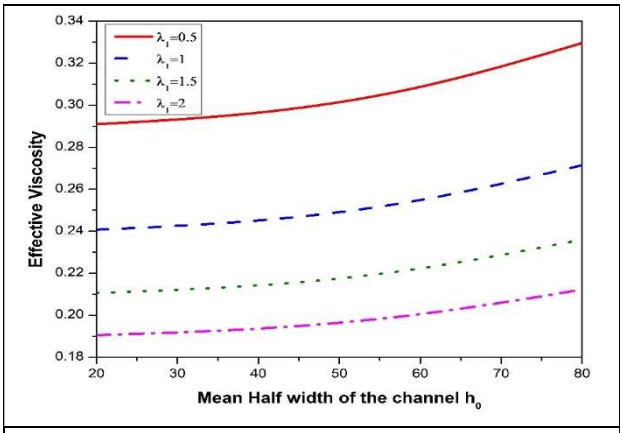

Fig 2: Impact of λ_1 upon μ_{eff} ($\delta_1 = 0.2, H_0 = 40\%$)

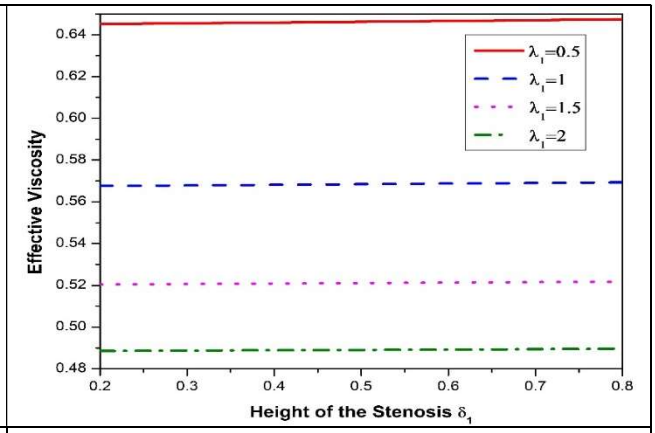

Fig 3: Impact of λ_1 upon μ_{eff} ($H_0 = 40\%$, $h_0 = 20$)

As the channel width increases, the flow becomes more laminar, and the interactions between blood cells and the channel walls diminish in significance. As a result, the effective viscosity of blood flow rises (Caro and Pedley [22]). This outcome is evident from Figures 2 and 4. Additionally, it shows the Fahraeus-Lindqvist effect. An abrupt decrease in the fluid's cross-sectional area occurs in a stenotic zone, which causes the fluid's velocity to rise. The fluid's shear rate, or velocity gradient, rises as it moves faster through the constriction. In Jeffrey Fluid, a more prominent shear-thinning behaviour may result from this increased shear rate, which would raise the effective viscosity. This outcome is seen in figures 3 and 5. Figures 6 to 9 illustrate that the mean haematocrit decreases with the Jeffrey parameter. The interaction between blood cells and the Jeffrey fluid can disrupt cell dispersion during blood flow. The elastic characteristics of the Jeffrey fluid may affect the deformation and orientation of blood cells, potentially resulting in alterations to the apparent haematocrit. The reduction in average haematocrit within a Jeffrey fluid flowing through a narrow conduit is likely due to the intricate interaction between

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the fluid's rheological properties and the behaviour of blood molecules in that particular flow context. From the figs 7 and 9 it is perceived that the mean hematocrit declines with the height of the stenosis. In areas of stenosis, RBCs may aggregate or clump together. This aggregation can cause a decrease in the effective volume occupied by RBCs, leading to a lower hematocrit. It is noticed through figures 10-13 that the core hematocrit diminutions with Jeffrey fluid. Figs. 10 and 12 depicted that the core hematocrit declines with the channel width. In a stenotic narrow channel, where the blood experiences higher shear rates, the viscosity of blood may decrease, potentially affecting the distribution of cellular elements like RBCs. This is the reason which causes the decrement in the core hematocrit. The 'Fahraeus-Lindqvist effect' describes the reduction in apparent viscosity and hematocrit in the core of rapidly flowing blood. In simpler terms, as blood flows more rapidly through a narrowed vessel, the deposition of RBC in the central flow stream (core) may appear to decrease. This result is observed in figures 11 and 12.

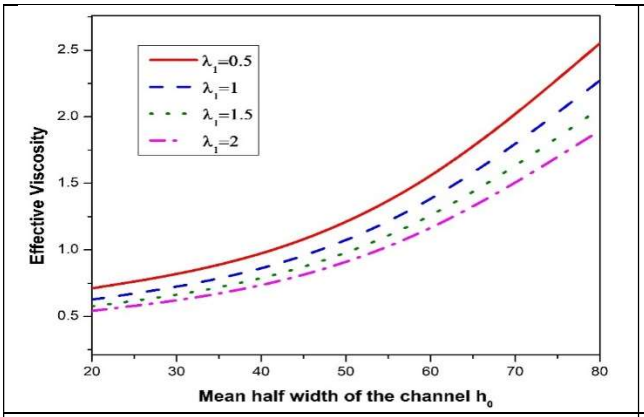

Fig 4: Impact of λ_1 upon μ_{eff} $(\delta_1 = 0.2, H_0 = 30\%)$

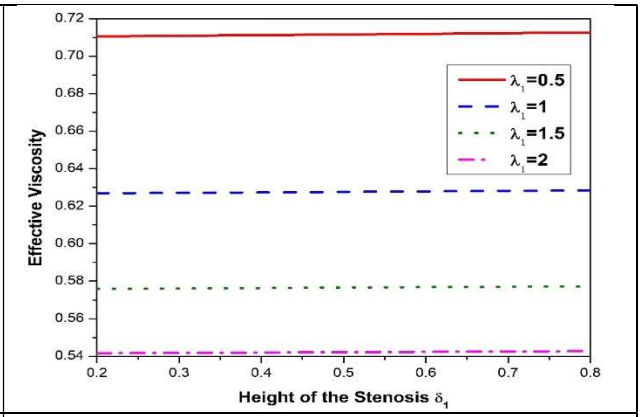

Fig 5: Impact of λ_1 upon μ_{eff} $(H_0 = 30\%, h_0 = 20)$

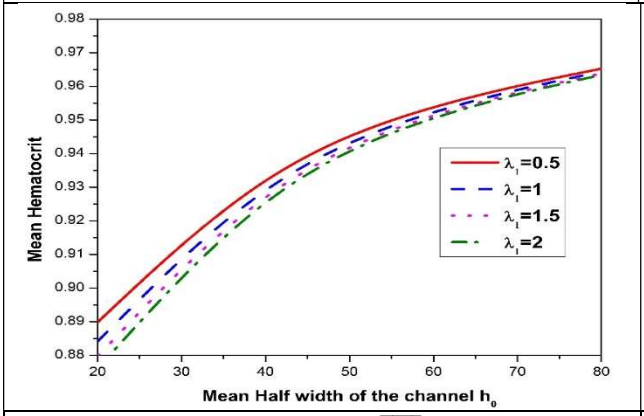

Fig 6: Impact of λ_1 upon $\overline{H_m}(\delta_1 = 0.2, H_0 = 40\%)$

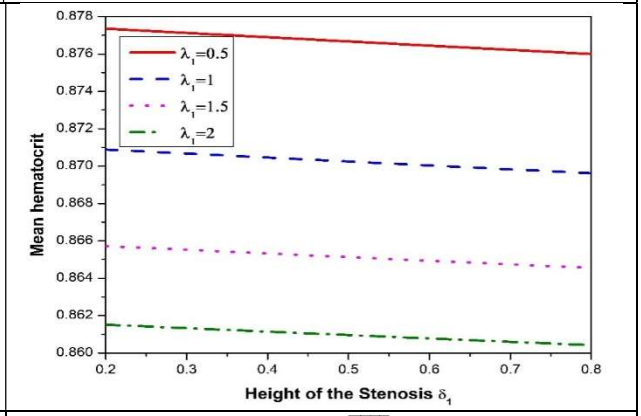

Fig 7: Impact of λ_1 upon $\overline{H_m}(H_0 = 40\%, h_0 = 20)$

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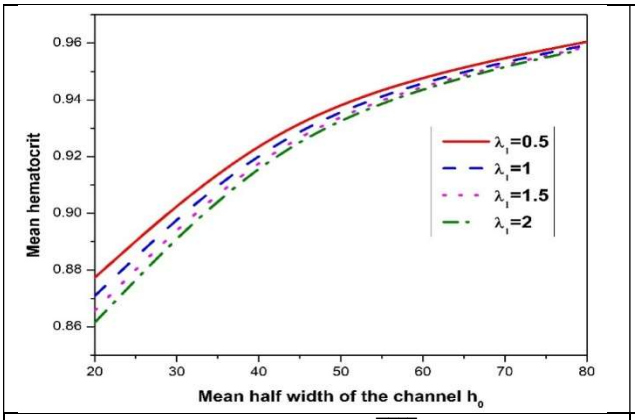

Fig 8: Impact of λ_1 upon $\overline{H_m}$ $(\delta_1 = 0.2, H_0 = 30\%)$

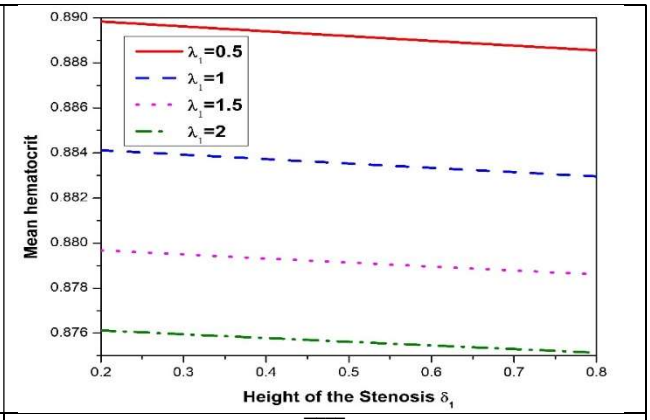

Fig 9: Impact of λ_1 upon $\overline{H_m}$ $(h_0 = 20, H_0 = 30\%)$

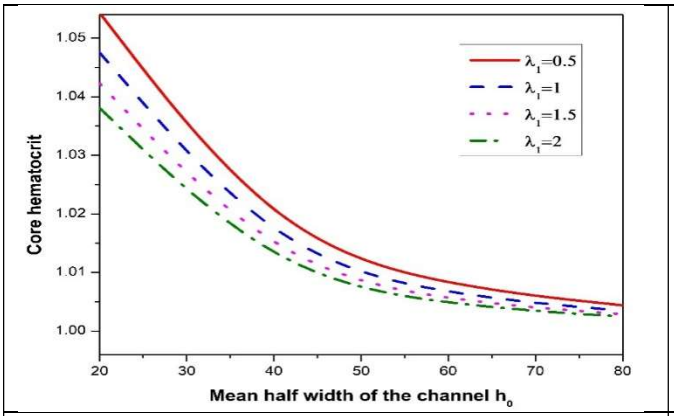

Fig 10: Impact of λ_1 upon $\overline{H_c}(\delta_1 = 0.2, H_0 = 40\%)$

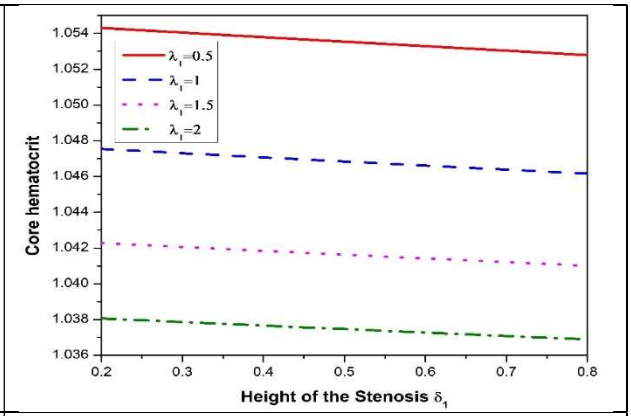

Fig 11: Impact of λ_1 upon $\overline{H_c}(h_0 = 20, H_0 = 40\%)$

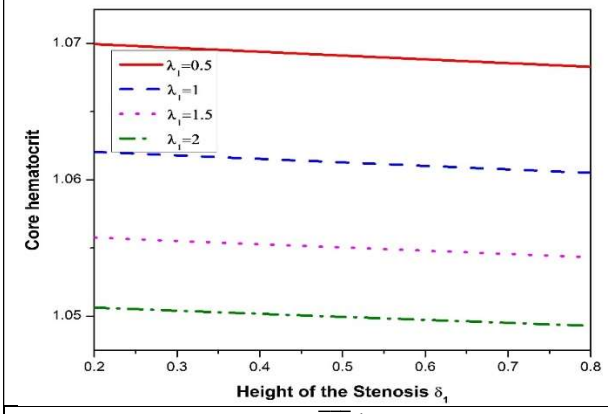

Fig 12: Impact of λ_1 upon $\overline{H_c}(h_0 = 20, H_0 = 30\%)$

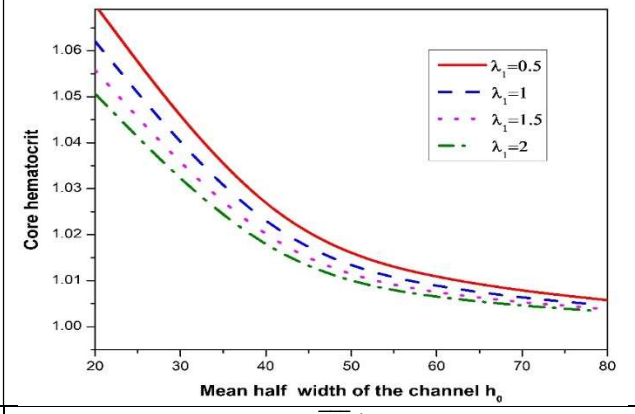

Fig 13: Impact of λ_1 upon $\overline{H_c}(\delta_1 = 0.2, H_0 = 30\%)$

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5. CONCLUSION

This paper examines the hemodynamic effects of stenosis on Jeffrey fluid transport through a constricted channel. Both fluid models are classified as Jeffrey fluids in the core region, whereas a Newtonian fluid is designated for the peripheral region. Upon simplifying the equations of motion, analytical results were derived for effective viscosity, flow velocities, mean hematocrit, and core hematocrit. The Mathematica software package was employed to execute a numerical computation on each of these flow-related variables. The effective viscosity increases with channel radius and stenosis height, while it decreases with the Jeffrey parameter. A larger channel radius and increased stenosis height correlate with a reduced core hematocrit, which diminishes as the Jeffrey parameter increases. An increase in the Jeffrey parameter corresponded with a decrease in mean hematocrit. An increase in channel radius correlates with a greater mean hematocrit associated with stenosis height.

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Outcomes of weakly compatible type(P) mappings in fuzzy menger space

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Abstract
In this paper we construct two fixed point results on fuzzy menger space (FMS) utilising weakly compatible type (P) mappings and reciprocally continuous, which are generalisations of Ruchi Sing et al.'s theorem.In addition to above two outcomes are substantiated by discussing proper example.

Keywords: Fuzzy menger space(FMS), reciprocally continuous mappings, weakly compatible mappings of type(P), self-mappings.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

The analysis is the core area of mathematics and increases its scope day by day increase by producing numerous results to all fields. Nowadays the fast growing area in analysis is fixed point theory.The probabilistic notion is well suited for searching physical quantities. In this era first foundation stone laydown by Menger [8]. He generalize the metric space by using probabilistic approach for distance function leads to the notion of menger space. Schweizer and Sklar [1] improved this concept by giving topological elements. The form of compatible mappings was introduced in menger spaces by Mishra [6] in addition to generate some fixed point results. The fuzzy version of compatible mappings introduced by Cho et.al and Sharma[2] and figure out some fixed point results in FMS. Sevet Kutukcu and Sushil Sharma[4] generated some fixed pointed by using the new concepts compatible type(P-1) as well as compatible type(P-2) in fuzzy metric space. Rajesh Shrivastav[9] generalize the fuzzy metric space by introducing the structure of fuzzy probabilistic metric space and generated some results using semi compatible mappings. Ruchi Singh.and A.D. Singh.et.al [11] using the form of weakly compatible mappings in FMS and provided some fixed point theorems

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Using expansion mappings in FMS, S. D. Divan et al. [3] were able to produce a number of fixed point outcomes. Ruchi Singh.et.al.[10] using the compatible type(P-1) mappings in FMS and obtained some fixed point results. One mores result on compatible type(P-1) mappings in FMS witnessed by Rashmi pathak.et.al[7].

Some more results on FMP like in[7],[8]. In this paper we define the weakly compatible type(P) mappings in FMS and provided fixed point theorem which is substantiated by by discussing proper illustration.

2. Preliminaries

Definition 2.1 [10] The continuous t- norm of a binary relation *: [0,1] → [0,1] is mentioned if * satisfies

- $*$ is associative
- $\mu * 1 = \mu \quad \forall \mu \in [0,1]$
- $*$ is commutative
- $*$ is continuous
- $\mu * \vartheta \leq \omega * \alpha$ whenever $\mu \leq \omega$ and $\vartheta \leq \alpha, \forall \mu, \vartheta, \omega, \alpha \in [0,1]$.

Definition 2.2 [10] A fuzzy probabilistic metric space (FPM space) is an order pair $(\Omega, \mathcal{F}_\alpha)$ involving of a non-empty set Ω and a mapping $\mathcal{F}_\alpha$ from $\Omega \times \Omega$ into the collection of all fuzzy distribution functions $\mathcal{F}_\alpha \in R \ \forall \ \alpha \in [0, 1]$,for $x, y \in \Omega$ we denote the fuzzy distribution functions $\mathcal{F}_\alpha (x, y)$ by $\mathcal{F}_{\alpha(x, y)}(v)$ is the value of $\mathcal{F}_{\alpha(x, y)}$ at v in R

The function $\mathcal{F}_{\alpha(x, y)}$ for all $\alpha \in [0, 1]$ being to hold the following properties.

- ◊ $\mathcal{F}_{\alpha(x, y)}(v) = 1 \ \forall v > 0 \Leftrightarrow x = y$
- ◊ $\mathcal{F}_{\alpha(x, y)}(0) = 0 \quad \forall x, y \in \Omega$
- ◊ $\mathcal{F}_{\alpha(x, y)} = \mathcal{F}_{\alpha(y, x)} \quad \forall x, y \in \Omega$
- ◊ If $\mathcal{F}_{\alpha(x, y)}(v) = 1$ and $\mathcal{F}_{\alpha(y, z)}(w) = 1 \Rightarrow \mathcal{F}_{\alpha(x, z)}(v + w) = 1 \ \forall x, y, z \in \Omega$ and $v, w > 0$.

Definition 2.3 [10] A triplet $(\Omega, \mathcal{F}_\alpha, *)$ is called as (FMS) if $(\Omega, \mathcal{F}_\alpha)$ is a FPM space, $*$ is continuous t- norm and the generalized triangle inequality

$$\mathcal{F}_{\alpha(x, z)}(v + w) \geq t\ (\mathcal{F}_{\alpha(x, z)}(v), \mathcal{F}_{\alpha(y, z)}(w))$$

true $\forall x, y, z \in \Omega \ v, w > 0$ and $\alpha \in [0, 1]$.

Definition 2.4 [10] In FMS $(\Omega, \mathcal{F}_\alpha, *)$ a sequence (μ_m)

- **converges** to α some point $\mu \in \Omega$ if $\lim_{m \to \infty} \mathcal{F}_{\alpha(\mu_m, \mu)}(t_\nabla) = 1 \ \forall \ t_\nabla > 0$ and $\alpha \in [0, 1]$
- **cauchy** if $lim_{n, m \to \infty} \mathcal{F}_{\alpha(\mu_m, \mu_n)}(t_\nabla) = 1 \ t_\nabla > 0$ and $\alpha \in [0,1]$.
- Further FMS $(\Omega, \mathcal{F}_\alpha, *)$ is **complete** if for each cauchy sequence converges in Ω.

Lemma 2.5 [10] Let $\{\mu_m\}$ be a general sequence in FMS $(\Omega, \mathcal{F}_\alpha, *)$ with continuous t- norm $*$ and $t_\nabla * t_\nabla \geq t_\nabla$. If there existing a constant $k \in (0.1)$ so that

$\mathcal{F}_{\alpha(\mu_m, \mu_{m+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(\mu_{m-1}, \mu_m)}(t_\nabla) \quad \forall t_\nabla > 0$ and $\forall m \geq 1$.

Resulting that $\{\mu_m\}$ is a Cauchy in Ω.

Lemma 2.6 [10] Let $\{\mu_m\}$ be a normal sequence in FMP $(\Omega, \mathcal{F}_\alpha, *)$ with continuous t- norm $*$ and $t_\nabla * t_\nabla \geq t_\nabla$. If there is a constant $k \in (0.1)$ in a manner that

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$\mathcal{F}_{\alpha(x,\ y)}(kt_\nabla) \geq \mathcal{F}_{\alpha(x,\ y)}(t_\nabla) \quad \forall\ x,\ y \in \Omega$ and $t_\nabla > 0$, then $x = y$.

Example 2.7. Let $\Omega = [0,1]$, the metric ρ is defined by $\rho(x,y) = |x-y|$ for each t_∇ define

$$\mathcal{F}_{\alpha(x,\ y)}(t_\nabla) = \begin{cases} 1, & if\ x = y \\ \mathcal{H}_\alpha(t_\nabla), & if\ x \neq y \end{cases} \text{ where } \mathcal{H}_\alpha(t_\nabla) = \begin{cases} 0, & t_\nabla \leq 0 \\ t_\nabla \alpha, & 0 < t_\nabla < 1 \\ 1, & t_\nabla \geq 1. \end{cases}$$

Then $(\Omega,\ \mathcal{F}_\alpha,\ *)$ with continuous t- norm $*$ and $t_\nabla * t_\nabla \geq t_\nabla$ forming a complete FMP, here $\alpha \in [0,1]$.

Definition 2.8 [10] A pair of self-mappings $A,\ P$ of a FMS $(\Omega,\ \mathcal{F}_\alpha,\ *)$ is said to **compatible type(P) [7]** if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1\ \forall\ t_\nabla > 0$, wherever the sequence (x_η) in Ω such way that $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Px_\eta = \omega$ for some $\omega \in \Omega$.

- **Compatible type(P-1) [10]** if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1\ \forall\ t_\nabla > 0$, whenever sequence (x_η) in Ω such that $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Px_\eta = \omega$ for some $\omega \in \Omega$.
- **Compatible type(P-2) [7]** if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1\ \forall\ t_\nabla > 0$, whenever sequence (x_η) in Ω such that $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Px_\eta = \omega$ for some $\omega \in \Omega$.
- **Weakly compatible type(P)** if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1\ \forall\ t_\nabla > 0$, whenever sequence (x_η) in Ω such that $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Px_\eta = \omega$ for some $\omega \in \Omega$.
- **Reciprocally continuous [8]** if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ A\omega)}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ P\omega)}(t_\nabla) = 1\ \forall\ t_\nabla > 0$, when the sequence (x_η) in Ω such a manner that $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Px_\eta = \omega$ for some $\omega \in \Omega$.

Remark:

(a) It can be noted that two mapping are compatible type (P) imply compatible type(P-1) imply weakly compatible type(P) however not conversely it can illustrated through the example (2.9).

(b) It can be noted that two mapping are continuous imply reciprocally continuous but not conversely. This can be illustrated by counter example (2.10).

Example 2.9 . Let FMS e $(\Omega,\ \mathcal{F}_\alpha,\ *)$ with continuous t- norm $*$, $\alpha \in [0,1]$ define

$$\mathcal{F}_{\alpha(\alpha,\ \beta)}(t_\nabla) = \begin{cases} e^{\frac{-|\alpha-\beta|}{t_\nabla}}, & t_\nabla > 0 \\ 0, & t_\nabla = 0 \end{cases}. \qquad (2.10.1)$$

Consider $\Omega = [0,\ 2\ ]$, further define the mappings , $\psi : \Omega \to \Omega$ as

$$\varphi(x) = \begin{cases} \frac{4}{3}x, & x \in [0,\ \frac{1}{3}] \\ \frac{25}{81}, & x \in (\frac{1}{3},\ \frac{4}{9}] \\ 0, & x \in (\frac{4}{9},\ 2] \end{cases} \qquad (2.10.2)$$

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$$\psi(x) = \begin{cases} (1-x)^2, & x \in [0,\ 1] \\ \frac{2}{9}, & x \in (1,\ 2]. \end{cases} \qquad (2.10.3)$$

Consider $(\mu_\eta) = \frac{1}{3} - \frac{\sqrt{2}}{\eta}\ \forall\ \eta \geq 1.$

Then from (2.10.2), (2.10.3)

$$\lim_{\eta\to\infty} \varphi\mu_\eta = \lim_{\eta\to\infty} \varphi\left(\frac{1}{3} - \frac{\sqrt{2}}{\eta}\right) = \lim_{\eta\to\infty} \frac{4}{3}\left(\frac{1}{3} - \frac{\sqrt{2}}{\eta}\right) = \lim_{\eta\to\infty} \left(\frac{4}{9} - \frac{4\sqrt{2}}{3\eta}\right) = \frac{4}{9},$$

(2.10.4)

$$\lim_{\eta\to\infty} \psi\mu_\eta = \lim_{\eta\to\infty} \psi\left(\frac{1}{3} - \frac{\sqrt{2}}{\eta}\right) = \lim_{\eta\to\infty} \left(1 - \left(\frac{1}{3} - \frac{\sqrt{2}}{\eta}\right)\right)^2 = \lim_{\eta\to\infty} \left(\frac{2}{3} + \frac{\sqrt{2}}{\eta}\right)^2 = \frac{4}{9}. \qquad (2.10.5)$$

From (2.10.4) and (2.10.5) $\lim_{\eta\to\infty} \varphi\mu_\eta = \lim_{\eta\to\infty} \psi\mu_\eta = \frac{4}{9}.$

$$\lim_{\eta\to\infty} \varphi\psi\mu_\eta = \lim_{\eta\to\infty} \varphi\left(\left(\frac{2}{3} + \frac{\sqrt{2}}{\eta}\right)^2\right) = 0. \qquad (2.10.6)$$

$$\lim_{\eta\to\infty} \psi\psi\mu_\eta = \lim_{\eta\to\infty} \psi\left(\left(\frac{2}{3} + \frac{\sqrt{2}}{\eta}\right)^2\right) = \lim_{\eta\to\infty} \left(1 - \left(\frac{2}{3} + \frac{\sqrt{2}}{\eta}\right)^2\right)^2 = \frac{25}{81}. \qquad (2.10.7)$$

$$\lim_{\eta\to\infty} \psi\varphi\mu_\eta = \lim_{\eta\to\infty} \psi\left(\frac{4}{9} - \frac{4\sqrt{2}}{3\eta}\right) = \lim_{\eta\to\infty} \left(1 - \left(\frac{4}{9} - \frac{4\sqrt{2}}{3\eta}\right)\right)^2 = \frac{25}{81}.$$

$$\lim_{\eta\to\infty} \varphi\varphi\mu_\eta = \lim_{\eta\to\infty} \varphi\left(\frac{4}{9} - \frac{4\sqrt{2}}{3\eta}\right) = \frac{25}{81}.$$

From (2.10.6) and (2.10.7)

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(\varphi\psi\mu_\eta,\ \psi\psi\mu_\eta)}(t_\nabla) \neq 1\ \forall\ t_\nabla > 0 \text{ and} \qquad (2.10.8)$$

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(\varphi\psi\mu_\eta,\ \psi\psi\mu_\eta)}(t_\nabla) = 1 \quad \forall\ t_\nabla > 0. \qquad (2.10.9)$$

From (2.10.8) , (2.10.9)

Turns out mappings $\varphi,\ \psi$ are weakly compatible type(P) but neither compatible type(P) nor compatible type(P -1).

Example 2.10. Let $(\Omega,\ \mathcal{F}_\alpha,\ *)$ be the as in example (2.10) where $\Omega = \mathbb{R}$.

Define the mappings $\varphi,\ \psi : \Omega \to \Omega$ as

$$\varphi(x) = \begin{cases} (\frac{5}{2} + x)^2, & x < \frac{5}{2} \\ 6, & x \geq \frac{5}{2} \end{cases} \qquad (2.11.1)$$

$$\psi(x) = \begin{cases} (\frac{5}{2} - x)^2, & x < \frac{5}{2} \\ 8, & x \geq \frac{5}{2} \end{cases} \qquad (2.11.2)$$

Consider a sequence $(\mu_m) = \frac{\sqrt{6}}{m}\ \forall\ m \geq 1$. Then from (2.11.1) and (2.11.2)

$$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \varphi\left(\frac{\sqrt{6}}{m}\right) = \lim_{m\to\infty} \left(\frac{5}{2} + \frac{\sqrt{6}}{m}\right)^2 = \frac{25}{4}, \qquad (2.11.3)$$

$$\lim_{m\to\infty} \psi\mu_m = \lim_{m\to\infty} \psi\left(\frac{\sqrt{6}}{m}\right) = \lim_{m\to\infty} \left(\frac{5}{2} - \frac{\sqrt{6}}{m}\right)^2 = \frac{25}{4}. \qquad (2.11.4)$$

From (2.11.3) and (2.11.4)

$$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = \frac{25}{4},$$

$$\lim_{m\to\infty} \varphi\psi\mu_m = \lim_{m\to\infty} \varphi\left(\left(\frac{5}{2} - \frac{\sqrt{6}}{m}\right)^2\right) = \lim_{m\to\infty} 6 = 6 = \varphi\left(\frac{25}{4}\right), \qquad (2.11.5)$$

$$\lim_{m\to\infty} \psi\varphi\mu_m = \lim_{m\to\infty} \psi\left(\left(\frac{5}{2} + \frac{\sqrt{6}}{m}\right)^2\right) = \lim_{m\to\infty} 8 = 8 = \psi\left(\frac{25}{4}\right). \qquad (2.11.6)$$

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From (2.11.5) and (2.11.6)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(\varphi\psi\mu_m,\ \varphi(\frac{25}{4}))}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(\psi\varphi\mu_m,\ \psi(\frac{25}{4}))}(t_\nabla) = 1 \ \forall\ t_\nabla > 0.$ (2.11.7)

Thus from (2.11.7) both the mappings $\varphi,\ \psi$ are reciprocally continuous however these are not continuous at $x = \frac{5}{2}$.

The next theorem was provided by Ruchi Singh et.al. [10]

"***Theorem*** *(A) Let A, P, Q and S be self-mappings of a complete* FMS *$(X, \mathcal{F}_\alpha, *)$ with continuous norm $*$ and $t * t \geq t,\ \forall t \in [0,1]$ fulfilling:*

(i) $P(X) \subseteq S(X),\ Q(X) \subseteq A(X)$

(ii) *there exists a constant $k \in (0,1)$ such that*

$$\mathcal{F}_{\alpha(Px,\ Qy)}(kt) \geq \mathcal{F}_{\alpha(Ax,\ Sy)}(t) * \mathcal{F}_{\alpha(Px,\ Ax)}(t) * \mathcal{F}_{\alpha(Qy,\ Sy)}(t) * \mathcal{F}_{\alpha(Px,\ Sy)}(t) * \mathcal{F}_{\alpha(Qy,\ Ax)}((2-\beta)t)$$

$\forall x, y \in X,\ \beta \in (0,2)$ *and* $t > 0$,

(iii)*either P or A is continuous*

(iv) *the pair (P, A) and (Q, S) are compatible type (P- 1)*

Then A, P, Q and S have a unique common fixed point in X.''

Now we generalize the above theorem as follows.

3. Main result

Theorem 3.1 *Let A, P, Q and S be self-mappings on a complete FMS $(\Omega, \mathcal{F}_\alpha, *)$ satisfying that*

(i) $P(\Omega) \subseteq S(\Omega),\ Q(\Omega) \subseteq A(\Omega)$;

(ii) *the pairs (P, A) and (Q, S) are weakly compatible type (P)*

(iii)*either P or A is continuous*

(iv)*there exists a constant $k \in (0,1)$ such that*

$$\mathcal{F}_{\alpha(Px,\ Qy)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax,\ Sy)}(t_\nabla) * \mathcal{F}_{\alpha(Px,\ Ax)}(t_\nabla) * \mathcal{F}_{\alpha(Qy,\ Sy)}(t_\nabla) * \mathcal{F}_{\alpha(Px,\ Sy)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qy,\ Ax)}((2-\beta)t_\nabla) \quad (3.1.1)$$

$\forall\ x, y \in X,\ \beta \in (0,2)$ *and* $t_\nabla > 0$.

.Then A, P, Q and S having a unique common fixed point in Ω.

Proof: The inclusion property (i) we can derive the sequences $\{x_\eta\}$ as well as $\{y_\eta\}$ in Ω as follows

$Px_{2\eta} = Sx_{2\eta+1} = y_{2\eta}$ and $Qx_{2\eta+1} = Ax_{2\eta+2} = y_{2\eta+1}\ \forall\ \eta \geq 0.$ (3.1.2)

Claim to show $\{y_\eta\}$ is Cauchy sequence in Ω.

Replace $x = x_{2\eta},\ y = x_{2\eta+1}$ in (3.1.1) for all $t_\nabla > 0$ and $\beta = 1 - q$ with $q \in (0,1)$ then

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta}\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sx_{2\eta+1})}((1-q)t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Ax_{2\eta})}((2-(1-q))t_\nabla).$$

Then from (3.1.2)

$$\mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta-1})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta})}((1-q)t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta-1})}((1+q)t_\nabla)$$

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$$\mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta-1})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla) * 1 * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta-1})}(qt_\nabla)$$

$$\mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta-1})}(qt_\nabla)$$

.

The $t-$ norm is continuous so letting $q \to 1$, we will get

$$\mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta-1})}(t_\nabla)$$

$$\mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta})}(t_\nabla)$$

$$\mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta-1},\ y_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(t_\nabla).$$

In the same way we have

$$\mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta+2})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{2\eta},\ y_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(y_{2\eta+1},\ y_{2\eta+2})}(t_\nabla).$$

In general we can write for all $\forall\ \eta \geq 0$

$$\mathcal{F}_{\alpha(y_\eta,\ y_{\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{\eta-1},\ y_\eta)}(t_\nabla) * \mathcal{F}_{\alpha(y_\eta,\ y_{\eta+1})}(t_\nabla). \qquad (3.1.3)$$

Resulting for $p = 1, 2, 3 \ldots$ we have from (3.1.3)

$$\mathcal{F}_{\alpha(y_\eta,\ y_{\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{\eta-1},\ y_\eta)}(t_\nabla) * \mathcal{F}_{\alpha(y_\eta,\ y_{\eta+1})}\left(\frac{t_\nabla}{k^p}\right)$$

as $p \to \infty$ we get

$$\mathcal{F}_{\alpha(y_\eta,\ y_{\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{\eta-1},\ y_\eta)}(t_\nabla) * 1$$

$$\mathcal{F}_{\alpha(y_\eta,\ y_{\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(y_{\eta-1},\ y_\eta)}(t_\nabla).$$

Then $\{y_\eta\}$ is Cauchy sequence in Ω by lemma(2.6).

Due to completeness of Ω the sequence $y_\eta \to z \in \Omega$.

This implies from (3.1.2)

$Px_{2\eta},\ Sx_{2\eta+1} \to z$ and $Qx_{2\eta+1},\ Ax_{2\eta+2} \to z \quad as\ \eta \to \infty.$

Thus we have

$$\lim_{\eta\to\infty} Px_{2\eta} = \lim_{\eta\to\infty} Ax_{2\eta} = z, \qquad (3.1.4)$$

$$\lim_{\eta\to\infty} Qx_{2\eta+1} = \lim_{\eta\to\infty} Sx_{2\eta+1} = z. \qquad (3.1.5)$$

The pair $(P,\ A)$ is weakly compatible type (P) implies from (3.1.4)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1.$

Case(i)

Assume $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1.$ (3.1.6)

Case(ia)

If A is continuous implies from (3.1.4)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(AAx_{2\eta},\ Az)}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ Az)}(t_\nabla) = 1.$ (3.1.7)

Substitute $x = Px_{2\eta},\ y = x_{2\eta+1}$ in (3.1.1)

$$\mathcal{F}_{\alpha(PPx_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(APx_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PPx_{2\eta},\ APx_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PPx_{2\eta},\ Sx_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ APx_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.5), (3.1.6) and (3.1.7) with $\beta = 1$

$$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Az,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$$

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$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla)$.
By lemma(2.7) $Az = z$.
Substitute $x = z,\ y = x_{2\eta+1}$ in (3.1.1)

$$\mathcal{F}_{\alpha(Pz,\ Q x_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ S x_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(Q x_{2\eta+1},\ S x_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ S x_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Q x_{2\eta+1},\ Az)}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.5), $Az = z$ with $\beta = 1$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$$

$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq 1 * \mathcal{F}_{\alpha(Pz,\ Az)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * 1$
$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla)$.
By lemma(2.7) we get $Pz = z$ this gives

$$Pz = Az = z. \tag{3.1.8}$$

Since $z = Az \in P(\Omega) \subseteq S(\Omega)$ resulting that there exists $w \in \Omega$ such that

$$z = Pz = Sw. \tag{3.1.9}$$

Substitute $x = x_{2\eta},\ y = w$ in (3.1.1)

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sw)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Ax_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.4), (3.1.9) with $\beta = 1$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$$

$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq 1 * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$.
Hence by lemma(2.7) $z = Qw$ resulting

$$z = Sw = Qw. \tag{3.1.10}$$

Since the pair $(Q,\ S)$ is weakly compatible type (P) by (3.1.5)
$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1$.
If $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ then $QSw = SSw$ implies from (3.1.8) $Qz = Sz$.
Also if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1$ then $SQw = QQw$ implies from (3.18) $Qz = Sz$.
In both the cases we have $Qz = Sz$.
Using $x = x_{2\eta},\ y = z$ in (3.1.1)

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sz)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Ax_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.4), $Qz = Sz$ with $\beta = 1$.

$$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$$

$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$.
By lemma(2.7) we get $Qz = z$. This consequences

$$Sz = Qz = z \tag{3.1.11}$$

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Thus we have from (3.1.8) and (3.1.11)

$z = Pz = Sz = Qz = Sz.$ (3.1.12)

Case(ib)

Assume $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1.$

If P is continuous implies from (3.1.4)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ Pz)}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PPx_{2\eta},\ Pz)}(t_\nabla) = 1.$ (3.1.13)

Substitute $x = Px_{2\eta}$, $y = x_{2\eta+1}$ in (3.1.1)

$$\mathcal{F}_{\alpha(PPx_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(APx_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PPx_{2\eta},\ APx_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PPx_{2\eta},\ Sx_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ APx_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.5), (3.1.6) and (3.1.13) with $\beta = 1$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Pz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Pz)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Pz)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla).$$

By lemma(2.7) $Pz = z.$

Since $z = Pz \in P(\Omega) \subseteq S(\Omega)$ resulting that there exists $w \in \Omega$ such that

$z = Pz = Sw.$ (3.1.14)

Substitute $x = x_{2\eta}$, $y = w$ in (3.1.1)

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sw)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Ax_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.4), (3.1.14) with $\beta = 1$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq 1 * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla).$$

Hence by lemma(2.7) $z = Qw$ resulting

$z = Sw = Qw.$ (3.1.15)

Since the pair $(Q,\ S)$ is weakly compatible type (P) by (3.1.5)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1.$

If $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ then $QSw = SSw$ implies from (3.1.9) $Qz = Sz.$

Also if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1$ then $SQw = QQw$ implies from (3.1.9) $Qz = Sz.$

In both the cases we have $Qz = Sz.$

Using $x = x_{2\eta}$, $y = z$ in (3.1.1)

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sz)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Ax_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.4), $Qz = Sz$ with $\beta = 1.$

$$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$$

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$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$.
By lemma(2.7) we get $Qz = z$. This consequences
$Sz = Qz = z$ (3.1.16)
Thus we have from (3.1.8) and (3.1.16)
$z = Pz = Qz = Sz$.
And $z = Qz \in Q(\Omega) \subseteq A(\Omega)$ implies there exist some u$\in \Omega$ such that $z = Qz = Au$
Put $x = u, y = z$
$\mathcal{F}_{\alpha(Pu,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Au,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Pu,\ Au)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) *$
$\mathcal{F}_{\alpha(Pu,\ Sz)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Au)}((2-\beta)t_\nabla)$
$\mathcal{F}_{\alpha(Pu,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pu,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pu,\ z)}(\beta t_\nabla) *$
$\mathcal{F}_{\alpha(z,\ z)}((2-\beta)t_\nabla)$
$\mathcal{F}_{\alpha(Pu,\ z)}(kt_\nabla) \geq 1 * \mathcal{F}_{\alpha(Pu,\ z)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Pu,\ z)}(\beta t_\nabla) * 1$ as $\beta = 1$
$\mathcal{F}_{\alpha(Pu,\ z)}(kt_\nabla) \geq 1 * \mathcal{F}_{\alpha(Pu,\ z)}(t_\nabla)$ implies $z = Pu$
Hence $z = Pu = Au$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1$ implies $Pz = Az$
Thus we have $Pz = Az = Qz = Sz = z$.

Case(ii)
Assume $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1$ (3.1.17)
Case(iia)
If A is continuous implies from (3.1.4)
$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(AAx_{2\eta},\ Az)}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ Az)}(t_\nabla) = 1$. (3.1.18)
Substitute $x = Ax_{2\eta}$, $y = x_{2\eta+1}$ in (3.1.1)
$\mathcal{F}_{\alpha(PAx_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(AAx_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla)$
$* \mathcal{F}_{\alpha(PAx_{2\eta},\ Sx_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ AAx_{2\eta})}((2-\beta)t_\nabla)$
as $\eta \to \infty$, using (3.1.5), (3.1.13) and (3.1.14) with $\beta = 1$
$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Az,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla)$
$* \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$
$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$
$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla)$.
By lemma $Az = z$. Here on words leads to the case(ia)
Case(iib)
Assume $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1$ and
if P is continuous implies from (3.1.4)
$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ Pz)}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PPx_{2\eta},\ Pz)}(t_\nabla) = 1$. (3.1.19)
Substitute $x = Px_{2\eta}$, $y = x_{2\eta+1}$ in (3.1.1)
$\mathcal{F}_{\alpha(PPx_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(APx_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PPx_{2\eta},\ APx_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla)$
$* \mathcal{F}_{\alpha(PPx_{2\eta},\ Sx_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ APx_{2\eta})}((2-\beta)t_\nabla)$
as $\eta \to \infty$, using (3.1.5), (3.1.6) and (3.1.19) with $\beta = 1$
$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Pz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla)$
$* \mathcal{F}_{\alpha(z,\ Pz)}(t_\nabla)$

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$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Pz)}(t_\nabla)$
$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla)$.
By lemma(2.7) $Pz = z$. From here this leads to case(ib)
In all cases turn out
$z = Pz = Sz = Qz = Az.$ (3.1.20)

Uniqueness

Let z_1 be some other common fixed point such that
$z_1 = Pz_1 = Sz_1 = Qz_1 = Az_1.$ (3.1.21)
Using $x = z_1,\ y = z$ in (3.1.1)
$\mathcal{F}_{\alpha(Pz_1,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az_1,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Pz_1,\ Az_1)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) *$
$\mathcal{F}_{\alpha(Pz_1,\ Sz)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Az_1)}((2-\beta)t_\nabla)$
From (3.1.19) and (3.1.21) with $\beta = 1$
$\mathcal{F}_{\alpha(z_1,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z_1,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z_1,\ z_1)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z_1,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z_1)}(t_\nabla)$
$\mathcal{F}_{\alpha(z_1,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z_1,\ z)}(t_\nabla)$.
Hence by lemma $z_1 = z$.
Hence uniqueness fulfilled.

3.2 Example

Let $(\Omega,\ \mathcal{F}_\alpha,\ *)$ be the FMS with the continuous t-norm and $\mathcal{F}_\alpha$ as in (2.10.1) where $\Omega = [0,\ 2]$.
Define the mappings , $\psi : \Omega \to \Omega$ as

$$P(x) = Q(x) = \begin{cases} 1-x, & x \in [0,\ 1] \\ |si\,n(\pi x)|, & x \in (1,\ 2] \end{cases} \quad (3.2.1)$$

$$A(x) = S(x) = \begin{cases} 3x, & x \in [0,\ \frac{1}{2}) \\ \frac{1}{\sqrt{2}}|\cos(\frac{\pi x}{2})|, & x \in [\frac{1}{2},\ \frac{3}{4}) \\ \frac{1}{2\sqrt{2}}|si\,n(\pi x)|, & x \in [\frac{3}{4},\ 2] \end{cases} \quad (3.2.2)$$

From (3.2.1) and (3.2.2)
$P(\Omega) = Q(\Omega) = [0,1]$, $A(\Omega) = S(\Omega) = [0, \frac{3}{2})$
This gives $P(\Omega) \subseteq S(\Omega),\ Q(\Omega) \subseteq A(\Omega)$.
Consider $(\mu_\eta) = \frac{1}{4} - \frac{\sqrt{11}}{\eta}\ \forall\ \eta \geq 1$. Then from (3.2.1), (3.2.2)

$$\lim_{\eta\to\infty} P\mu_\eta = \lim_{\eta\to\infty} P\left(\frac{1}{4} - \frac{\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty} 1 - \left(\frac{1}{4} - \frac{\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty}\left(\frac{3}{4} + \frac{\sqrt{11}}{\eta}\right) = \frac{3}{4}, \quad (3.2.3)$$

$$\lim_{\eta\to\infty} A\mu_\eta = \lim_{\eta\to\infty} A\left(\frac{1}{4} - \frac{\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty} 3\left(\frac{1}{4} - \frac{\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty}\left(\frac{3}{4} - \frac{3\sqrt{11}}{\eta}\right) = \frac{3}{4}. \quad (3.2.4)$$

From (3.2.3) and (3.2.4)
$\lim_{\eta\to\infty} P\mu_\eta = \lim_{\eta\to\infty} A\mu_\eta = \frac{3}{4}$.

$$\lim_{\eta\to\infty} PA\mu_\eta = \lim_{\eta\to\infty} P\left(\frac{3}{4} - \frac{3\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty} 1 - \left(\frac{3}{4} - \frac{3\sqrt{11}}{\eta}\right) = \frac{1}{4}, \quad (3.2.5)$$

$$\lim_{\eta\to\infty} AA\mu_\eta = \lim_{\eta\to\infty} A\left(\frac{3}{4} - \frac{3\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty} \frac{1}{\sqrt{2}}\left|\cos\pi\left(\frac{3}{4} - \frac{3\sqrt{11}}{\eta}\right)\right| = \frac{1}{2}, \quad (3.2.6)$$

$$\lim_{\eta\to\infty} AP\mu_\eta = \lim_{\eta\to\infty} A\left(\frac{3}{4} + \frac{\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty} \frac{1}{2\sqrt{2}}\left|si\,n\left(\pi\left(\frac{3}{4} + \frac{\sqrt{11}}{\eta}\right)\right)\right| = \frac{1}{4}, \quad (3.2.7)$$

$$\lim_{\eta\to\infty} PP\mu_\eta = \lim_{\eta\to\infty} P\left(\frac{3}{4} + \frac{\sqrt{11}}{\eta}\right) = \lim_{\eta\to\infty} 1 - \left(\frac{3}{4} + \frac{\sqrt{11}}{\eta}\right) = \frac{1}{4}. \quad (3.2.8)$$

From (3.2.5), (3.2.6), (3.2.7) and (3.2.8)
$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PA\mu_\eta,\ AA\mu_\eta)}(t_\nabla) \neq 1\ \forall\ t_\nabla > 0$ and (3.2.9)

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$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(AP\mu_\eta,\ PP\mu_\eta)}(t_\nabla) = 1 \quad \forall\ t_\nabla > 0. \tag{3.2.10}$$

From (3.2.9) and (3.2.10)

The mappings $P,\ A$ are weakly compatible type(P) but none of them is compatible type(P) or compatible type(P -1).

Further mapping P is continuous. And also at $x = \frac{1}{2}$, $A(\frac{1}{2}) = P\left(\frac{1}{2}\right) = S(\frac{1}{2}) = Q\left(\frac{1}{2}\right) = \frac{1}{2}$.

Hence the mappings P, A, Q and S have the unique fixed point $x = \frac{1}{2}$ which is common and satisfy every condition of the Thoeorem (3.1).

The other generalization of the Theorem(A) as following.

Theorem 3.3 *Let $A,\ P,\ Q$ and S be self-mappings on a complete FMS $(\Omega,\ \mathcal{F}_\alpha,\ *)$ satisfying that*

(i) $P(\Omega) \subseteq S(\Omega),\ Q(\Omega) \subseteq A(\Omega)$;

(ii) *the pairs $(P,\ A)$ and $(Q,\ S)$ are weakly compatible type (P)*

(iii) *the pair $(P,\ A)$ is reciprocally continuous*

(iv) *there exists a constant $k \in (0,1)$ such that*

$$\mathcal{F}_{\alpha(Px,\ Qy)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax,\ Sy)}(t_\nabla) * \mathcal{F}_{\alpha(Px,\ Ax)}(t_\nabla) * \mathcal{F}_{\alpha(Qy,\ Sy)}(t_\nabla)$$
$$* \mathcal{F}_{\alpha(Px,\ Sy)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qy,\ Ax)}((2-\beta)t_\nabla) \tag{3.3.1}$$

$\forall\ x, y \in X,\ \beta \in (0,2)$ *and* $t_\nabla > 0$.

Then $A,\ P,\ Q$ and Sshare a unique common fixed point.

Proof: From inclusion condition (i) we can derive the sequences $\{x_\eta\}$ as well as $\{y_\eta\}$ in Ω as follows

$$Px_{2\eta} = Sx_{2\eta+1} = y_{2\eta} \text{ and } Qx_{2\eta+1} = Ax_{2\eta+2} = y_{2\eta+1}\ \forall\ \eta \geq 0. \tag{3.3.2}$$

And $\{y_\eta\}$ is Cauchy sequence in Ω from Theorem(3.1)

Due to completeness of Ω the sequence $y_\eta \to z \in \Omega$.

This implies from (3.3.2)

$Px_{2\eta},\ Sx_{2\eta+1} \to z$ and $Qx_{2\eta+1},\ Ax_{2\eta+2} \to z$ *as* $\eta \to \infty$.

Thus we have

$$\lim_{\eta\to\infty} Px_{2\eta} = \lim_{\eta\to\infty} Ax_{2\eta} = z, \tag{3.3.4}$$

$$\lim_{\eta\to\infty} Qx_{2\eta+1} = \lim_{\eta\to\infty} Sx_{2\eta+1} = z. \tag{3.3.5}$$

The pair $(P,\ A)$ is weakly compatible type (P) implies from (3.3.4)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1$.

Case(i)

Assume $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ PPx_{2\eta})}(t_\nabla) = 1$. (3.3.6)

And also the pair $(P,\ A)$ is reciprocally continuous implies from (3.2.4)

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ Pz)}(t_\nabla) = 1 \text{ and } \lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ Az)}(t_\nabla) = 1. \tag{3.3.7}$$

Substitute $x = Px_{2\eta},\ y = x_{2\eta+1}$ in (3.3.1)

$$\mathcal{F}_{\alpha(PPx_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla)$$
$$\geq \mathcal{F}_{\alpha(APx_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PPx_{2\eta},\ APx_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla)$$
$$* \mathcal{F}_{\alpha(PPx_{2\eta},\ Sx_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ APx_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.3.5), (3.3.6) and (3.3.7) with $\beta = 1$

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$$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Az,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(Az,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla).$$

By lemma(2.7) $Az = z$.

Substitute $x = z,\ y = x_{2\eta+1}$ in (3.3.1)

$$\mathcal{F}_{\alpha(Pz,\ Q x_{2\eta+1})}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ S x_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(Q x_{2\eta+1},\ S x_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ S x_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Q x_{2\eta+1},\ Az)}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.3.5), $Az = z$ with $\beta = 1$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Az,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Az)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Az)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq 1 * \mathcal{F}_{\alpha(Pz,\ Az)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * 1$$

$$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla).$$

By lemma(2.7) we get $Pz = z$ this gives

$$Pz = Az = z. \tag{3.3.8}$$

Since $z = Az \in P(\Omega) \subseteq S(\Omega)$ resulting that there exists $w \in \Omega$ such that

$$z = Pz = Sw. \tag{3.3.9}$$

Substitute $x = x_{2\eta},\ y = w$ in (3.3.1)

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sw)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Ax_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.3.4), (3.3.9) with $\beta = 1$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq 1 * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$$

$$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla).$$

Hence by lemma(2.7) $z = Qw$ resulting

$$z = Sw = Qw. \tag{3.3.10}$$

Since the pair $(Q,\ S)$ is weakly compatible type (P) by (3.3.5)

$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1$.

If $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ then $QSw = SSw$ implies from (3.3.10) $Qz = Sz$.

Also if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1$ then $SQw = QQw$ implies from (3.3.10) $Qz = Sz$.

In both the cases we have $Qz = Sz$.

Using $x = x_{2\eta},\ y = z$ in (3.3.1)

$$\mathcal{F}_{\alpha(Px_{2\eta},\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sz)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Ax_{2\eta})}((2-\beta)t_\nabla)$$

as $\eta \to \infty$, using (3.1.4), $Qz = Sz$ with $\beta = 1$.

$$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$$

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$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla).$
By lemma(2.7) we get $Qz = z$. This consequences
$Sz = Qz = z$ (3.3.11)
Thus we have from (3.1.8) and (3.3.11)
$z = Pz = Sz = Qz = Sz.$ (3.3.12)
Case(ii)
Assume $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1$ (3.3.13)
And also the pair $(P,\ A)$ is reciprocally continuous implies from (3.3.4)
$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ Pz)}(t_\nabla) = 1$ and $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(APx_{2\eta},\ Az)}(t_\nabla) = 1.$ (3.3.14)
Substitute $x = Ax_{2\eta},\ y = x_{2\eta+1}$ in (3.3.1)
$\mathcal{F}_{\alpha(PAx_{2\eta},\ Qx_{2\eta+1})}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(AAx_{2\eta},\ Sx_{2\eta+1})}(t_\nabla) * \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ Sx_{2\eta+1})}(t_\nabla)$
$* \mathcal{F}_{\alpha(PAx_{2\eta},\ Sx_{2\eta+1})}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qx_{2\eta+1},\ AAx_{2\eta})}((2-\beta)t_\nabla)$
as $\eta \to \infty$, using (3.3.5), (3.3.13) and (3.1.14) with $\beta = 1$
$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ Pz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla)$
$* \mathcal{F}_{\alpha(z,\ Pz)}(t_\nabla)$
$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Pz)}(t_\nabla)$
$\mathcal{F}_{\alpha(Pz,\ z)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pz,\ z)}(t_\nabla).$
By lemma $Pz = z.$
Since $z = Pz \in P(\Omega) \subseteq S(\Omega)$ resulting that there exists $w \in \Omega$ such that
$z = Pz = Sw.$ (3.3.15)
Substitute $x = x_{2\eta},\ y = w$ in (3.3.1)
$\mathcal{F}_{\alpha(Px_{2\eta},\ Qw)}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sw)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Sw)}(t_\nabla)$
$* \mathcal{F}_{\alpha(Px_{2\eta},\ Sw)}(\beta t_\nabla) * \mathcal{F}_{\alpha(Qw,\ Ax_{2\eta})}((2-\beta)t_\nabla)$
as $\eta \to \infty$, using (3.3.4), (3.3.15) with $\beta = 1$
$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq 1 * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qw)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qw,\ z)}(t_\nabla).$
Hence by lemma $z = Qw$ resulting
$z = Pz = Sw = Qw.$ (3.3.16)
Since the pair $(Q,\ S)$ is weakly compatible type (P) by (3.3.5)
$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ or $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1.$
If $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(QSx_{2\eta+1},\ SSx_{2\eta+1})}(t_\nabla) = 1$ then $QSw = SSw$ implies from (3.1.16) $Qz = Sz.$
Also if $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(SQx_{2\eta+1},\ QQx_{2\eta+1})}(t_\nabla) = 1$ then $SQw = QQw$ implies from (3.1.16) $Qz = Sz.$
Using $x = x_{2\eta},\ y = z$ in (3.3.1)
$\mathcal{F}_{\alpha(Px_{2\eta},\ Qz)}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(Ax_{2\eta},\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Ax_{2\eta})}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Px_{2\eta},\ Sz)}(\beta t_\nabla)$
$* \mathcal{F}_{\alpha(Qz,\ Ax_{2\eta})}((2-\beta)t_\nabla)$
as $\eta \to \infty$, using (3.1.4), $Qz = Sz$ with $\beta = 1.$

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$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla)$
$\geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ z)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla)$
$* \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * 1 * 1 * \mathcal{F}_{\alpha(z,\ Qz)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$
$\mathcal{F}_{\alpha(z,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Qz,\ z)}(t_\nabla)$.
By lemma we get $Qz = z$.
Thus we have
$z = Sz = Qz.$ (3.3.17)
$z = Sz = Qz \in Q(\Omega) \subseteq A(\Omega)$
This implies there exist v $\in$ Ω such that
$z = Sz = Qz = Av.$ (3.3.18)
Claim $Pv = Av$. Put $x = v,\ y = z$ in (3.3.1)
$\mathcal{F}_{\alpha(Pv,\ Qz)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Av,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Pv,\ Av)}(t_\nabla) * \mathcal{F}_{\alpha(Qz,\ Sz)}(t_\nabla) * \mathcal{F}_{\alpha(Pv,\ Sz)}(\beta t_\nabla) *$
$\mathcal{F}_{\alpha(Qz,\ Av)}((2-\beta)t_\nabla$
)
using (3.3.18) with $\beta = 1$
$\mathcal{F}_{\alpha(Pv,\ Av)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Av,\ Av)}(t_\nabla) * \mathcal{F}_{\alpha(Pv,\ Av)}(t_\nabla) * \mathcal{F}_{\alpha(Av,\ Av)}(t_\nabla) * \mathcal{F}_{\alpha(Pv,\ Av)}(t_\nabla) *$
$\mathcal{F}_{\alpha(Av,\ Av)}(t_\nabla$
)
$\mathcal{F}_{\alpha(Pv,\ Av)}(kt_\nabla) \geq 1 * \mathcal{F}_{\alpha(Pv,\ Av)}(t_\nabla) * 1 * \mathcal{F}_{\alpha(Pv,\ Av)}(t_\nabla) * 1$
$\mathcal{F}_{\alpha(Pv,\ Av)}(kt_\nabla) \geq \mathcal{F}_{\alpha(Pv,\ Av)}(t_\nabla)$.
By lemma we get $Pv = Av$.
And from (3.3.13) $\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PAx_{2\eta},\ AAx_{2\eta})}(t_\nabla) = 1$ implies $PAv = AAv$ resulting $Pz = Az$.
Therefore from (3.3.18)
$z = Pz = Sz = Qz = Az.$ (3.3.19)
Uniqueness proved in Theorem(3.1).

3.4 Example

Let $(\Omega,\ \mathcal{F}_\alpha,\ *)$ be the FMS with the continuous t-norm, $\mathcal{F}_\alpha$ as in (2.10.1) where $\Omega = [0,\ 2]$.
Define the mappings , $\psi : \Omega \to \Omega$ as

$$P(x) = Q(x) = \begin{cases} \frac{9}{5}x, & x \in [0,\ \frac{1}{5}] \\ \frac{49}{625}, & x \in (\frac{1}{5},\ \frac{9}{25}) \\ 2, & x \in [\frac{9}{25},\ 2] \end{cases} \quad (3.4.1)$$

$$A(x) = S(x) = \begin{cases} (1-2x)^2, & x \in [0,\ 1] \\ 2sin\left(\frac{\pi}{4}x\right), & x \in (1,\ 2] \end{cases} \quad (3.4.2)$$

From (3.4.1) and (3.4.2)
$P(\Omega) = Q(\Omega) = \left[0, \frac{9}{25}\right] U \{2\},\ A(\Omega) = S(\Omega) = [0,1]\ U\ (\sqrt{2}\ ,\ 2]$.
This gives $P(\Omega) \subseteq S(\Omega),\ Q(\Omega) \subseteq A(\Omega)$.
Consider $(\mu_\eta) = \frac{1}{5} - \frac{\sqrt{5}}{\eta}\ \forall\ \eta \geq 1$. Then from (3.4.1), (3.4.2)
$\lim_{\eta\to\infty} P\mu_\eta = \lim_{\eta\to\infty} P\left(\frac{1}{5} - \frac{\sqrt{5}}{\eta}\right) = \lim_{\eta\to\infty} \frac{9}{5}\left(\frac{1}{5} - \frac{\sqrt{5}}{\eta}\right) = \lim_{\eta\to\infty} \left(\frac{9}{25} - \frac{9\sqrt{5}}{5\eta}\right) = \frac{9}{25}$,
(3.4.3)
$\lim_{\eta\to\infty} A\mu_\eta = \lim_{\eta\to\infty} A\left(\frac{1}{5} - \frac{\sqrt{5}}{\eta}\right) = \lim_{\eta\to\infty} \left(1 - 2\left(\frac{1}{5} - \frac{\sqrt{5}}{\eta}\right)\right)^2 = \lim_{\eta\to\infty} \left(\frac{3}{5} + \frac{2\sqrt{5}}{\eta}\right)^2 = \frac{9}{25}$.

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(3.4.4)

From (3.4.3) and (3.3.4)

$$\lim_{\eta\to\infty} P\mu_\eta = \lim_{\eta\to\infty} A\mu_\eta = \frac{9}{25}.$$

$$\lim_{\eta\to\infty} PA\mu_\eta = \lim_{\eta\to\infty} P\left(\left(\frac{3}{5}+\frac{2\sqrt{5}}{\eta}\right)^2\right) = \lim_{\eta\to\infty} 2sin\left(\frac{\pi}{4}\left(\left(\frac{3}{5}+\frac{2\sqrt{5}}{\eta}\right)^2\right)\right) = 2 \quad (3.3.5)$$

$$\lim_{\eta\to\infty} AA\mu_\eta = \lim_{\eta\to\infty} A\left(\left(\frac{3}{5}+\frac{\sqrt{5}}{\eta}\right)^2\right) = \lim_{\eta\to\infty} (1-2\left(\frac{3}{5}+\frac{\sqrt{5}}{\eta}\right)^2)^2 = \frac{49}{625}. \quad (3.3.6)$$

$$\lim_{\eta\to\infty} AP\mu_\eta = \lim_{\eta\to\infty} A\left(\frac{9}{25}-\frac{9\sqrt{5}}{5\eta}\right) = \lim_{\eta\to\infty} (1-2(\frac{9}{25}-\frac{9\sqrt{5}}{5\eta}))^2 = \frac{49}{625}. \quad (3.3.7)$$

$$\lim_{\eta\to\infty} PP\mu_\eta = \lim_{\eta\to\infty} P\left(\frac{9}{25}-\frac{9\sqrt{5}}{5\eta}\right) = \frac{49}{625}. \quad (3.3.8)$$

From (3.3.5), (3.3.6), (3.3.7) and (3.3.8)

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PA\mu_\eta,\ AA\mu_\eta)}(t_\nabla) \neq 1 \ \forall\ t_\nabla > 0 \text{ and} \quad (3.3.9)$$

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(AP\mu_\eta,\ PP\mu_\eta)}(t_\nabla) = 1 \quad \forall\ t_\nabla > 0. \quad (3.3.10)$$

From (3.3.9) and (3.3.10)

The mappings P, A are weakly compatible type(P) but neither compatible type(P) nor compatible type(P -1).

Further from (3.3.5) and (3.3.6)

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(PA\mu_\eta,\ P(\frac{9}{25}))}(t_\nabla) = 1 \ \forall\ t_\nabla > 0 \text{ and} \quad (3.3.11)$$

$$\lim_{\eta\to\infty} \mathcal{F}_{\alpha(AP\mu_\eta,\ A(\frac{9}{25}))}(t_\nabla) = 1 \quad \forall\ t_\nabla > 0 \quad (3.3.12).$$

Thus from (3.3.11) and (3.3.12) the mappings P, A are reciprocally continuous but not continuous. At $x = 2$, $A(2) = P(2) = S(2) = Q(2) = 2$.

Hence the mappings P, A, Q and S having a unique fixed point $x = 2$ which is common and satisfy necessary conditions of the Theorem(3.3).

4. Conclusion

Theorem (A) is generalized within this paper through two different approaches, which are outlined in the subsequent sections with the following ways.

- By utilising condition such as weakly compatible type(P) instead of compatible type(P-1) in Theorem (3.1).
- By using another weaker condition such as weakly compatible type(P), reciprocally continuous instead of compatible type(P-1) and continuity in Theorem (3.3).

In addition to these two results are supported by relevant illustrations.

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CONCERNING MULTIPLICATIVE (GENERALIZED) - (ϑ, φ) - REVERSE DERIVATIONS IN PRIME RINGS

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Abstract: "A mapping G, D : R → R (not necessarily additive) is called multiplicative (generalized) - (ϑ, φ) - reverse derivation if there exists a map (neither necessarily additive or derivation) g : R→ R such that(s.t) G(xy) = G(y)ϑ(x)+ φ(y)g(x) ∀ x, y ∈ R and d : R→ R s.t D(xy) = D(y)ϑ(x)+ φ(y)d(x) ∀ x, y ∈ R , where ϑ and φ are automorphisms on R." The main purpose of this paper is to study some algebraic identities with multiplicative (generalized) - (ϑ, φ) - reverse derivations. Also, study some algebraic identities with multiplicative (generalized) (ϑ, φ) – reverse derivations on the left ideal of a prime ring R. (i) G(xy) = D(xy) (ii) G(x) ± D (y) = ϑ(xy) (iii) G(xy) = G(x)D(y) (iv) G(xy) = D(x)G(y) ∀ x, y in an appropriate subset of R.

Keywords: Prime ring, multiplicative (generalized) derivation, multiplicative (generalized) reverse derivation, Multiplicative (generalized) - (ϑ, φ) - derivation, Multiplicative (generalized) - (ϑ, φ) - reverse derivation.

1. "Introduction and preliminaries

Throughout at R will represent an associative ring. A ring R is n-torsion free, if nx = 0 implies x = 0 ∀ x ∈ R, where n >1 is an integer. A ring R is called prime ring if aRb = (0) implies either a = 0 or b = 0. An additive mapping δ: R→ R is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ holds ∀ pairs x, y ∈ R and is said to be a Jordan derivation if $\delta(x^2) = \delta(x)x + x\delta(x)$ is fulfilled ∀ x ∈ R." Following [1], an additive mapping F: R→ R is said to be a generalized derivation if there exits an associated derivation F : R→ R s.t F (xy) = F(x)y + x δ(y) holds ∀ x, y ∈ R. "Inspired by Martindale [8], in 1991, Daif [2] has given a generalization of derivation as multiplicative derivation which is defined as: a mapping d: R→ R (not necessarily additive) is said to be multiplicative derivation if d(xy) = d(x)y + xd(y) ∀ x, y ∈ R. Similar type of notation is defined in [7]. Later, in [3] Daif and Tammam have extended this notation to multiplicative generalized derivation as follows: a mapping G: R→ R is a multiplicative generalized derivation if there exists a derivation g such that G(xy) = G(x)y + xg(y) ∀ x, y ∈ R. G is known as multiplicative (generalised) derivation, which was first shown by Dhara and Ali in [4], if we define g to be any map that is neither necessarily derivative nor additive. Multiplicative generalised derivation and

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multiplicative centraliser (if g = 0) are both included in the notion of multiplicative (generalised) derivation. A mapping f from R to R is centralizing if $[f(x), x] \in Z(R)$ and particularly commuting if $[f(x), x] = 0 \ \forall \ x \in R$. The study of the commutativity of rings with derivations has been started by Posner [9]. In specifics, he demonstrated that if a non-zero derivation δ centralises on a prime ring R, then R is commutative. This conclusion was expanded in various ways by a number of authors (see [5, 6, 11]). Further, Dhara and Ali considered the following identities with multiplicative (generalized) – derivation in semiprime ring: (i) $F(xy) \pm xy = 0$, (ii) $F(xy) \pm yx = 0$, (iii) $F(x)F(y) \pm xy \in Z(R)$, (iv) $F(xy) \pm yx \in Z(R)$ ∀ x and y in suitable subset of R. Rehman et al. [10] then generalise this kind of work by presenting a novel idea of multiplicative (generalised) (ϑ, φ)-reverse derivation in prime rings. In this study, we expand several results inspired by the work in [10].

Main Results:

Let's start with the subsequent theorem:

Theorem 1:- Let R be a prime ring, I be a non zero left ideal of R. Suppose that G, D: R→ R are multiplicative generalized (ϑ, φ) - reverse derivations associated with the maps g, d respectively. If G and D satisfy $G(xy) = D(xy) \ \forall$ x, y ∈ I, then G and D are identical on I, where ϑ and φ are automorphisms on R."

Proof: We have

$G(xy) = D(xy) \ \forall \ x, y \in I$ 1

Replacing x by yx

$G(yxy) = D(yxy) \ \forall \ x, y \in I$

$G(xy)\vartheta(y) + \varphi(xy)g(y) = D(xy)\vartheta(y) + \varphi(xy)d(y) \quad \forall \ x, y \in I$

$\varphi(xy)g(y) = \varphi(xy)d(y) \quad \forall \ x, y \in I$ 2

$\varphi(xy)g(y) - \varphi(xy)d(y) = 0 \quad \forall \ x, y \in I$

$\varphi(xy)[g(y) - d(y)] = 0 \quad \forall \ x, y \in I$

"Replacing x by xr

$\varphi(xry)[g(y) - d(y)] = 0 \quad \forall \ x, y \in I$ and $r \in R$

$\varphi(x)\varphi(r)\varphi(y)[g(y) - d(y)] = 0 \quad \forall \ x, y \in I$ and $r \in R$

Replacing r by $\varphi^{-1}(r)$

$\varphi(x)\varphi(\varphi^{-1}(r))\varphi(y)[g(y) - d(y)] = 0 \quad \forall \ x, y \in I$ and $r \in R$

$\varphi(x)\ r\ \varphi(y)[g(y) - d(y)] = 0 \quad \forall \ x, y \in I$ and $r \in R$

Primeness of R implies that either $\varphi(x) = 0$ or $\varphi(y)[g(y) - d(y)] = 0 \quad \forall \ x, y \in I$ and $r \in R$.

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If $\varphi(x) = 0 \ \forall \ x \in I$, then $\varphi(I) = (0)$.

Since I is nonzero left ideal of R, then we get a contradiction.

So, $\varphi(y)[g(y) - d(y)] = 0 \quad \forall \ y \in I$,

then $\varphi(y)g(y) = \varphi(y)d(y) \quad \forall \ x, y \in I$

Again from 1

$G(xy) = D(xy) \ \forall \ x, y \in I$

$G(y)\vartheta(x) + \varphi(y)g(x) = D(y)\vartheta(x) + \varphi(y)d(x)) \quad \forall \ x, y \in I$ 3

Using 2, we get"

$G(y)\vartheta(x) = D(y)\vartheta(x) \quad \forall \ x, y \in I$

$G(y)\vartheta(x) - D(y)\vartheta(x) = 0 \quad \forall \ x, y \in I$

$[G(y) - D(y)]\vartheta(x) = 0 \quad \forall \ x, y \in I$ Replacing x by rx

$[G(y) - D(y)]\vartheta(rx) = 0 \quad \forall \ x, y \in I$ Replacing x by rx and $r \in R$

$[G(y) - D(y)]\vartheta(r)\vartheta(x) = 0 \quad \forall \ x, y \in I$ Replacing r by $\vartheta^{-1}(r)$ and $r \in R$

$[G(y) - D(y)]\vartheta(\vartheta^{-1}(r))\vartheta(x) = 0 \quad \forall \ x, y \in I$ and $r \in R$

$[G(y) - D(y)]\ r\ \vartheta(x) = 0 \quad \forall \ x, y \in I$ and $r \in R$.

"Using Primeness of R, we get either $\vartheta(x) = 0$ or $[G(y) - D(y)] = 0 \quad \forall \ x, y \in I$.

Again $\vartheta(x) = 0$ gives a contradiction, hence G = D on I.

Theorem 2:- Let R be a Prime ring, I be a non zero left ideal of R. Suppose that G, D : R→ R are multiplicative generalized (ϑ, φ) - reverse derivations associated with the maps g, d respectively. If G and D satisfy $G(x)D(y) = \vartheta(xy) \ \forall \ x, y \in I$, then $\varphi(I)\ g(I) = (0)$, where ϑ and φ are automorphisms on R.

Proof: We have

$G(x)\ D(y) = \vartheta(xy) \ \forall \ x, y \in I$ 4

Replacing y by xy

$G(x)\ D(xy) = \vartheta(xxy) \ \forall \ x, y \in I$

$G(x)[D(y)\vartheta(x) + \varphi(y)d(x)] = \vartheta(xxy) \quad \forall \ x, y \in I$

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$G(x)D(y)\vartheta(x) + G(x)\varphi(y)d(x) = \vartheta(xxy) \quad \forall\ x, y \in I$

$G(x)\varphi(y)d(x) = 0 \ \forall\ x, y \in I$ 5

Replacing y by ry"

$G(x)\varphi(ry)d(x) = 0 \ \forall\ x, y \in I$ and $r \in R$

$G(x)\varphi(r)\varphi(y)d(x) = 0 \ \forall\ x, y \in I$ and $r \in R$ 6

Replacing r by $\varphi^{-1}(r)$

$G(x)r\varphi(y)d(x) = 0 \ \forall\ x, y \in I$ and $r \in R$ 7

Primeness of R implies that either $G(x) = 0$ or $\varphi(y)d(x) = 0 \quad \forall\ x, y \in I$ and $r \in R$.

If $G(x) = 0 \ \forall\ x \in I$, then $\vartheta(xy) = 0 \ \forall\ x, y \in I$

Replacing y by ry

$\vartheta(xry) = 0 \ \forall\ x, y \in I$ and $r \in R$

$\vartheta(x)\ \vartheta(r)\ \vartheta(y) = 0 \ \forall\ x, y \in I$ and $r \in R$

"Replacing r by $\vartheta^{-1}(r)$

$\vartheta(x)\ r\ \vartheta(y) = 0 \ \forall\ x, y \in I$ and $r \in R$

Using Primeness of R, we get either $\vartheta(x) = 0$ or $\vartheta(y) = 0 \ \forall\ x, y \in I$.

In other words $\vartheta(I) = (0)$, gives a contradiction, hence $\varphi(y)d(x) = 0 \quad \forall\ x, y \in I$.

In other words $\varphi(I)d(I) = 0 \quad \forall\ x, y \in I$.

Theorem 3:- Let R be a Prime ring, I be a non zero left ideal of R. Suppose that G,D : R→ R are multiplicative generalized (ϑ, φ) - reverse derivations associated with the maps g, d respectively. If G and D satisfy $G(xy) = G(x)D(y)$ $\forall\ x, y \in I$, then either $D(xy) = \varphi(x)D(y)$ or $G(xy) = G(x)\ \vartheta(y) \ \forall\ x, y \in I$, where ϑ and φ are automorphisms on R."

Proof: We have

$G(xy) = G(x)D(y) \ \forall\ x, y \in I$. Replacing x by wx 8

$G(wxy) = G(wx)D(y) \ \forall\ x, y \in I$

$G(xy)\vartheta(w) + \varphi(xy)g(w) = [G(x)\vartheta(w) + \varphi(x)g(w)]\ D(y) \quad \forall\ x, y \in I$ (since $G(xy) = G(x)D(y)$)

$\varphi(xy)g(w) = \varphi(x)g(w)\ D(y) \quad \forall\ x, y \in I$ (since $\varphi(I)d(I) = 0 \quad \forall\ x, y \in I$,) 9

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Replacing y by wy and x by xw and then subtract

$\varphi(xwy)g(w) = \varphi(x)g(w)\, D(wy) \quad \forall\ x, y, w \in I$ 10

$\varphi(xwy)g(w) = \varphi(xw)g(w)\, D(y) \quad \forall\ x, y, w \in I$ (since $\varphi(I)d(I) = 0 \quad \forall\ x, y \in I$,) 11

$D(wy)\, \varphi(x)g(w) - D(y)\varphi(xw)g(w) = 0 \quad \forall\ x, y, w \in I$

"$[D(wy) - \varphi(w)\, D(y)]\, \varphi(x)g(w) = 0 \quad \forall\ x, y, w \in I$

Replacing x by rx

$[D(wy) - \varphi(w)\, D(y)]\, \varphi(rx)g(w) = 0 \quad \forall\ x, y, w \in I$

$[D(wy) - \varphi(w)\, D(y)]\, \varphi(r)\, \varphi(x)g(w) = 0 \quad \forall\ x, y, w \in I$

Replacing r by $\varphi^{-1}(r)$

$[D(wy) - \varphi(w)\, D(y)]\, \varphi(\varphi^{-1}(r))\, \varphi(x)g(w) = 0 \quad \forall\ x, y, w \in I$

$[D(wy) - \varphi(w)\, D(y)]\, r\, \varphi(x)g(w) = 0 \quad \forall\ x, y, w \in I$

Primeness of R implies that either $[D(wy) - \varphi(w)\, D(y)]$ or $\varphi(x)g(w) = 0 \quad \forall\ x, y, w \in I$ and $r \in R$. If $\varphi(x)g(w) = 0$, then $D(wy) = \varphi(w)\, D(y) \ \forall\ w, y \in I$.

Theorem 4:- Let R be a Prime ring, I be a non zero left ideal of R. Suppose that G, D : R→ R are multiplicative generalized (ϑ, φ) - reverse derivations associated with the maps g, d respectively. If G and D satisfy $G(xy) = D(x)\, G(y)$ $\forall\ x, y \in I$, then $D(xy) = \varphi(x)D(y) \ \forall\ x, y \in I$, where ϑ and φ are automorphisms on R."

Proof: We have

$G(xy) = D(x)\, G(y) \quad \forall\ x, y \in I$ 12

Replacing x by wx

$G(wxy) = D(wx)G(y) \quad \forall\ x, y \in I$

$G(xy)\vartheta(w) + \varphi(xy)g(w) = [D(x)\vartheta(w) + \varphi(x)d(w)]\, G(y) \quad \forall\ x, y \in I$

$\varphi(xy)g(w) = \varphi(x)d(w)\, G(y) \quad \forall\ x, y \in I$

"Replacing y by wy and x by xw and then subtract

$\varphi(xwy)g(w) = \varphi(x)d(w)\, G(wy) \quad \forall\ x, y, w \in I$

$\varphi(xwy)g(w) = \varphi(xw)d(w)\, G(y) \quad \forall\ x, y, w \in I$ (since $\varphi(I)d(I) = 0 \quad \forall\ x, y \in I$,)

$G(wy)\, \varphi(x)g(w) - G(y)\varphi(xw)d(w) = 0 \quad \forall\ x, y, w \in I$

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$[G(wy) - \varphi(w) G(y)] \varphi(x)d(w) = 0 \ \forall \ x, y, w \in I$

Replacing x by rx

$[G(wy) - \varphi(w) G(y)] \varphi(xr)d(w) = 0 \ \forall \ x, y, w \in I$

$[G(wy) - \varphi(w) G(y)] \varphi(r) \varphi(x)d(w) = 0 \ \forall \ x, y, w \in I$

Replacing r by $\varphi^{-1}(r)$

$[G(wy) - \varphi(w) G(y)] \varphi(\varphi^{-1}(r)) \varphi(x)d(w) = 0 \ \forall \ x, y, w \in I$

$[G(wy) - \varphi(w) G(y)] r \varphi(x)d(w) = 0 \ \forall \ x, y, w \in I.$"

Primeness of R implies that either $[G(wy) - \varphi(w) G(y)]$ or $\varphi(x)d(w) = 0 \ \forall \ x, y, w \in I$ and $r \in R$. If $\varphi(x)d(w) = 0$, then $G(wy) = \varphi(w) G(y) \ \forall \ w, y \in I$.

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INFLUENCE OF MAGNETIC FIELD ON FREE AND FORCED CONVECTION FLOW IN A VERTICAL CHANNEL BOUNDED BY POROUS MEDIA

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Abstract: A vertical channel limited by porous media with varied permeabilities is used to study the effect of a magnetic field on forced and free convection flow. The existence of a magnetic field is used to derive the equations for bulk temperature and velocity. The numerical study examines the impact on velocity and bulk temperature of several factors, such as the ratio of wall temperature differences (rT), the magnetic field parameter (M), the Reynolds number (Gr/Re), the Grashoff number, the pressure gradient α=-dp/dx, and the porosity parameters σ1 and σ2.

KeyWords:MHD,Porous parameter, Combined Convection, bulk temperature,FDF.

1. INTRODUCTION

Heat transfer problems have several utilizations and applications in engineering sciences in various areas, eg., designing cooling systems for motors, generators and transformers. Also, in this study, the chemical industries are interested in the evaporation, condensation, heating and cooling of fluids. Knowledge of loss of heat flow is important to the civil engineers in the designing and building of dams and structures. Cost estimation for the viability and the tools required to transfer desired heat in the required time requires a thorough study of heat transfer should be carrie out. The proportions of boilers, heaters, refrigerators and exchange of heat rely on the amount of heat to be transmitted and also on the rate at which heat should be transmitted within the parameters. In all branches of engineering, the problems of heat transfer exist that can be solved by thermodynamic reasoning and an analysis based on the science of heat transfer. Either Darcy's or non-Darcy's law govern the flow in a porous medium. Muskat [9], Yah[16], Sheidegger[13], Bear[6] and several more analyzed various flows into porous media using Darcy's law. For flows through high porosity porous bodies such as fibreglass, Darcy's law is not valid. Moreover, Darcy's law cannot explain the phenomena of flow near a solid boundary or near an interface. Both free and forced convection heat transfer in channels was explored by Tao[15]. The buoyancy influence on hydrodynamic and thermal characteristics in the upward laminar flow of a viscous fluid in a parallel plate channel was numerically solved by Aung and Worku [1]. Another study by Aung and Worku [2] looked at how mixed convection changed the flow in a vertical channel with parallel plates and different wall temperatures. If you look at what a magnetic field does to fully developed mixed convection flows in a vertical channel, Reddy [10] also found closed-form solutions. In a vertical channel with non-uniform wall temperatures, Reddy investigated the effect of a magnetic field on the onset and reversal of flow [11]. Reddy [11] figured out how a magnetic field affects combined convection in ducts with uneven wall heat fluxes.

2. Problem

Consider a mixed convection flow that is constant and laminar, operating in a vertical channel with porous beds as its boundaries. Assume that y=0 and y=b are the locations of the walls, and let 'b' represent the distance between them. Figure 1.1 depicts the geometry. The fluid used here has a consistent x-direction vertical stream velocity at the channel entry. Porous material encased the channel walls. Applied in the y-direction is a constant transverse magnetic field of intensity B_0.

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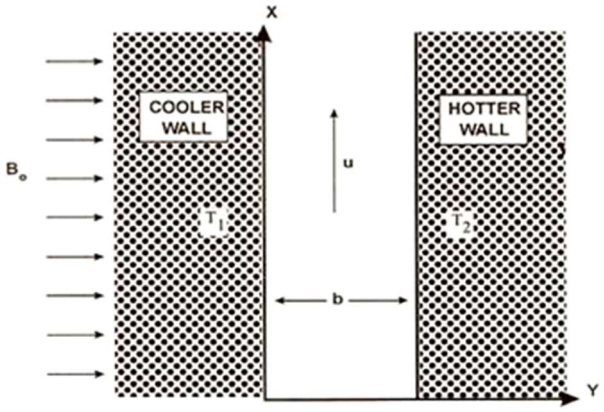

Fig.1.1Geometric Model

The flow is divided into three zones.

The governing equations in these zones are:

Zone 1: Darcy's's law

$$\overline{u_1} = -k_1 \frac{\left(\nabla p - \overline{J} \times \overline{B}\right)}{\mu} \qquad (1.1)$$

$$\nabla . \overline{u_1} = 0 \qquad (1.2)$$

$$\nabla . \overline{B} = 0 \qquad (1.3)$$

Zone 2:

$$\nabla . \overline{q} = 0 \qquad (1.4)$$

$$\rho \frac{D\overline{q}}{Dt} = -\nabla p^1 + \mu \nabla^2 \overline{q} - \rho\, g + \overline{J} \times \overline{B} \qquad (1.5)$$

$$\rho\, C_p \left(\overline{q} . \nabla\right) T = \kappa \nabla^2 T + \phi \qquad (1.6)$$

where

$$\overline{J} = \sigma\left(\overline{E} + \mu_e\, \overline{q} \times \overline{H}\right) \qquad (1.7)$$

and ϕ = Dissipation function

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$$= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$\bar{q}$ = (u, v, 0) and $\overline{H}$ = (0, H_0, 0)

Zone 3: Darcy's's law

$$\overline{u_2} = -k_2\frac{\left(\nabla p - \bar{J}\times\bar{B}\right)}{\mu} \qquad (1.8)$$

$$\nabla.\overline{u_2} = 0 \qquad (1.9)$$

Where B = $\mu_e \overline{H}$

The following assumptions are made:

(i) A consistent, viscous, incompressible and completely developed flow was observed.
(ii) The height of the channel is more than the channel spacing. Hence, the velocity component v in y- direction is considered as zero in the total cross section of flow.
(iii) The induced magnetic field and electric field ($\overline{E}$) are omitted. [12, 14].
(iv) Dissipation of energy is ignored.
(v) Temperature in Zone 1 and Zone 3 are ignored.

Based on the above, the governing equations become

Zone 1:

$$u_1 = \frac{-k_1}{\mu}\left(\frac{dp^1}{dx}+\sigma\mu_e^2 h_0^2 u_{b1}\right) \qquad (1.10)$$

Zone 2:

$$\frac{\partial u}{\partial x} = 0 \qquad (1.11)$$

$$0 = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{dp^1}{dx} - g - \frac{\sigma\mu_0^2 H_0^2 u}{\rho} \qquad (1.12)$$

$$\frac{\partial p^1}{\partial y} = 0 \qquad (1.13)$$

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Hence the completely developed temperature profile for uniform wall temperature is $\frac{\partial T}{\partial x} = 0$.

The equation of energy becomes

$$\frac{d^2T}{dy^2} = 0 \qquad (1.14)$$

Zone 3:

$$u_1 = \frac{-k_2}{\mu}\left(\frac{dp^1}{dx} + \sigma \mu_e^2 h_0^2 u_{b2}\right) \qquad (1.15)$$

Boundary conditions are

$u = u_{b1}$; $T = T_1$

$$\frac{du}{dy} = \frac{\alpha^1(u_{b1} - u_1)}{\sqrt{k_1}}$$

at y = 0 (1.16)

$u = u_{b2}$; $T = T_2$

$$\frac{du}{dy} = \frac{-\alpha^1(u_{b2} - u_2)}{\sqrt{k_2}}$$

at y = b (1.17)

The non-dimensional quantities are defined as follows:

$U = u/u_0$; $V = vb/\nu$; $X = x/(b\,Re)$; $Y = y/b$

$$p = \frac{(p^1 - p^{11})}{\rho u_0^2}, \quad Pr = \frac{\mu C_p}{\kappa}; \quad Re = b\,u_0/\nu;$$

$$Gr = g\beta\frac{(T_2 - T_0)b^3}{\nu^2}, \quad \theta = \frac{(T - T_0)}{(T_2 - T_0)} \qquad (1.18)$$

$U_1 = u_1/u_0$; $U_2 = u_2/u_0$; $\alpha_1 = \frac{b}{\sqrt{k_1}}$;

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$$\alpha_2 = \frac{b}{\sqrt{k_2}}; \qquad U_{b1} = \frac{u_{b1}}{u_0}; \qquad U_{b2} = \frac{u_{b2}}{u_0}$$

The non-dimensional version of the governing equations is

Zone 1:

$$U_1 = \frac{\alpha^1}{\left(M^2 + \sigma_1^2\right)} \tag{1.19}$$

Zone 2:

Continuity equation

$$\frac{dU}{dX} = 0 \tag{1.20}$$

X-momentum equation

$$0 = \alpha + \frac{Gr}{\mathrm{Re}}\theta + \frac{d^2U}{dY^2} - M^2U \tag{1.21}$$

Y-momentum equation

$$\frac{dP}{dY} = 0 \tag{1.22}$$

Energy equation

$$0 = \frac{d^2\theta}{dY^2} \tag{1.23}$$

Zone 3:

$$U_2 = \frac{\alpha^1}{\left(M^2 + \sigma_2^2\right)} \tag{1.24}$$

Where

$$M^2 = \sigma\frac{\mu_e^2 H_0^2 b^2}{\mu}, \text{ and } \alpha = \frac{dP}{dX}, \text{ the pressure gradient} \tag{1.25}$$

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In equation (2.12), Boussinesq equation of state, $\rho-\rho_0 = -\rho\beta(T-T_0)$, and by the definition p = p1- p11, (p11 is the pressure at any stream wise position). If the temperature were T_0 is applied, the definition gives $\frac{dp''}{dx} = -\rho_0 g$. Hence

$$-\frac{dp^1}{dx} - \rho\, g = -\frac{dp}{dx} + \rho\, g\, \beta\,(T - T_0)$$

The dimensionless pressure becomes $P = \frac{(p^1 - p^{11})}{\rho\, u_0^2}$; for a horizontal channel, $P = \frac{p^1}{\rho\, u_0^2}$ (standard definition in pure forced flow).

Conditions at boundary are:

$U = U_{b1}$; $\theta = r_T$

$$\frac{dU}{dY} = \alpha^1 \sigma_1^2 (U_{b1} - U_1) \qquad \text{at Y = 0} \qquad (1.26)$$

$U = U_{b2}$; $\theta = 1$

$$\frac{dU}{dY} = -\alpha^1 \sigma_2^2 (U_{b2} - U_2) \qquad \text{at Y = 1} \qquad (1.27)$$

Using the equations (1.26) and (1.27) for θ, integrating equation (1.23) we get the fully developed temperature profile.

$$\theta = (1 - r_T)Y + r_T \qquad (1.28)$$

Solving equation (1.21), using equations (1.26), (1.27) and (1.28) we get

$$U = C_1 Cosh\, MY + C_2 Sinh\, MY - \frac{(AY + B)}{M^2} \qquad \textbf{(1.29)}$$

Where A = $-(1 - r_T)\frac{Gr}{\mathrm{Re}}$, $B = -\alpha - r_T \frac{Gr}{\mathrm{Re}}$

$$C_1 = \frac{N_1}{D} \text{ and } C_2 = \frac{N_2}{D}$$

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Where D = M(M Sinh M + α1σ2 2 Cosh M) +α1σ1 2 (M Cosh M+α1σ2 2 Sinh M)

$$N_1 = \frac{\left[\begin{array}{l}\left(M\,Cosh\,M + \alpha^1\sigma_2^2 Sinh\,M\right)\left\{\left(r_T - 1\right)\frac{Gr}{\mathrm{Re}} + \alpha^1\sigma_1^2\left(r_T\frac{Gr}{\mathrm{Re}} + \sigma_1^2 U_1\right)\right\} - \\ M\left\{\left(r_T - 1\right)\frac{Gr}{\mathrm{Re}} + \alpha^1\sigma_2^2\left(\frac{Gr}{\mathrm{Re}} + \sigma_2^2 U_2\right)\right\}\end{array}\right]}{M^2}$$

$$N_2 = \frac{-\left[\left(M\,Sinh\,M + \alpha^1\sigma_2^2 Cosh\,M\right)\left\{\left(r_T - 1\right)\frac{Gr}{\mathrm{Re}} + \alpha^1\sigma_1^2\left(r_T\frac{Gr}{\mathrm{Re}} + \sigma_1^2 U_1\right)\right\} + \alpha^1\sigma_1^2\left\{\left(r_T - 1\right)\frac{Gr}{\mathrm{Re}} + \alpha^1\sigma_2^2\left(\frac{Gr}{\mathrm{Re}} + \sigma_2^2 U_2\right)\right\}\right]}{M^2}$$

To evaluate α at any cross section in the channel, we need an expression of global conservation of mass. Let this equation be [1].

$$\int_0^1 U\,dY = 1 \qquad (1.30)$$

is dimensionless.

Substituting (2.29) in (2.30), we get

$$\alpha = -\frac{dP}{dX} = \frac{M^2 - N_{11}M\,Sinh\,M + N_{21}M\left(1 - Cosh\,M\right) - \frac{1}{2}\left(r_T + 1\right)\frac{Gr}{\mathrm{Re}}}{1 + N_{12}M\,Sinh\,M - N_{22}M\left(1 - Cosh\,M\right)} \qquad (1.31)$$

Where

$$N_{11} = \left[\left(M\,Cosh\,M + \alpha^1\sigma_2^2 Sinh\,M\right)\left\{r_T\left(1 + \alpha^1\sigma_1^2\right) - 1\right\} - M\left(r_T - 1 + \alpha^1\sigma_2^2\right)\right]\frac{\left(\frac{Gr}{\mathrm{Re}}\right)}{M^2 D}$$

$$N_{12} = \frac{\left[\left(M\,Cosh\,M + \alpha^1\sigma_2^2 Sinh\,M\right)\alpha^1\sigma_1^4 U_{11} - M\alpha^1\sigma_2^4 U_{12}\right]}{M^2 D}$$

$$N_{21} = -\left[\left(M\,Sinh\,M + \alpha^1\sigma_2^2 Cosh\,M\right)\left\{r_T\left(1 + \alpha^1\sigma_1^2\right) - 1\right\} + \alpha^1\sigma_1^2\left(r_T - 1 + \alpha^1\sigma_2^2\right)\right]\frac{\left(\frac{Gr}{\mathrm{Re}}\right)}{M^2 D}$$

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$$N_{22} = \frac{-\left[\left(M\,Sinh\,M + \alpha^{1}\sigma_2^2 Cosh\,M\right)\alpha^{1}\sigma_1^4 U_{11} + \alpha^{1^2}\sigma_2^4 U_{12}\right]}{M^2 D}$$

$$U_{11} = \frac{1}{\left(M^2 + \sigma_1^2\right)} \text{ and } U_{12} = \frac{1}{\left(M^2 + \sigma_2^2\right)}$$

using (2.31) in (2.29), we get the velocity distribution for all Gr/Re, r_T and M (≠0).

$U = C_1 \text{ Cosh MY} + C_2 \text{ Sinh MY} + [\{(1\text{-}r_T)Y + r_T\}Gr/Re]/M^2 +$

$$\frac{M^2 - N_{11}M\,Sinh\,M + N_{21}M\left(1 - Cosh\,M\right) - \left(r_T + 1\right)\frac{Gr}{2\,\mathrm{Re}}}{M^2\left[1 + N_{12}M\,Sinh\,M - N_{22}M\left(1 - Cosh\,M\right)\right]}$$

(1.32)

The velocity distribution and α for a natural forced flow and asymmetrically heated uniform wall temperature, become

$$U = \frac{N_{12}Cosh\,MY + N_{22}Sinh\,MY + 1}{N_{12}M\,Sinh\,M - N_{22}\,M\left(1 - Cosh\,M\right) + 1}$$

(1.33)

$$\alpha = -\frac{dP}{dX} = \frac{M^2}{N_{12}M\,Sinh\,M - N_{22}M\left(1 - Cosh\,M\right) + 1}$$

(1.34)

Bulk Temperature:

The bulk temperature is defined as [1]

$$\theta_b = \frac{\int_0^1 U\theta\,dY}{\int_0^1 U\,dY}$$

(1.35)

Substituting (1.28) and (1.32) in (1.35), the expression for bulk temperature θ_b becomes

θ_b= TN/TD

(1.36)

Where

TN= $[C_1\{M \text{ Sinh } M + (r_T\text{-}1)(-1+\text{Cosh } M)\} + C_2\{M(\text{Cosh } M\text{-}r_T)$
$+(r_T\text{-}1) \text{ Sinh } M + (r_T^2 + r_T + 1)\, Gr/(3Re) + (r_T+1)\,\sigma/2]$

TD = $[M\{C_1 \text{ Sinh } M + C_2 (\text{Cosh } M\text{-}1)\} + (r_T+1)\, Gr/(2Re) + \alpha]$

The bulk temperature θ_b^* for natural forced flow is obtained by letting Gr/Re = 0 in (2.36).

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$\theta_b^* = TN_1/TD_1$

(1.37)

where

$TN_1 = [C_{11}\{M \text{ Sinh } M + (r_T-1)(-1+\text{Cosh } M)\} + C_{21}\{M(\text{Cosh } M - r_T) + (r_T-1)\text{ Sinh } M\} + (r_T+1)\alpha^*/2]$

$TD_1 = [M\{C_{11}\text{ Sinh } M + C_{21}(\text{Cosh } M-1)\} + \alpha^*]$

Where

$$C_{11} = \frac{\left[\left(M\,Cosh M + \alpha^1\sigma_2^2 Sinh M\right)\alpha^1\sigma_1^4 U_{11} - M\alpha^1\sigma_2^4 U_{21}\right]}{M^2 D}$$

$$C_{21} = \frac{-\left[\left(M\,Sinh M + \alpha^1\sigma_2^2 Cosh M\right) + \alpha^1\,\sigma_1^4 U_1 + \alpha^1\sigma_1^2\sigma_2^4 U_2\right]}{M^2 D}$$

$$\alpha^* = \frac{M^2}{N_{12} M\,Sinh M - N_{22} M\left(1 - Cosh M\right) + 1}$$

Velocity Distribution in the lack of Pressure Forces:

The first two terms on the right side of equation (1.21) represent the buoyancy and pressure forces in that order. Both the forces have same significance. The velocity distribution for combined convection in the absence of pressure force is acquired by putting $\alpha = 0$ in (1.29) and is given by

$U = C_{31}\text{ Cosh } MY + C_{32}\text{ Sinh } MY + [(1-r_T)Y + r_T]\,(Gr/Re)/M^2$ (1.38)

Where

$$C_{31} = \frac{\left[\left(M\ Cosh\ M + \alpha^1\sigma_2^2 Sinh\ M\right)\{r_T\left(1+\alpha^1\sigma_1^2\right) - 1\} - M\left(\alpha^1\sigma_1^2 + r_T - 1\right)\right](Gr/\text{Re})}{M^2 D}$$

$$C_{32} = \frac{-\left[\left(M\,Sinh M + \alpha^1\sigma_2^2 Cosh M\right)\{r_T\left(1+\alpha^1\sigma_1^2\right) - 1\} + \alpha^1\sigma_1^2\left(\alpha^1\sigma_1^2 + r_T - 1\right)\right](Gr/\text{Re})}{M^2 D}$$

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3. RESULTS AND DISCUSSIONS :

A graphical representation of stream-wise velocity profile, from (1.32) is given in the model 1.1 for different values of wall temperature r_T ratios, magnetic field parameter (M), porosity parameters σ_1 and σ_2 and Gr/Re. Figures 1.2(a) to 1.7(c), show that for fixed r_T, M, σ_1 and σ_2 values and increase in Gr/Re, the profiles of velocity turn increasingly skewed. The increased positive velocity near the hot wall (i.e. Y=1) and reduced velocity near the cold wall (i.e., Y=0) characterize the skew-ness.

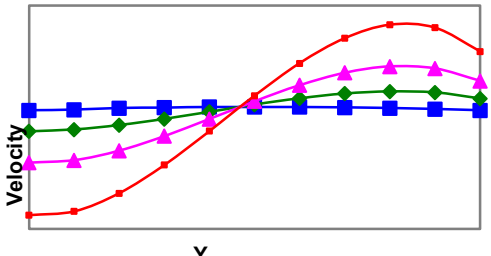

Figure1.2(a): Velocity profiles for fixed M=1, r_T=0 and σ_1= σ_2=5

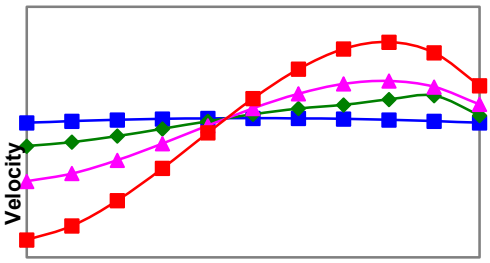

Figure 1.2(b): Velocity profiles for fixed M=1, r_T=0.3 and σ_1= σ_2=5

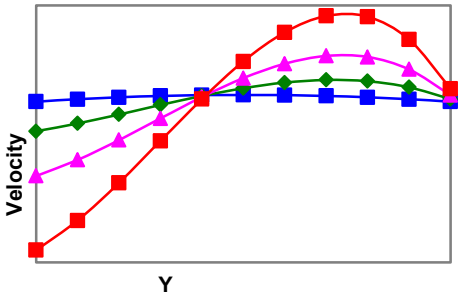

Figure 1.2(c): Velocity profiles for fixed M=1, r_T=0.5 and σ_1= σ_2=5

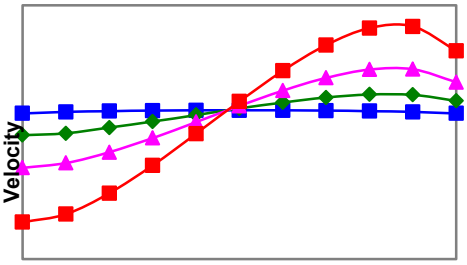

Figure 1.3(a): Velocity profiles for fixed M=5, r_T=0 and σ_1= σ_2=5

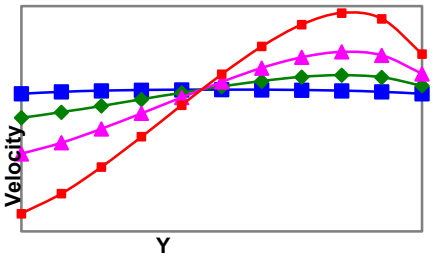

Figure 1.3(b): Velocity profiles for fixed M=5, r_T=0.3 and σ_1= σ_2=5

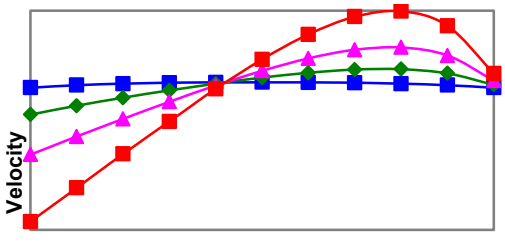

Figure 1.3(c): Velocity profiles for fixed M=5, r_T=0.5 and σ_1= σ_2=5

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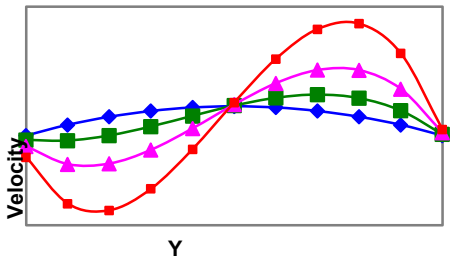

Figure 1.4(a): Velocity profiles for fixed M=1, r_T=0 and σ_1= σ_2=25

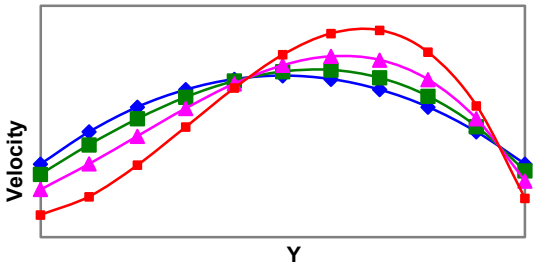

Figure 1.4(b): Velocity profiles for fixed M=1, r_T=0.3 and σ_1= σ_2=25

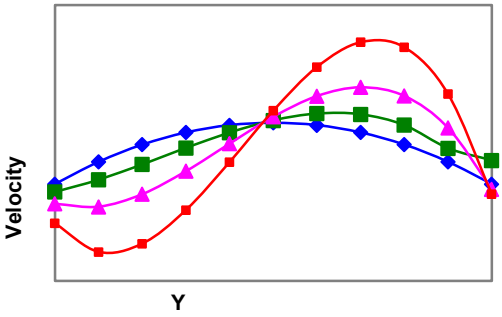

Figure 1.4(c): Velocity profiles for fixed M=1, r_T=0.5 and σ_1= σ_2=25

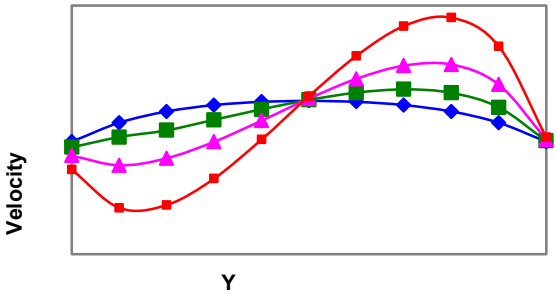

Figure 1.5(a): Velocity profiles for fixed M=5, r_T=0 and σ_1= σ_2=25

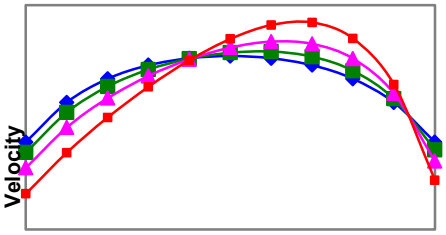

Figure 1.5(b): Velocity profiles for fixed M=5, r_T=0.3 and σ_1= σ_2=25

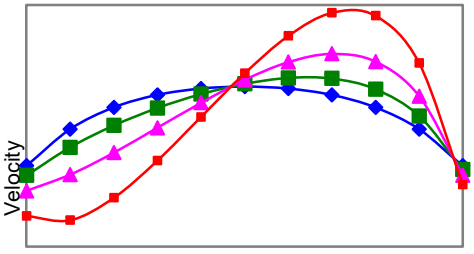

Figure 1.5(c): Velocity profiles for fixed M=5, rT=0.5 and s1= s2=25

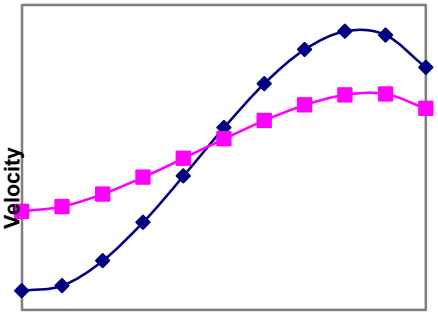

Figure 1.6(a): Velocity profiles for fixed Gr/Re=500, r_T=0 and σ_1= σ_2=5

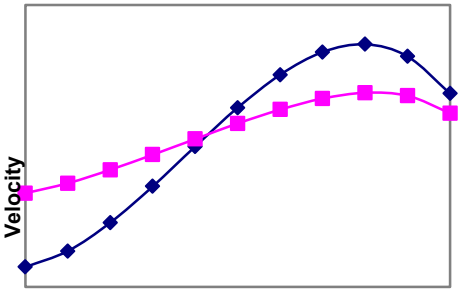

Figure 1.6(b): Velocity profiles for fixed Gr/Re=500, r_T=0.3 and σ_1= σ_2=5

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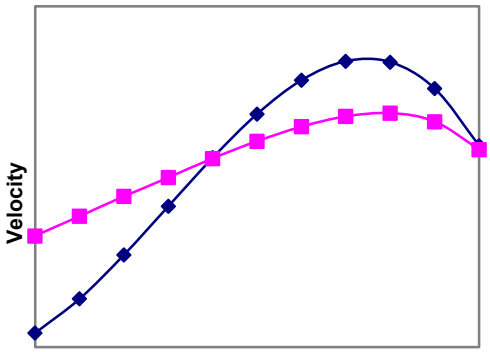

Figure 1.6(c): Velocity profiles for fixed Gr/Re=500, r_T=0.5 and σ_1= σ_2=5

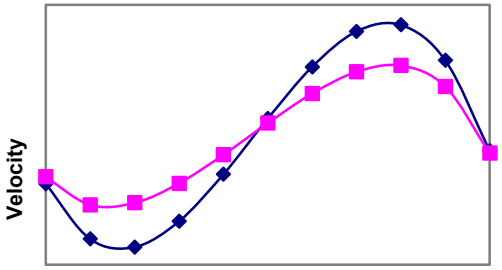

Figure 1.7(a): Velocity profiles for fixed Gr/Re=500, r_T=0 and σ_1= σ_2=25

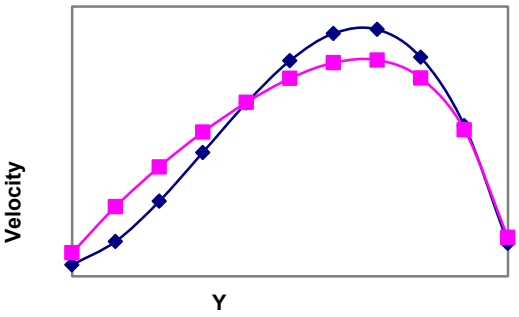

Figure 1.7(b): Velocity profiles for fixed Gr/Re=500, r_T=0.3 and σ_1= σ_2=25

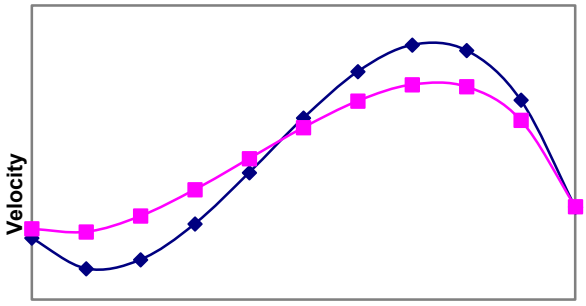

Figure 1.7(c): Velocity profiles for fixed Gr/Re=500, r_T=0 and σ_1= σ_2=25

The figures 1.2(a) to 1.3(c) show that the velocity profiles for Gr/Re=0 and for small values of σ_1 and σ_2 are nearly straight lines at constant velocity. When σ_1 and σ_2 are sufficiently large and Gr/Re=0 the velocity profiles develop into parabolic (figures 1.4(a) to 1.5(c)) and attain the maximum velocity near y=0.5. In light of this, the velocity profiles in a natural forced flow or asymmetric heated wall temperature channel are always positive, regardless of the magnitude of the magnetic field parameter (M), the porosity parameters (σ_1 and σ_2), and any other relevant factors. There is no flow reversal in fully developed flow (FDF) in both magnetic and non-magnetic case.

Figures 1.6(a) to 1.7(c) show that increasing the magnetic field and porosity parameters σ_1 and $\sigma_2$can lessen the negativeness of the velocity profile (flow reversal) close to the wall. The negativity of the velocity profile close to the cold wall gradually diminishes as rT rises.

When the magnetic field parameter (M) is increased, the pressure gradient values go up, as shown in figures 1.8 and 1.9. This is because rT = 1 and σ_1 and σ_2 are always the same.

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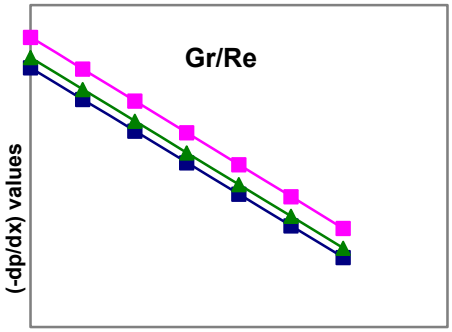

Figure 1.8: Pressure Gradient for fixed $\sigma_1 = \sigma_2 = 5$ and $r_T = 1$, $\alpha_1 = 0.01$

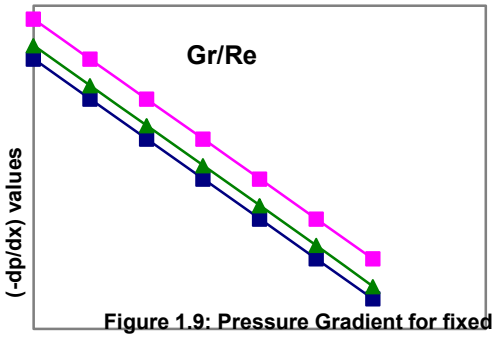

Figure 1.9: Pressure Gradient for fixed $\sigma_1 = \sigma_2 = 25$ and $r_T = 1$, $\alpha_1 = 0.01$

Graphs 1.10 and 1.11 are drawn for bulk temperature θ_b at fixed porosity parameters σ_1 and σ_2 and for various values of r_T and M. It is seen that introduction of magnetic field enhances the bulk temperature θ_b but it starts declining as the magnetic parameter (M) increases for constant porosity parameters. θ_b enhances with increasing r_T (≤ 1). θ_b is gradually increasing with Gr/Re for $r_T \leq 0.8$ for $0.8 \leq r_T \leq 1$, the bulk temperature θ_b profile almost becomes a straight line.

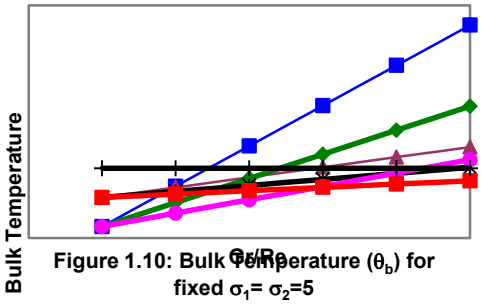

Figure 1.10: Bulk Temperature (θ_b) for fixed $\sigma_1 = \sigma_2 = 5$

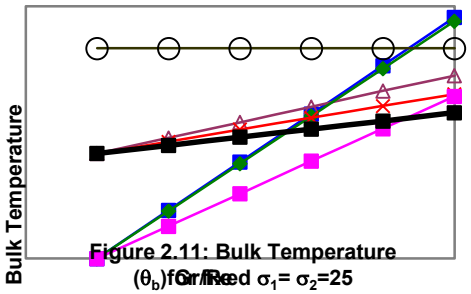

Figure 2.11: Bulk Temperature (θ_b) for fixed $\sigma_1 = \sigma_2 = 25$

4. CONCLUSIONS

Mixed convection flow in a vertical channel bordered by porous media with variable permeabilities is investigated in this work as a function of magnetic field strength. In a magnetic example, the equations for bulk temperature and velocity may be expressed. It also looks at how the pressure gradient α=-dp/dx, the porosity parameters π1 and π2, the ratio of wall temperature differences (rT), the Grashoff number and Reynolds number (Gr/Re), and the magnetic field parameter (M) affect the speed and temperature of the bulk.

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HANKEL AND TOEPLITZ DETERMINANTS FOR STARLIKE AND CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS RELATED TO THREE LEAF DOMAIN

S. SAMBASIVA RAO[1,*], R. BHARAVI SHARMA[2], R. RUDRANI[3], K. GANESH[4]

ABSTRACT. The aim of this paper is to estimate upper bounds for Hankel and Toeplitz determinants of starlike functions with respect to symmetric points associated with three leaf function that maps the open unit disk in the complex plane onto a three leaf domain with suitable illustrative examples in support of sharpness of certain proven inequalities.

1. INTRODUCTION

1.1. **Preliminaries.** Let $\mathcal{A}$ be the family of analytic functions f defined on the open unit disk Δ in the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. The collection of univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. The well known classes of starlike, convex and bounded turning functions are respectively denoted by $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$, are subclasses of $\mathcal{S}$. The Maclaurin series expansion of f be of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ for all } z \in \Delta. \tag{1.1}$$

The q^{th} Hankel determinant of index $n \geq 1$ for $f \in \mathcal{A}$, is denoted by$H_{q,n}(f)$ or $H_q(n)(f)$, and is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}, \tag{1.2}$$

where $q \geq 2$ and $a_1 = 1$ (see [2]), whereas q^{th} symmetric Toeplitz determinant $T_q(n)$ for $f \in \mathcal{A}$ defined as (see [13]),

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix}. \tag{1.3}$$

Let B_o be the collection of analytic functions $w : \Delta \to \mathbb{C}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \Delta$. The members of B_o are called Schwarz functions. $f \in \mathcal{A}$ subordinate to $g \in \mathcal{A}$ if there exists a $w \in B_0$ such that $f(z) = g(w(z))$ for all

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$z \in \Delta$. In this case, we write $f \prec g$. If g is univalent, then $f \prec g$ if and only if $g(0) = f(0)$ and $f(\Delta) \subset g(\Delta)$.

1.2. **Literature Review and Motivation.** The study of Hankel and Toeplitz determinants associated with $f \in \mathcal{S}$ play a vital role in the field of Geometric Function theory (see [18]). The pioneering works of Pommerenke, Hayman, Babalola, Zaprawa, Kowalezyk and Oh Sang Kown concerning Hankel determinants for $f \in \mathcal{S}$ motivated numerous other researchers to investigate $H_2(2)$ and $H_3(1)$ for various other subclasses of $\mathcal{S}$ (see [21] and references therein). Similar type of studies concerning Toeplitz determinant can be seen in ([3], [13], [23]).
The concept of starlike and convex functions with respect to symmetric points were introduced by Sakaguchi ([9]) and Das et al.([20]). In continuation to these notions, utilizing the concept of subordination, Ravichandran ([24]) introduced classes

$$\mathcal{S}_s^*(\varphi) = \{f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z)\},$$

$$\mathcal{C}_s(\varphi) = \{f \in \mathcal{S} : \frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec \varphi(z)\}$$

where $\varphi \in \mathcal{A}$ such that $\Re\{\varphi(z))\} > 0$, $\varphi'(0) > 0$, $\varphi(\Delta)$ is symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$.

Bharavi Sharma et al. estimated an upper bound of Hankel determinants for f in $\mathcal{S}_s^*(\varphi)$ and in $\mathcal{C}_s(\varphi)$ for various choices of φ (see [19],[4] and [17]).

Recently, Gandhi ([22]) introduced and studied radius estimates for $f \in \mathcal{S}$ subordinate to $\varphi_{3L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$ which maps Δ onto the interior of three leaf domain as shown in the Figure 1 (generated by using online software provided by www.singsurf.org). Further, Lei Shi et al. and Murugusundaramoorthy et al. studied upper bound of $H_3(1)$ for f in $\mathcal{S}^*(\varphi_{3L})$, $\mathcal{C}(\varphi_{3L})$ and $\mathcal{R}(\varphi_{3L})$ (see ([11]) and [5]).

Motivated by earlier mentioned research work, upper bounds of initial coefficients, $H_3(1)$ and $T_3(1)$ for f in $\mathcal{S}_s^*(\varphi_{3L})$, $\mathcal{C}_s^*(\varphi_{3L})$ are obtained in this paper. These classes are defined as

$$S_s^*(\varphi_{3L}) = \{f \in \mathcal{A} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi_{3L}(z)\},$$

$$C_s(\varphi_{3L}) = \{f \in \mathcal{A} : \frac{2\big(zf'(z)\big)'}{f(z) + f(-z)} \prec \varphi_{3L}(z)\}.$$

2. A Set of Useful Lemmas

The collection $\mathcal{P}$ consists of analytic functions $p : \Delta \to \mathbb{C}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$. The following lemmas concerning $p \in \mathcal{P}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ for all } z \in \Delta. \tag{2.1}$$

are useful in the sequel.

Lemma 2.1. *([16]) $|c_n| \leq 2$ for any positive integer n. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$.*

Lemma 2.2. *([12]) for any $\mu \in \mathbb{C}$, $|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.*

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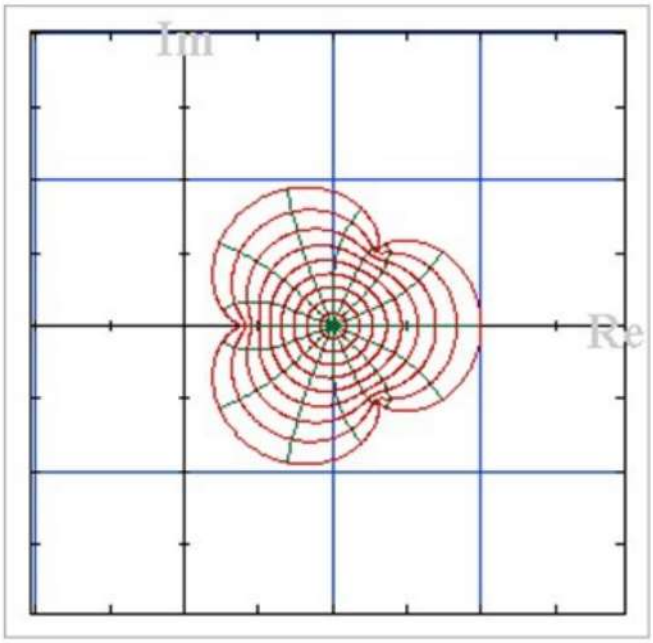

FIGURE 1. The Image of Δ under $\varphi_{3L}(z)$

Lemma 2.3. *([14]) $|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|$ for real numbers J, K and L.*

Lemma 2.4. *([10])*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.5. *([25]) If m, n, l and the inequalities $0 < m < 1$, $0 < r < 1$ and*
(2.2)
$$8r(1-r)\big((mn-2l)^2+(m(r+m)-n)^2\big)+m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2r(1-r).$$

hold. If $p(z) \in \mathcal{P}$ is of the form (2.1). then

$$|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4| \leq 2. \tag{2.3}$$

Lemma 2.6. *(Ravichandran et al.,2015) For all positive integers n, m*

$$|\mu c_n c_m - c_{n+m}| = \begin{cases} 2 & \text{if } 0 \leq \mu \leq 1 \\ 2|2\mu - 1| & \text{otherwise.} \end{cases}$$

This inequality is sharp.

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3. Main Results

If $f \in S_s^*(\varphi_{3L})$, then there exists $w \in B_0$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = \varphi_{3L}(w(z))$$

for all $z \in \Delta$. If we take $p(z) = \frac{1+w(z)}{1-w(z)}$ for all $z \in \Delta$, then $p \in \mathcal{P}$ and $w(z) = \frac{p(z)-1}{p(z)+1}$ so that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4 \text{ for all } z \in \Delta. \tag{3.1}$$

$$\begin{aligned}\text{But, } \frac{2zf'(z)}{f(z)-f(-z)} =& 1 + 2a_2z + 3a_3z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 \\ &+ (6a_6 - 4a_4a_3 - 2a_2a_5 + 2a_2a_3^2)z^5 + (6a_7 - 6a_3a_5 + 2a_3^2)z^6 + \dots, \end{aligned} \tag{3.2}$$

$$\begin{aligned}1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4 =& 1 + \frac{2}{5}c_1z + \frac{2}{5}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{10}\left(c_1^3 - 4c_1c_2 + 4c_3\right)z^3 \\ &+ \frac{2}{5}\left(c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{3}{32}c_1^4\right)z^4 + \dots \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3), we obtain

$$a_2 = \frac{c_1}{5}, \tag{3.4}$$

$$a_3 = \frac{1}{5}\left(c_2 - \frac{c_1^2}{2}\right), \tag{3.5}$$

$$a_4 = \frac{1}{200}\left(3c_1^3 - 16c_1c_2 + 20c_3\right), \tag{3.6}$$

$$a_5 = \frac{-1}{10}\left(\frac{7}{160}c_1^4 + \frac{3}{10}c_2^2 + c_1c_3 - \frac{11}{20}c_1^2c_2 - c_4\right). \tag{3.7}$$

$$a_6 = \frac{-1}{150}\left(\frac{47}{160}c_1^5 + \frac{23}{10}c_1c_2^2 - \frac{41}{10}c_1^3c_2 - c_1c_4\right) + \frac{1}{75}c_2c_3 \tag{3.8}$$

$$\begin{aligned}a_7 =& \frac{1}{24000}(935c_1^6 - 18504c_1^2c_2^2 - 8000c_3^2 - 1245c_1^4c_2 + 18000c_1c_2c_3 - 5230c_1^3c_3 \\ &+ 1920c_2^3 - 11200c_2c_4 + 12000c_1^2c_3 + 1000c_1^3c_2 - 16000c_1c_2 + 16000c_6)\end{aligned} \tag{3.9}$$

Example 3.1. *For each positive integer n, $w(z) = z^n$ are in B_0 and hence the functions $f_n(z)$ satisfying $\left(\frac{2zf_n'(z)}{f_n(z)-f_n(-z)}\right) = \varphi_{3L}(z^n)$ for all $z \in \Delta$ are in $S_s^*(\varphi_{3L})$. In particular,*

1. $f_1(z) = z + \frac{2}{5}z^2 + \frac{2}{5}z^3 - \frac{1}{20}z^5 + \dots,$
2. $f_2(z) = z + \frac{2}{5}z^3 + \frac{2}{25}z^5 + \dots,$
3. $f_3(z) = z + \frac{z^4}{5} + \dots,$
4. $f_4(z) = z + \frac{1}{5}z^5 + \frac{1}{50}z^9 + \dots$

are few members of $S_s^*(\varphi_{3L})$.

The proofs of the following results can be seen in [6].

Theorem 3.1. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_2| \le \frac{2}{5}$, $|a_3| \le \frac{2}{5}$, $|a_4| \le \frac{1}{5}$ and $|a_5| \le \frac{1}{5}$. These inequalities are sharp.*

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Theorem 3.2. *Let $f \in S_s^*(\varphi_{3L})$. Then for any $\mu \in \mathbb{C}$,*

$$|a_3 - \mu a_2^2| \leq \frac{2}{5} \max\{1, \frac{2|\mu|}{5}\}.$$

this inequality is sharp.

Theorem 3.3. *Let $f \in S_s^*(\varphi_{3L})$. Then $|H_2(2)| \leq \frac{4}{25}$ and $f_1(z)$ as in Example (3.1) is an extremal function for this inequality.*

Theorem 3.4. *Let $f \in S_s^*(\varphi_{3L})$. Then $|H_2(3)| \leq 0.044$.*

Theorem 3.5. *Let $f \in S_s^*(\varphi_{3L})$. Then $|H_3(1)| \leq 0.047$.*

The authors proved the following results in [7]

Theorem 3.6. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_6| \leq \frac{138}{375} = 0.368$.*

Theorem 3.7. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_7| \leq \frac{480960}{240000} = 22.004$.*

Theorem 3.8. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_4 - a_2 a_3| \leq \frac{1}{5}$. The result is sharp.*

Theorem 3.9. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|a_5 - a_3^2| \leq \frac{1}{5}$ and $|a_7 - a_4^2| \leq \frac{29069}{11250}$*

Theorem 3.10. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|a_5 - a_2 a_4| \leq \frac{1}{5}$ and $|a_6 - a_3 a_4| \leq \frac{1651}{1500}$.*

Theorem 3.11. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|H_{4,1}(f)| \leq 0.2305096$.*

Theorem 3.12. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|T_4(1)| \leq \frac{1318}{625} \approxeq 2.1088$.*

Unless otherwise stated in what follows, series expansions of $f \in S_s^*(\varphi_{3L})$ is of the form (1.1), $p \in \mathcal{P}$ of the form (2.1) and the initial coefficients a_2, a_3, a_4, a_5,a_6 and a_7 of $f \in S_s^*(\varphi_{3L})$ are as in the Equations (3.4), (3.5), (3.6), (3.7), (3.6) and (3.7) respectively. Thus, we obtain

Theorem 3.13. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{2,4}(f)| \leq \frac{92243}{101250} \approxeq 0.911041973$.*

Proof. Let $f \in S_s^*(\varphi_{3L})$. A simple computation by using (3.6), (3.7),and (3.6) we obtain

$$\begin{aligned}
|H_{2,4}(f)| &= |a_4 a_6 - a_5^2| \\
&= \frac{1}{1500}\Big|\frac{1863}{25600}c_1^8 + \frac{73}{80}c_1^4 c_2^2 - \frac{503}{320}c_1^6 c_2 - \frac{679}{100}c_1^2 c_2^3 \\
&\quad + \frac{27}{20}c_2^4 + 15c_1^2 c_3^2 + 15c_4^2 + \frac{163}{160}c_1^5 c_3 - 31c_1 c_3 c_4 - 9c_2^2 c_4 \\
&\quad - \frac{4}{5}c_1^2 c_2 c_4 + \frac{39}{10}c_1 c_2^2 c_3 - 2c_2 c_3^2 - \frac{110}{25}c_1^3 c_2 c_3 - \frac{3}{20}c_1^4 c_4\Big| \\
|H_{2,4}(f)| &\leq \frac{1}{1500}\Big(\frac{503}{320}|c_1|^6\left|c_2 - \frac{1863}{40240}c_1^2\right| + \frac{3}{20}|c_1|^4\left|c_4 - \frac{73}{12}c_2^2\right| \\
&\quad + \frac{110}{25}|c_1|^3|c_3|\left|c_2 - \frac{163}{704}c_1^2\right| + \frac{27}{20}|c_2|^3\left|c_2 - \frac{679}{135}c_1^2\right| \\
&\quad + 31|c_1||c_3|\left|c_4 - \frac{15}{31}c_1 c_3\right| + 9|c_2|^2\left|c_4 - \frac{39}{90}c_1 c_3\right| \\
&\quad + 15|c_4|^2 + \frac{4}{5}|c_1|^2||c_2||c_4| + 2|c_2||c_3|^2\Big)
\end{aligned}$$

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By grouping the terms as above and utilizing Lemma 2.1, Lemma 2.2 and Lemma 2.6 we obtain

$$|H_{2,4}(f)| \leq \frac{92243}{101250} \approxeq 0.911041973.$$

□

Theorem 3.14. *If $f \in S_s^*(\varphi_{3L})$, then $|T_2(1)| \leq \frac{29}{25}$ and $|T_2(2)| \leq \frac{7}{50}$. First inequality is sharp.*

Proof. As $f \in \mathcal{S}_s^*(\varphi_{3L})$, we have

$$|a_2| \leq \frac{5}{12} \text{ and } |a_3| \leq \frac{5}{12}$$

Applying the Triangle inequality and using Lemma 2.1, we obtain

$$\begin{aligned} |T_2(1)| = |1 - a_2^2| &\leq 1 + |a_2|^2 \\ &\leq 1 + \left(\frac{2}{5}\right)^2 = \frac{29}{25}. \end{aligned}$$

The inequality is sharp for $f_5(z) = z + \frac{2i}{5}z^2 + \frac{1}{20}z^5 + \cdots \in \mathcal{S}_s^*(\varphi_{3L})$. (Which is obtained by taking $w(z) = iz$ in the Equation (3.2))
On substituting a_2 and a_3 in $|T_2(2)| = |a_2^2 - a_3^2|$ followed by expressing values of c_2 and c_1 as in Lemma 2.4, we get

$$\begin{aligned} |T_2(2)| &= \frac{1}{100}\left|4c_1^2 - x^2(4 - c_1^2)\right| \\ &\leq \frac{1}{100}\left(4c^2 + t^2(4 - c^2)\right), \end{aligned}$$

where $t = |x| \in [0, 1]$ and $c = |c_1| \in [0, 2]$. It is clear that $F(c, t) = 4c^2 + t^2(4 - c^2)$ attains its maximum at the point $(2, 1)$ and hence $|T_2(2)| \leq F(1, 2) = \frac{7}{50}$. □

3.1. Upper Bound of Third-Order Hankel and Toeplitz determinants for $f \in S_s^*(\varphi_{3L})$.

Theorem 3.15. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{3,2}(f)| \leq 0.10368$.*

Proof. Let $f \in \mathcal{S}_s^*(\varphi_{3L})$. By using (3.4) - (3.7), we obtain

$$\begin{aligned} |a_2a_5 - a_3a_4| &= \frac{1}{1000}\left|-10c_1c_2^2 - \frac{5}{8}c_1^5 + 10c_1^2c_3 + 20c_2c_3 - 20c_1c_4\right| \\ &\leq \frac{1}{1000}\left(\frac{5}{8}|c_1|^2 + 20|c_2|\left|c_3 - \frac{1}{2}c_1c_2\right| + 20|c_1|\left|c_4 - \frac{1}{2}c_1c_3\right|\right) \\ &\leq \frac{1}{50} + \frac{2}{25} + \frac{2}{25} = \frac{9}{50}. \end{aligned}$$

followed by Lemmas 2.1 and 2.5. A simple computation by using the fact that $|a_2a_5 - a_3a_4| \leq \frac{9}{50}$ and using the bounds obtained in Theorem 3.1, Theorem 3.3,

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Theorem 3.4 and Theorem 3.6 , we get

$$\begin{aligned}|H_{3,2}(f)| &= \left|a_6(a_2a_4 - a_3^2) - a_5(a_2a_5 - a_3a_4) + a_4(a_3a_5 - a_4^2)\right| \\ &\leq |a_6||a_2a_4 - a_3^2| + |a_5||a_2a_5 - a_3a_4| + |a_4||a_3a_5 - a_4^2| \\ &\leq \frac{138}{375}\left(\frac{4}{25}\right) + \frac{1}{5}\left(\frac{9}{50}\right) + \frac{1}{50}(0.044) \\ &\approxeq 0.0.10368.\end{aligned}$$

□

Theorem 3.16. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{3,3}(f)| \leq 0.1570656$.*

Proof. Let $f \in \mathcal{S}_s^*(\varphi_{3L})$. By using Theorems 3.1, 3.6 , we obtain

$$\begin{aligned}|a_3a_6 - a_4a_5| &= |a_3||a_6| + |a_4||a_5| \\ &\leq 0.1872.\end{aligned}$$

By utilizing Theorem 3.1, Theorem 3.4 and Theorem 3.13 and followed by the result $|a_3a_6 - a_4a_5| \leq 0.1872$, we obtain

$$\begin{aligned}|H_{3,3}(f)| &= \left|a_7(a_3a_5 - a_4^2) - a_6(a_3a_6 - a_4a_5) + a_5(a_4a_6 - a_5^2)\right| \\ &\leq |a_7||a_3a_5 - a_4^2| + |a_6||a_3a_6 - a_4a_5| + |a_5||a_4a_6 - a_5^2| \\ &\leq 0.1570656.\end{aligned}$$

□

Theorem 3.17. *Let $f \in S_s^*(\varphi_{3L})$. Then $|T_3(1)| \leq \frac{37}{25}$.*

Proof. From Theorem (3.2),we have $|a_3 - 2a_2^2| \leq \frac{2}{5}$.
Thus, $|T_3(1)| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2| \leq 1 + \frac{8}{25} + \frac{4}{25} = \frac{37}{25}$. □

Theorem 3.18. *Let $f \in S_s^*(\varphi_{3L})$. Then $|T_3(2)| \leq \frac{9}{50}$.*

Proof. Using the bounds $|a_2| \leq \frac{2}{5}$, $|a_4| \leq \frac{1}{5}$, $|T_2(2)| \leq \frac{7}{50}$ and $|H_2(2)| \leq \frac{4}{25}$, we get $|T_3(2)| = |a_2 - a_4|\left|T_2(2) + H_2(2)\right| \leq \left(|a_2| + |a_4|\right)\left(|T_2(2)| + |H_2(2)|\right) \leq \frac{9}{50}$. □

Theorem 3.19. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|T_4(2)| \leq 0.162736$.*

Proof. By definition of $T_4(2)$ for $f \in \mathcal{A}$ of the form (1.1), we have

$$T_4(2) = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_3 \\ a_5 & a_4 & a_3 & a_2 \end{vmatrix} \tag{3.10}$$

Upon simplification, we get

$$T_4(2) = \left(a_2^2 - a_3^2\right)^2 - \left(a_3a_4 - a_2a_5\right)^2 + \left(a_4^2 - a_3a_5\right)^2 - \left(a_2a_3 - a_3a_4\right)^2 + 2\left(a_3^2 - a_2a_4\right)\left(a_2a_4 - a_3a_5\right)$$

Since $f \in \mathcal{S}_s^*(\varphi_{3L})$, in view of Theorems 3.4, 3.3 , 3.15 and 3.14 we have $|T_2(2)| \leq \frac{7}{50}$, $|H_{2,3}(f)| \leq 0.044$, $|H_{2,2}(f)| \leq \frac{4}{25}$ and $|a_3a_4 - a_2a_5| \leq \frac{9}{50}$ respectively.
Further,

$$|a_2a_3 - a_3a_4| \leq |a_3|(|a_2| + |a_4|) \leq \frac{6}{25},$$

$$|a_2a_4 - a_3a_5| \leq |a_2||a_4| + |a_3||a_5| \leq \frac{4}{25}.$$

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Therefore, we obtain

$$\begin{aligned}|T_4(2)| &\leq |T_2(2)|^2 + |a_3a_4 - a_2a_5|^2 + |H_{2,3}(f)|^2 + |a_2a_3 - a_3a_4|^2 \\ &\quad + 2|H_{2,2}(f)||a_2a_4 - a_3a_5| \\ &\leq \left(\frac{7}{50}\right)^2 + \left(\frac{9}{50}\right)^2 + \left(\frac{44}{1000}\right)^2 + \left(\frac{6}{25}\right)^2 + 2\left(\frac{4}{25}\right)\left(\frac{4}{25}\right). \\ &\leq 0.162736. \end{aligned}$$ □

This completes the proof.

4. Initial Coefficient and Determinant inequalities for $f \in C_s(\varphi_{3L})$

Remark 4.1. *It is clear that $g \in C_s(\varphi_{3L})$ if and only if $f(z) = zg'(z) \in S_s^*(\varphi_{3L})$. Let $g \in C_s(\varphi_{3L})$ be of the form*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \text{ for all } z \in \Delta. \tag{4.1}$$

Then there exists $f(z) \in S_s^*(\varphi_{3L})$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = zg'(z) = z + \sum_{n=2}^{\infty} n b_n z^n$$

for all $z \in \Delta$. Hence, $b_n = \frac{a_n}{n}$ for all n.

Example 4.1. *For each positive integer n, $w(z) = z^n$ are in B_0 and hence the functions $g_n(z)$ satisfying $\frac{2\left(zg_n'(z)\right)'}{g_n(z)+g_n(-z)} = \varphi_{3L}(z^n)$ for all $z \in \Delta$ are in $C_s(\varphi_{3L})$. In particular,*

(1) $g_1(z) = z + \frac{1}{5}z^2 + \frac{2}{15}z^3 - \frac{1}{100}z^5 + \ldots,$
(2) $g_2(z) = z + \frac{2}{15}z^3 + \frac{1}{100}z^5 + \ldots,$
(3) $g_3(z) = z + \frac{z^4}{20} + \ldots$
(4) $g_4(z) = z + \frac{1}{25}z^5 + \frac{1}{450}z^9 + \ldots$

are in $C_s(\varphi_{3L})$.

Throughout this section $g \in C_s(\varphi_{3L})$, we mean $g(z)$ is of the form (4.1). We now prove the following

Theorem 4.1. *If $g \in C_s(\varphi_{3L})$, then*

(1) $|b_2| \leq \frac{1}{5}$, $|b_3| \leq \frac{2}{15}$, $|b_4| \leq \frac{1}{20}$ *and* $|b_5| \leq \frac{1}{25}$.
(2) $|b_3 - \mu b_2^2| \leq \frac{2}{15}\max\{1, |\frac{3\mu}{2} - 1|\}$ *for any complex parameter μ and the inequality is sharp.*
(3) $|H_2(1)| \leq \frac{2}{15}$.

Proof. Proof of (1) and (2) follows by applying Theorems (3.1) and (3.2) respectively, in view of Remark (4.1). If $\mu = 1$ in (2), then $|H_2(1)| \leq \frac{2}{15}$. □

Remark 4.2. *The functions $g_1(z)$ to $g_4(z)$ as in Example (4.1) are extremal functions for the inequalities $|b_2| \leq \frac{1}{5}$, $|b_3| \leq \frac{2}{15}$, $|b_4| \leq \frac{1}{20}$ and $|b_5| \leq \frac{1}{25}$ respectively.*

Theorem 4.2. *If $g \in C_s(\varphi_{3L})$, then $|H_2(2)| \leq \frac{4}{225}$ and the function $g_2(z)$ as in Example (4.1) is an extremal function for this inequalities.*

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Proof. In view of Remark (4.1) and expressing $a_i's$ in terms of $c_i's$ as in Lemma(2.4), we get

$$\begin{aligned}|H_{2,2}(g)| &= \frac{1}{8}|a_2a_4 - \frac{8}{9}a_3^2| \\ &= \frac{1}{72000}|-53c_1^4 + 176c_1^2c_2 + 180c_1c_3 - 320c_2^2| \\ &\leq \frac{4-c^2}{72000}\Big(18c^2t + 45c^2t^2 + 80(4-c^2)t^2 + 90c(1-t^2)\Big) \\ &= F(c,t)((say)),\end{aligned}$$

where $c = |c_1| \in [0,2]$, $t = |x| \in [0,1]$. By applying the technique of finding maxima-minima, one can show that $F(c,t)$ attains its maximum at $c = 0$ and $t = 1$. Hence, $|H_2(2)| \leq F(0,1) = \frac{4}{225}$. □

Theorem 4.3. *If $g \in C_s(\varphi_{3L})$, then $|H_3(1)| \leq \frac{37}{2000}$.*

Proof. In view of Remark (4.1), substituting b_i's in terms of $a_i's$ followed by expressing them in terms c_i's as computed earlier in

$$H_3(1) = |b_5(b_3 - b_2^2) + 2b_2b_3b_4 - b_4^2 - b_4b_3^2|$$

and grouping the terms, we get

$$\begin{aligned}|H_3(1)| &= \Big|\frac{1}{750}\Big(c_2 - \frac{13}{20}c_1^2\Big)\Big(\Big(\frac{c_1}{1440}\big(143c_1^3 - 1112c_1c_2 + 1440c_3\big) - \Big(c_4 - \frac{47}{90}c_2^2\Big)\Big) \\ &\quad - \frac{1}{5760000}\big(17c_1^3 - 64c_1c_2 + 60c_3\big)^2\Big| \\ &\leq \frac{1}{750}\Big|c_2 - \frac{13}{20}c_1^2\Big|\Big(\frac{|c_1|}{1440}\big|143c_1^3 - 1112c_1c_2 + 1440c_3\big| + \Big|c_4 - \frac{47}{90}c_2^2\Big|\Big) \\ &\quad + \frac{1}{5760000}\big|17c_1^3 - 64c_1c_2 + 60c_3\big|^2.\end{aligned}$$

In view of Lemmas 2.1, 2.2 and 2.3, $\big|c_2 - \frac{13}{20}c_1^2\big| \leq 2$, $|c_1| \leq 2$, $\big|143c_1^3 - 1112c_1c_2 + 1440c_3\big| \leq 2880$, $\big|c_4 - \frac{47}{90}c_2^2\big| \leq 2$ and $\big|17c_1^3 - 64c_1c_2 + 60c_3\big| \leq 120$ follows. Hence, $|H_3(1)| \leq \frac{37}{2000} \approxeq 0.0185$. □

Theorem 4.4. *Let $g \in C_s(\varphi_{3L})$. Then $|T_2(1)| \leq \frac{26}{25}$ and $|T_2(2)| \leq \frac{4}{225}$ and the first result is sharp.*

Proof.

$$\begin{aligned}|T_2(1)| = |1 - a_2^2| &\leq 1 + |a_2|^2 \\ &\leq 1 + \left(\frac{1}{5}\right)^2 = \frac{26}{25}.\end{aligned}$$

The inequality is sharp for $g_5(z) = z + \frac{2i}{10}z^2 + \frac{1}{100}z^5 + \cdots \in \mathcal{S}_s^*(\varphi_{3L})$.
On substituting $b_2 = \frac{a_2}{2}$ and $b_3 = \frac{a_3}{3}$ in $|T_2(2)| = \frac{1}{36}|9a_2^2 - 4a_3^2|$ followed by expressing values of c_2 and c_1 as in Lemma 2.4, we get

$$\begin{aligned}|T_2(2)| &= \frac{1}{900}\Big|9c_1^4 - x^2\big(4 - c_1^2\big)^2\Big| \\ &\leq \frac{1}{900}\Big(9c^4 + t^2\big(4 - c^2\big)^2\Big) = F(c,t)\,(\text{say}),\end{aligned}$$

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where $t = |x| \in [0,1]$ and $c = |c_1| \in [0,2]$. It is clear that $F(c,t)$ attains its maximum at $(0,1)$ and hence $|T_2(2)| \leq F(0,1) = \frac{4}{225}$. □

Theorem 4.5. *Let $f \in C_S(\varphi_{3L})$. Then $|T_3(1)| \leq \frac{251}{225}$.*

Proof. If we choose $\mu = 2$ in (2) of Theorem (4.1), then $|b_3 - 2b_2^2| \leq \frac{4}{15}$.
Thus, $|T_3(1)| \leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2| \leq 1 + \frac{2}{25} + \frac{8}{225} = \frac{251}{225}$. □

Theorem 4.6. *Let $f \in C_s(\varphi_{3L})$. Then $|T_3(2)| \leq \frac{2}{225}$.*

Proof. Using the bounds $|b_2| \leq \frac{1}{5}$, $|b_4| \leq \frac{1}{20}$, $|T_2(2)| \leq \frac{4}{225}$ and $|H_2(2)| \leq \frac{2}{225}$, we get
$|T_3(2)| = |b_2 - b_4||T_2(2) + H_2(2)| \leq (|b_2| + |b_4|)(|T_2(2)| + |H_2(2)|) \leq \frac{2}{225}$. □

Theorem 4.7. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{4,1}(f)| \leq 0.00633595371$.*

Proof. we have

$$|H_{4,1}(f)| \leq |b_7||H_{3,1}| + 2|b_4||b_6||b_2b_4 - b_3^2| + 2|b_5||b_6|\,|b_2b_3 - b_4| + |b_6|^2\,|b_3 - b_2^2| + |b_5|^2|b_2b_4 - b_3^2| + |b_5|^2|b_2b_4 + 2b_3^2| + |b_5|^3 + |b_4|^4 + 3|b_3||b_4|^2|b_5|.$$

As $f \in S_s^*(\varphi_{3L})$, $|b_2b_4 + 2b_3^2| \leq |b_2|\,|b_4| + 2\,|b_3|^2 \leq \frac{125}{288}$ (In view of Theorem 3.1). Thus,

$$\begin{aligned}|H_{4,1}(f)| \leq & \frac{2.004}{7}\left(\frac{37}{2000}\right) + 2\left(\frac{1}{20}\right)\left(\frac{0.368}{6}\right)\left(\frac{4}{225}\right) + 2\left(\frac{1}{25}\right)\left(\frac{0.368}{6}\right)\left(\frac{1}{20}\right) \\ & + \left(\frac{0.368}{6}\right)^2\left(\frac{2}{15}\right) + \left(\frac{1}{25}\right)^2\left(\frac{4}{225}\right) + \left(\frac{1}{25}\right)^2\left(\frac{41}{900}\right) + \left(\frac{1}{25}\right)^3 \\ & + \left(\frac{1}{20}\right)^4 + 3\left(\frac{0.368}{6}\right)\left(\frac{1}{20}\right)^2\left(\frac{1}{25}\right) \\ & \leq 0.00633595371.\end{aligned}$$

follows by using the bounds obtained in Theorems 3.1, 3.3, 3.8 and 3.9. □

Theorem 4.8. *If $f \in S_s^*(\varphi_{3L})$, then $|T_4(1)| \leq 1.174016$*

Proof. By definition of $T_4(1)$ for $f \in \mathcal{A}$ of the form (1.1), we have

$$T_4(1) = \begin{vmatrix} 1 & b_2 & b_3 & b_4 \\ b_2 & 1 & b_2 & b_3 \\ b_3 & b_2 & 1 & b_2 \\ b_4 & b_3 & b_2 & 1 \end{vmatrix} \tag{4.2}$$

Upon simplification, we get

$T_4(1) = (1 - b_2^2)^2 - (b_2b_3 - b_4)^2 + (b_3^2 - b_2b_4)^2 - (b_2 - b_2b_3)^2 + 2(b_2^2 - b_3)(b_3 - b_2b_4)$.
Let $f \in S_s^*(\varphi_{3L})$. Then $|b_3^2 - b_2b_4| \leq \frac{4}{225}$ from Theorem 3.3.

$$\text{Also, } |1 - b_2^2| \leq (1 + \frac{1}{25}) = \frac{26}{25},$$
$$|b_2 - b_2b_3| \leq \frac{1}{5}(1 + \frac{2}{15}) = \frac{17}{75},$$

and

$$|b_3 - b_2b_4| \leq \frac{2}{15} + \frac{1}{5}\left(\frac{1}{20}\right) \leq \frac{43}{300}.$$

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Using these bounds along with bounds obtained in the Theorem 3.2 and Theorem 3.5 in

$$\begin{aligned}|T_4(1)| \leq & |(1-b_2^2)^2| + |b_2b_3 - b_4|^2 + |b_3^2 - b_2b_4|^2 + |b_2 - b_2b_3|^2 + 2|b_2^2 - b_3||b_3 - b_2b_4| \\ & \leq \left(\frac{26}{25}\right)^2 + \left(\frac{1}{20}\right)^2 + \left(\frac{4}{225}\right)^2 + \left(\frac{17}{75}\right)^2 + 2\left(\frac{2}{15}\right)\left(\frac{43}{300}\right) \\ & \leq 1.174016.\end{aligned}$$

This completes the proof. □

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HANKEL AND TOEPLITZ DETERMINANTS FOR STARLIKE AND CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS RELATED TO THREE LEAF DOMAIN

S. SAMBASIVA RAO[1,*], R. BHARAVI SHARMA[2], R. RUDRANI[3], K. GANESH[4]

ABSTRACT. The aim of this paper is to estimate upper bounds for Hankel and Toeplitz determinants of starlike functions with respect to symmetric points associated with three leaf function that maps the open unit disk in the complex plane onto a three leaf domain with suitable illustrative examples in support of sharpness of certain proven inequalities.

1. INTRODUCTION

1.1. **Preliminaries.** Let $\mathcal{A}$ be the family of analytic functions f defined on the open unit disk Δ in the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. The collection of univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. The well known classes of starlike, convex and bounded turning functions are respectively denoted by $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$, are subclasses of $\mathcal{S}$. The Maclaurin series expansion of f be of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ for all } z \in \Delta. \tag{1.1}$$

The q^{th} Hankel determinant of index $n \geq 1$ for $f \in \mathcal{A}$, is denoted by$H_{q,n}(f)$ or $H_q(n)(f)$, and is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}, \tag{1.2}$$

where $q \geq 2$ and $a_1 = 1$ (see [2]), whereas q^{th} symmetric Toeplitz determinant $T_q(n)$ for $f \in \mathcal{A}$ defined as (see [13]),

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix}. \tag{1.3}$$

Let B_o be the collection of analytic functions $w : \Delta \to \mathbb{C}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \Delta$. The members of B_o are called Schwarz functions. $f \in \mathcal{A}$ subordinate to $g \in \mathcal{A}$ if there exists a $w \in B_0$ such that $f(z) = g(w(z))$ for all

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$z \in \Delta$. In this case, we write $f \prec g$. If g is univalent, then $f \prec g$ if and only if $g(0) = f(0)$ and $f(\Delta) \subset g(\Delta)$.

1.2. **Literature Review and Motivation.** The study of Hankel and Toeplitz determinants associated with $f \in \mathcal{S}$ play a vital role in the field of Geometric Function theory (see [18]). The pioneering works of Pommerenke, Hayman, Babalola, Zaprawa, Kowalezyk and Oh Sang Kown concerning Hankel determinants for $f \in \mathcal{S}$ motivated numerous other researchers to investigate $H_2(2)$ and $H_3(1)$ for various other subclasses of $\mathcal{S}$ (see [21] and references therein). Similar type of studies concerning Toeplitz determinant can be seen in ([3], [13], [23]).
The concept of starlike and convex functions with respect to symmetric points were introduced by Sakaguchi ([9]) and Das et al.([20]). In continuation to these notions, utilizing the concept of subordination, Ravichandran ([24]) introduced classes

$$\mathcal{S}_s^*(\varphi) = \{f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z)\},$$

$$\mathcal{C}_s(\varphi) = \{f \in \mathcal{S} : \frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec \varphi(z)\}$$

where $\varphi \in \mathcal{A}$ such that $\Re\{\varphi(z))\} > 0$, $\varphi'(0) > 0$, $\varphi(\Delta)$ is symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$.

Bharavi Sharma et al. estimated an upper bound of Hankel determinants for f in $\mathcal{S}_s^*(\varphi)$ and in $\mathcal{C}_s(\varphi)$ for various choices of φ (see [19],[4] and [17]).

Recently, Gandhi ([22]) introduced and studied radius estimates for $f \in \mathcal{S}$ subordinate to $\varphi_{3L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$ which maps Δ onto the interior of three leaf domain as shown in the Figure 1 (generated by using online software provided by www.singsurf.org). Further, Lei Shi et al. and Murugusundaramoorthy et al. studied upper bound of $H_3(1)$ for f in $\mathcal{S}^*(\varphi_{3L})$, $\mathcal{C}(\varphi_{3L})$ and $\mathcal{R}(\varphi_{3L})$ (see ([11]) and [5]).

Motivated by earlier mentioned research work, upper bounds of initial coefficients, $H_3(1)$ and $T_3(1)$ for f in $\mathcal{S}_s^*(\varphi_{3L})$, $\mathcal{C}_s^*(\varphi_{3L})$ are obtained in this paper. These classes are defined as

$$S_s^*(\varphi_{3L}) = \{f \in \mathcal{A} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi_{3L}(z)\},$$

$$C_s(\varphi_{3L}) = \{f \in \mathcal{A} : \frac{2\big(zf'(z)\big)'}{f(z) + f(-z)} \prec \varphi_{3L}(z)\}.$$

2. A Set of Useful Lemmas

The collection $\mathcal{P}$ consists of analytic functions $p : \Delta \to \mathbb{C}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$. The following lemmas concerning $p \in \mathcal{P}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ for all } z \in \Delta. \tag{2.1}$$

are useful in the sequel.

Lemma 2.1. *([16]) $|c_n| \leq 2$ for any positive integer n. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$.*

Lemma 2.2. *([12]) for any $\mu \in \mathbb{C}$, $|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.*

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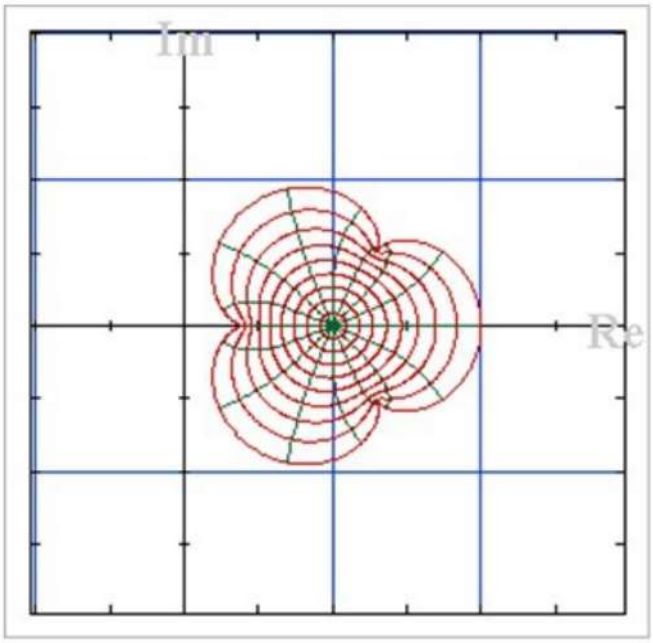

FIGURE 1. The Image of Δ under $\varphi_{3L}(z)$

Lemma 2.3. *([14]) $|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|$ for real numbers J, K and L.*

Lemma 2.4. *([10])*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.5. *([25]) If m, n, l and the inequalities $0 < m < 1$, $0 < r < 1$ and*
(2.2)
$$8r(1-r)\big((mn-2l)^2+(m(r+m)-n)^2\big)+m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2r(1-r).$$

hold. If $p(z) \in \mathcal{P}$ is of the form (2.1). then

$$|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4| \leq 2. \tag{2.3}$$

Lemma 2.6. *(Ravichandran et al.,2015) For all positive integers n, m*

$$|\mu c_n c_m - c_{n+m}| = \begin{cases} 2 & \text{if } 0 \leq \mu \leq 1 \\ 2|2\mu - 1| & \text{otherwise.} \end{cases}$$

This inequality is sharp.

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3. Main Results

If $f \in S_s^*(\varphi_{3L})$, then there exists $w \in B_0$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = \varphi_{3L}(w(z))$$

for all $z \in \Delta$. If we take $p(z) = \frac{1+w(z)}{1-w(z)}$ for all $z \in \Delta$, then $p \in \mathcal{P}$ and $w(z) = \frac{p(z)-1}{p(z)+1}$ so that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4 \text{ for all } z \in \Delta. \tag{3.1}$$

$$\begin{aligned}\text{But, } \frac{2zf'(z)}{f(z)-f(-z)} =& 1 + 2a_2z + 3a_3z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 \\ &+ (6a_6 - 4a_4a_3 - 2a_2a_5 + 2a_2a_3^2)z^5 + (6a_7 - 6a_3a_5 + 2a_3^2)z^6 + \dots, \end{aligned} \tag{3.2}$$

$$\begin{aligned}1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4 =& 1 + \frac{2}{5}c_1z + \frac{2}{5}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{10}\left(c_1^3 - 4c_1c_2 + 4c_3\right)z^3 \\ &+ \frac{2}{5}\left(c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{3}{32}c_1^4\right)z^4 + \dots \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3), we obtain

$$a_2 = \frac{c_1}{5}, \tag{3.4}$$

$$a_3 = \frac{1}{5}\left(c_2 - \frac{c_1^2}{2}\right), \tag{3.5}$$

$$a_4 = \frac{1}{200}\left(3c_1^3 - 16c_1c_2 + 20c_3\right), \tag{3.6}$$

$$a_5 = \frac{-1}{10}\left(\frac{7}{160}c_1^4 + \frac{3}{10}c_2^2 + c_1c_3 - \frac{11}{20}c_1^2c_2 - c_4\right). \tag{3.7}$$

$$a_6 = \frac{-1}{150}\left(\frac{47}{160}c_1^5 + \frac{23}{10}c_1c_2^2 - \frac{41}{10}c_1^3c_2 - c_1c_4\right) + \frac{1}{75}c_2c_3 \tag{3.8}$$

$$\begin{aligned}a_7 =& \frac{1}{24000}(935c_1^6 - 18504c_1^2c_2^2 - 8000c_3^2 - 1245c_1^4c_2 + 18000c_1c_2c_3 - 5230c_1^3c_3 \\ &+ 1920c_2^3 - 11200c_2c_4 + 12000c_1^2c_3 + 1000c_1^3c_2 - 16000c_1c_2 + 16000c_6)\end{aligned} \tag{3.9}$$

Example 3.1. *For each positive integer n, $w(z) = z^n$ are in B_0 and hence the functions $f_n(z)$ satisfying $\left(\frac{2zf_n'(z)}{f_n(z)-f_n(-z)}\right) = \varphi_{3L}(z^n)$ for all $z \in \Delta$ are in $S_s^*(\varphi_{3L})$. In particular,*

1. $f_1(z) = z + \frac{2}{5}z^2 + \frac{2}{5}z^3 - \frac{1}{20}z^5 + \dots,$
2. $f_2(z) = z + \frac{2}{5}z^3 + \frac{2}{25}z^5 + \dots,$
3. $f_3(z) = z + \frac{z^4}{5} + \dots,$
4. $f_4(z) = z + \frac{1}{5}z^5 + \frac{1}{50}z^9 + \dots$

are few members of $S_s^*(\varphi_{3L})$.

The proofs of the following results can be seen in [6].

Theorem 3.1. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_2| \le \frac{2}{5}$, $|a_3| \le \frac{2}{5}$, $|a_4| \le \frac{1}{5}$ and $|a_5| \le \frac{1}{5}$. These inequalities are sharp.*

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Theorem 3.2. *Let $f \in S_s^*(\varphi_{3L})$. Then for any $\mu \in \mathbb{C}$,*

$$|a_3 - \mu a_2^2| \leq \frac{2}{5} \max\{1, \frac{2|\mu|}{5}\}.$$

this inequality is sharp.

Theorem 3.3. *Let $f \in S_s^*(\varphi_{3L})$. Then $|H_2(2)| \leq \frac{4}{25}$ and $f_1(z)$ as in Example (3.1) is an extremal function for this inequality.*

Theorem 3.4. *Let $f \in S_s^*(\varphi_{3L})$. Then $|H_2(3)| \leq 0.044$.*

Theorem 3.5. *Let $f \in S_s^*(\varphi_{3L})$. Then $|H_3(1)| \leq 0.047$.*

The authors proved the following results in [7]

Theorem 3.6. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_6| \leq \frac{138}{375} = 0.368$.*

Theorem 3.7. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_7| \leq \frac{480960}{240000} = 22.004$.*

Theorem 3.8. *Let $f \in S_s^*(\varphi_{3L})$ be of the form (1.1). Then $|a_4 - a_2 a_3| \leq \frac{1}{5}$. The result is sharp.*

Theorem 3.9. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|a_5 - a_3^2| \leq \frac{1}{5}$ and $|a_7 - a_4^2| \leq \frac{29069}{11250}$*

Theorem 3.10. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|a_5 - a_2 a_4| \leq \frac{1}{5}$ and $|a_6 - a_3 a_4| \leq \frac{1651}{1500}$.*

Theorem 3.11. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|H_{4,1}(f)| \leq 0.2305096$.*

Theorem 3.12. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|T_4(1)| \leq \frac{1318}{625} \approxeq 2.1088$.*

Unless otherwise stated in what follows, series expansions of $f \in S_s^*(\varphi_{3L})$ is of the form (1.1), $p \in \mathcal{P}$ of the form (2.1) and the initial coefficients a_2, a_3, a_4, a_5,a_6 and a_7 of $f \in S_s^*(\varphi_{3L})$ are as in the Equations (3.4), (3.5), (3.6), (3.7), (3.6) and (3.7) respectively. Thus, we obtain

Theorem 3.13. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{2,4}(f)| \leq \frac{92243}{101250} \approxeq 0.911041973$.*

Proof. Let $f \in S_s^*(\varphi_{3L})$. A simple computation by using (3.6), (3.7),and (3.6) we obtain

$$\begin{aligned}
|H_{2,4}(f)| &= |a_4 a_6 - a_5^2| \\
&= \frac{1}{1500}\Big|\frac{1863}{25600}c_1^8 + \frac{73}{80}c_1^4 c_2^2 - \frac{503}{320}c_1^6 c_2 - \frac{679}{100}c_1^2 c_2^3 \\
&\quad + \frac{27}{20}c_2^4 + 15c_1^2 c_3^2 + 15c_4^2 + \frac{163}{160}c_1^5 c_3 - 31c_1 c_3 c_4 - 9c_2^2 c_4 \\
&\quad - \frac{4}{5}c_1^2 c_2 c_4 + \frac{39}{10}c_1 c_2^2 c_3 - 2c_2 c_3^2 - \frac{110}{25}c_1^3 c_2 c_3 - \frac{3}{20}c_1^4 c_4\Big| \\
|H_{2,4}(f)| &\leq \frac{1}{1500}\Big(\frac{503}{320}|c_1|^6\left|c_2 - \frac{1863}{40240}c_1^2\right| + \frac{3}{20}|c_1|^4\left|c_4 - \frac{73}{12}c_2^2\right| \\
&\quad + \frac{110}{25}|c_1|^3|c_3|\left|c_2 - \frac{163}{704}c_1^2\right| + \frac{27}{20}|c_2|^3\left|c_2 - \frac{679}{135}c_1^2\right| \\
&\quad + 31|c_1||c_3|\left|c_4 - \frac{15}{31}c_1 c_3\right| + 9|c_2|^2\left|c_4 - \frac{39}{90}c_1 c_3\right| \\
&\quad + 15|c_4|^2 + \frac{4}{5}|c_1|^2||c_2||c_4| + 2|c_2||c_3|^2\Big)
\end{aligned}$$

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By grouping the terms as above and utilizing Lemma 2.1, Lemma 2.2 and Lemma 2.6 we obtain

$$|H_{2,4}(f)| \leq \frac{92243}{101250} \approxeq 0.911041973.$$

□

Theorem 3.14. *If $f \in S_s^*(\varphi_{3L})$, then $|T_2(1)| \leq \frac{29}{25}$ and $|T_2(2)| \leq \frac{7}{50}$. First inequality is sharp.*

Proof. As $f \in \mathcal{S}_s^*(\varphi_{3L})$, we have

$$|a_2| \leq \frac{5}{12} \text{ and } |a_3| \leq \frac{5}{12}$$

Applying the Triangle inequality and using Lemma 2.1, we obtain

$$\begin{aligned} |T_2(1)| = |1 - a_2^2| &\leq 1 + |a_2|^2 \\ &\leq 1 + \left(\frac{2}{5}\right)^2 = \frac{29}{25}. \end{aligned}$$

The inequality is sharp for $f_5(z) = z + \frac{2i}{5}z^2 + \frac{1}{20}z^5 + \cdots \in \mathcal{S}_s^*(\varphi_{3L})$. (Which is obtained by taking $w(z) = iz$ in the Equation (3.2))
On substituting a_2 and a_3 in $|T_2(2)| = |a_2^2 - a_3^2|$ followed by expressing values of c_2 and c_1 as in Lemma 2.4, we get

$$\begin{aligned} |T_2(2)| &= \frac{1}{100}\left|4c_1^2 - x^2(4 - c_1^2)\right| \\ &\leq \frac{1}{100}\left(4c^2 + t^2(4 - c^2)\right), \end{aligned}$$

where $t = |x| \in [0, 1]$ and $c = |c_1| \in [0, 2]$. It is clear that $F(c, t) = 4c^2 + t^2(4 - c^2)$ attains its maximum at the point $(2, 1)$ and hence $|T_2(2)| \leq F(1, 2) = \frac{7}{50}$. □

3.1. Upper Bound of Third-Order Hankel and Toeplitz determinants for $f \in S_s^*(\varphi_{3L})$.

Theorem 3.15. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{3,2}(f)| \leq 0.10368$.*

Proof. Let $f \in \mathcal{S}_s^*(\varphi_{3L})$. By using (3.4) - (3.7), we obtain

$$\begin{aligned} |a_2a_5 - a_3a_4| &= \frac{1}{1000}\left|-10c_1c_2^2 - \frac{5}{8}c_1^5 + 10c_1^2c_3 + 20c_2c_3 - 20c_1c_4\right| \\ &\leq \frac{1}{1000}\left(\frac{5}{8}|c_1|^2 + 20|c_2|\left|c_3 - \frac{1}{2}c_1c_2\right| + 20|c_1|\left|c_4 - \frac{1}{2}c_1c_3\right|\right) \\ &\leq \frac{1}{50} + \frac{2}{25} + \frac{2}{25} = \frac{9}{50}. \end{aligned}$$

followed by Lemmas 2.1 and 2.5. A simple computation by using the fact that $|a_2a_5 - a_3a_4| \leq \frac{9}{50}$ and using the bounds obtained in Theorem 3.1, Theorem 3.3,

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Theorem 3.4 and Theorem 3.6 , we get

$$\begin{aligned}|H_{3,2}(f)| &= \left|a_6(a_2a_4 - a_3^2) - a_5(a_2a_5 - a_3a_4) + a_4(a_3a_5 - a_4^2)\right| \\ &\leq |a_6||a_2a_4 - a_3^2| + |a_5||a_2a_5 - a_3a_4| + |a_4||a_3a_5 - a_4^2| \\ &\leq \frac{138}{375}\left(\frac{4}{25}\right) + \frac{1}{5}\left(\frac{9}{50}\right) + \frac{1}{50}(0.044) \\ &\approxeq 0.0.10368.\end{aligned}$$

□

Theorem 3.16. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{3,3}(f)| \leq 0.1570656$.*

Proof. Let $f \in \mathcal{S}_s^*(\varphi_{3L})$. By using Theorems 3.1, 3.6 , we obtain

$$\begin{aligned}|a_3a_6 - a_4a_5| &= |a_3||a_6| + |a_4||a_5| \\ &\leq 0.1872.\end{aligned}$$

By utilizing Theorem 3.1, Theorem 3.4 and Theorem 3.13 and followed by the result $|a_3a_6 - a_4a_5| \leq 0.1872$, we obtain

$$\begin{aligned}|H_{3,3}(f)| &= \left|a_7(a_3a_5 - a_4^2) - a_6(a_3a_6 - a_4a_5) + a_5(a_4a_6 - a_5^2)\right| \\ &\leq |a_7||a_3a_5 - a_4^2| + |a_6||a_3a_6 - a_4a_5| + |a_5||a_4a_6 - a_5^2| \\ &\leq 0.1570656.\end{aligned}$$

□

Theorem 3.17. *Let $f \in S_s^*(\varphi_{3L})$. Then $|T_3(1)| \leq \frac{37}{25}$.*

Proof. From Theorem (3.2),we have $|a_3 - 2a_2^2| \leq \frac{2}{5}$.
Thus, $|T_3(1)| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2| \leq 1 + \frac{8}{25} + \frac{4}{25} = \frac{37}{25}$. □

Theorem 3.18. *Let $f \in S_s^*(\varphi_{3L})$. Then $|T_3(2)| \leq \frac{9}{50}$.*

Proof. Using the bounds $|a_2| \leq \frac{2}{5}$, $|a_4| \leq \frac{1}{5}$, $|T_2(2)| \leq \frac{7}{50}$ and $|H_2(2)| \leq \frac{4}{25}$, we get $|T_3(2)| = |a_2 - a_4|\left|T_2(2) + H_2(2)\right| \leq \left(|a_2| + |a_4|\right)\left(|T_2(2)| + |H_2(2)|\right) \leq \frac{9}{50}$. □

Theorem 3.19. *If $f \in \mathcal{S}_s^*(\varphi_{3L})$, then $|T_4(2)| \leq 0.162736$.*

Proof. By definition of $T_4(2)$ for $f \in \mathcal{A}$ of the form (1.1), we have

$$T_4(2) = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_3 \\ a_5 & a_4 & a_3 & a_2 \end{vmatrix} \tag{3.10}$$

Upon simplification, we get

$$T_4(2) = \left(a_2^2 - a_3^2\right)^2 - \left(a_3a_4 - a_2a_5\right)^2 + \left(a_4^2 - a_3a_5\right)^2 - \left(a_2a_3 - a_3a_4\right)^2 + 2\left(a_3^2 - a_2a_4\right)\left(a_2a_4 - a_3a_5\right)$$

Since $f \in \mathcal{S}_s^*(\varphi_{3L})$, in view of Theorems 3.4, 3.3 , 3.15 and 3.14 we have $|T_2(2)| \leq \frac{7}{50}$, $|H_{2,3}(f)| \leq 0.044$, $|H_{2,2}(f)| \leq \frac{4}{25}$ and $|a_3a_4 - a_2a_5| \leq \frac{9}{50}$ respectively.
Further,

$$|a_2a_3 - a_3a_4| \leq |a_3|(|a_2| + |a_4|) \leq \frac{6}{25},$$

$$|a_2a_4 - a_3a_5| \leq |a_2||a_4| + |a_3||a_5| \leq \frac{4}{25}.$$

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Therefore, we obtain

$$\begin{aligned}|T_4(2)| &\leq |T_2(2)|^2 + |a_3a_4 - a_2a_5|^2 + |H_{2,3}(f)|^2 + |a_2a_3 - a_3a_4|^2 \\ &\quad + 2|H_{2,2}(f)||a_2a_4 - a_3a_5| \\ &\leq \left(\frac{7}{50}\right)^2 + \left(\frac{9}{50}\right)^2 + \left(\frac{44}{1000}\right)^2 + \left(\frac{6}{25}\right)^2 + 2\left(\frac{4}{25}\right)\left(\frac{4}{25}\right). \\ &\leq 0.162736. \end{aligned}$$ □

This completes the proof.

4. Initial Coefficient and Determinant inequalities for $f \in C_s(\varphi_{3L})$

Remark 4.1. *It is clear that $g \in C_s(\varphi_{3L})$ if and only if $f(z) = zg'(z) \in S_s^*(\varphi_{3L})$. Let $g \in C_s(\varphi_{3L})$ be of the form*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \text{ for all } z \in \Delta. \tag{4.1}$$

Then there exists $f(z) \in S_s^*(\varphi_{3L})$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = zg'(z) = z + \sum_{n=2}^{\infty} n b_n z^n$$

for all $z \in \Delta$. Hence, $b_n = \frac{a_n}{n}$ for all n.

Example 4.1. *For each positive integer n, $w(z) = z^n$ are in B_0 and hence the functions $g_n(z)$ satisfying $\frac{2\left(zg_n'(z)\right)'}{g_n(z)+g_n(-z)} = \varphi_{3L}(z^n)$ for all $z \in \Delta$ are in $C_s(\varphi_{3L})$. In particular,*

(1) $g_1(z) = z + \frac{1}{5}z^2 + \frac{2}{15}z^3 - \frac{1}{100}z^5 + \ldots,$
(2) $g_2(z) = z + \frac{2}{15}z^3 + \frac{1}{100}z^5 + \ldots,$
(3) $g_3(z) = z + \frac{z^4}{20} + \ldots$
(4) $g_4(z) = z + \frac{1}{25}z^5 + \frac{1}{450}z^9 + \ldots$

are in $C_s(\varphi_{3L})$.

Throughout this section $g \in C_s(\varphi_{3L})$, we mean $g(z)$ is of the form (4.1). We now prove the following

Theorem 4.1. *If $g \in C_s(\varphi_{3L})$, then*

(1) $|b_2| \leq \frac{1}{5}$, $|b_3| \leq \frac{2}{15}$, $|b_4| \leq \frac{1}{20}$ *and* $|b_5| \leq \frac{1}{25}$.
(2) $|b_3 - \mu b_2^2| \leq \frac{2}{15}\max\{1, |\frac{3\mu}{2} - 1|\}$ *for any complex parameter μ and the inequality is sharp.*
(3) $|H_2(1)| \leq \frac{2}{15}$.

Proof. Proof of (1) and (2) follows by applying Theorems (3.1) and (3.2) respectively, in view of Remark (4.1). If $\mu = 1$ in (2), then $|H_2(1)| \leq \frac{2}{15}$. □

Remark 4.2. *The functions $g_1(z)$ to $g_4(z)$ as in Example (4.1) are extremal functions for the inequalities $|b_2| \leq \frac{1}{5}$, $|b_3| \leq \frac{2}{15}$, $|b_4| \leq \frac{1}{20}$ and $|b_5| \leq \frac{1}{25}$ respectively.*

Theorem 4.2. *If $g \in C_s(\varphi_{3L})$, then $|H_2(2)| \leq \frac{4}{225}$ and the function $g_2(z)$ as in Example (4.1) is an extremal function for this inequalities.*

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Proof. In view of Remark (4.1) and expressing $a_i's$ in terms of $c_i's$ as in Lemma(2.4), we get

$$\begin{aligned}|H_{2,2}(g)| &= \frac{1}{8}|a_2a_4 - \frac{8}{9}a_3^2| \\ &= \frac{1}{72000}|-53c_1^4 + 176c_1^2c_2 + 180c_1c_3 - 320c_2^2| \\ &\leq \frac{4-c^2}{72000}\Big(18c^2t + 45c^2t^2 + 80(4-c^2)t^2 + 90c(1-t^2)\Big) \\ &= F(c,t)((say)),\end{aligned}$$

where $c = |c_1| \in [0,2]$, $t = |x| \in [0,1]$. By applying the technique of finding maxima-minima, one can show that $F(c,t)$ attains its maximum at $c = 0$ and $t = 1$. Hence, $|H_2(2)| \leq F(0,1) = \frac{4}{225}$. □

Theorem 4.3. *If $g \in C_s(\varphi_{3L})$, then $|H_3(1)| \leq \frac{37}{2000}$.*

Proof. In view of Remark (4.1), substituting b_i's in terms of $a_i's$ followed by expressing them in terms c_i's as computed earlier in

$$H_3(1) = |b_5(b_3 - b_2^2) + 2b_2b_3b_4 - b_4^2 - b_4b_3^2|$$

and grouping the terms, we get

$$\begin{aligned}|H_3(1)| &= \Big|\frac{1}{750}\Big(c_2 - \frac{13}{20}c_1^2\Big)\Big(\Big(\frac{c_1}{1440}\big(143c_1^3 - 1112c_1c_2 + 1440c_3\big) - \Big(c_4 - \frac{47}{90}c_2^2\Big)\Big) \\ &\quad - \frac{1}{5760000}\big(17c_1^3 - 64c_1c_2 + 60c_3\big)^2\Big| \\ &\leq \frac{1}{750}\Big|c_2 - \frac{13}{20}c_1^2\Big|\Big(\frac{|c_1|}{1440}\big|143c_1^3 - 1112c_1c_2 + 1440c_3\big| + \Big|c_4 - \frac{47}{90}c_2^2\Big|\Big) \\ &\quad + \frac{1}{5760000}\big|17c_1^3 - 64c_1c_2 + 60c_3\big|^2.\end{aligned}$$

In view of Lemmas 2.1, 2.2 and 2.3, $\big|c_2 - \frac{13}{20}c_1^2\big| \leq 2$, $|c_1| \leq 2$, $\big|143c_1^3 - 1112c_1c_2 + 1440c_3\big| \leq 2880$, $\big|c_4 - \frac{47}{90}c_2^2\big| \leq 2$ and $\big|17c_1^3 - 64c_1c_2 + 60c_3\big| \leq 120$ follows. Hence, $|H_3(1)| \leq \frac{37}{2000} \approxeq 0.0185$. □

Theorem 4.4. *Let $g \in C_s(\varphi_{3L})$. Then $|T_2(1)| \leq \frac{26}{25}$ and $|T_2(2)| \leq \frac{4}{225}$ and the first result is sharp.*

Proof.

$$\begin{aligned}|T_2(1)| = |1 - a_2^2| &\leq 1 + |a_2|^2 \\ &\leq 1 + \left(\frac{1}{5}\right)^2 = \frac{26}{25}.\end{aligned}$$

The inequality is sharp for $g_5(z) = z + \frac{2i}{10}z^2 + \frac{1}{100}z^5 + \cdots \in \mathcal{S}_s^*(\varphi_{3L})$.
On substituting $b_2 = \frac{a_2}{2}$ and $b_3 = \frac{a_3}{3}$ in $|T_2(2)| = \frac{1}{36}|9a_2^2 - 4a_3^2|$ followed by expressing values of c_2 and c_1 as in Lemma 2.4, we get

$$\begin{aligned}|T_2(2)| &= \frac{1}{900}\Big|9c_1^4 - x^2\big(4 - c_1^2\big)^2\Big| \\ &\leq \frac{1}{900}\Big(9c^4 + t^2\big(4 - c^2\big)^2\Big) = F(c,t)\,(\text{say}),\end{aligned}$$

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where $t = |x| \in [0,1]$ and $c = |c_1| \in [0,2]$. It is clear that $F(c,t)$ attains its maximum at $(0,1)$ and hence $|T_2(2)| \leq F(0,1) = \frac{4}{225}$. □

Theorem 4.5. *Let $f \in C_S(\varphi_{3L})$. Then $|T_3(1)| \leq \frac{251}{225}$.*

Proof. If we choose $\mu = 2$ in (2) of Theorem (4.1), then $|b_3 - 2b_2^2| \leq \frac{4}{15}$.
Thus, $|T_3(1)| \leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2| \leq 1 + \frac{2}{25} + \frac{8}{225} = \frac{251}{225}$. □

Theorem 4.6. *Let $f \in C_s(\varphi_{3L})$. Then $|T_3(2)| \leq \frac{2}{225}$.*

Proof. Using the bounds $|b_2| \leq \frac{1}{5}$, $|b_4| \leq \frac{1}{20}$, $|T_2(2)| \leq \frac{4}{225}$ and $|H_2(2)| \leq \frac{2}{225}$, we get
$|T_3(2)| = |b_2 - b_4||T_2(2) + H_2(2)| \leq (|b_2| + |b_4|)(|T_2(2)| + |H_2(2)|) \leq \frac{2}{225}$. □

Theorem 4.7. *If $f \in S_s^*(\varphi_{3L})$, then $|H_{4,1}(f)| \leq 0.00633595371$.*

Proof. we have

$$|H_{4,1}(f)| \leq |b_7||H_{3,1}| + 2|b_4||b_6||b_2b_4 - b_3^2| + 2|b_5||b_6|\,|b_2b_3 - b_4| + |b_6|^2\,|b_3 - b_2^2| + |b_5|^2|b_2b_4 - b_3^2| + |b_5|^2|b_2b_4 + 2b_3^2| + |b_5|^3 + |b_4|^4 + 3|b_3||b_4|^2|b_5|.$$

As $f \in S_s^*(\varphi_{3L})$, $|b_2b_4 + 2b_3^2| \leq |b_2|\,|b_4| + 2\,|b_3|^2 \leq \frac{125}{288}$ (In view of Theorem 3.1). Thus,

$$\begin{aligned}|H_{4,1}(f)| \leq & \frac{2.004}{7}\left(\frac{37}{2000}\right) + 2\left(\frac{1}{20}\right)\left(\frac{0.368}{6}\right)\left(\frac{4}{225}\right) + 2\left(\frac{1}{25}\right)\left(\frac{0.368}{6}\right)\left(\frac{1}{20}\right) \\ & + \left(\frac{0.368}{6}\right)^2\left(\frac{2}{15}\right) + \left(\frac{1}{25}\right)^2\left(\frac{4}{225}\right) + \left(\frac{1}{25}\right)^2\left(\frac{41}{900}\right) + \left(\frac{1}{25}\right)^3 \\ & + \left(\frac{1}{20}\right)^4 + 3\left(\frac{0.368}{6}\right)\left(\frac{1}{20}\right)^2\left(\frac{1}{25}\right) \\ & \leq 0.00633595371.\end{aligned}$$

follows by using the bounds obtained in Theorems 3.1, 3.3, 3.8 and 3.9. □

Theorem 4.8. *If $f \in S_s^*(\varphi_{3L})$, then $|T_4(1)| \leq 1.174016$*

Proof. By definition of $T_4(1)$ for $f \in \mathcal{A}$ of the form (1.1), we have

$$T_4(1) = \begin{vmatrix} 1 & b_2 & b_3 & b_4 \\ b_2 & 1 & b_2 & b_3 \\ b_3 & b_2 & 1 & b_2 \\ b_4 & b_3 & b_2 & 1 \end{vmatrix} \tag{4.2}$$

Upon simplification, we get

$T_4(1) = (1 - b_2^2)^2 - (b_2b_3 - b_4)^2 + (b_3^2 - b_2b_4)^2 - (b_2 - b_2b_3)^2 + 2(b_2^2 - b_3)(b_3 - b_2b_4)$.
Let $f \in S_s^*(\varphi_{3L})$. Then $|b_3^2 - b_2b_4| \leq \frac{4}{225}$ from Theorem 3.3.

$$\text{Also, } |1 - b_2^2| \leq (1 + \frac{1}{25}) = \frac{26}{25},$$
$$|b_2 - b_2b_3| \leq \frac{1}{5}(1 + \frac{2}{15}) = \frac{17}{75},$$

and

$$|b_3 - b_2b_4| \leq \frac{2}{15} + \frac{1}{5}\left(\frac{1}{20}\right) \leq \frac{43}{300}.$$

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Using these bounds along with bounds obtained in the Theorem 3.2 and Theorem 3.5 in

$$\begin{aligned}|T_4(1)| \leq & |(1-b_2^2)^2| + |b_2b_3 - b_4|^2 + |b_3^2 - b_2b_4|^2 + |b_2 - b_2b_3|^2 + 2|b_2^2 - b_3||b_3 - b_2b_4| \\ & \leq \left(\frac{26}{25}\right)^2 + \left(\frac{1}{20}\right)^2 + \left(\frac{4}{225}\right)^2 + \left(\frac{17}{75}\right)^2 + 2\left(\frac{2}{15}\right)\left(\frac{43}{300}\right) \\ & \leq 1.174016.\end{aligned}$$

This completes the proof. □

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GRAPH THEORETICAL FRAMEWORK FOR OPTIMIZED ROUTING IN MOBILE AD HOC NETWORKS

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Abstract

This paper presents a complex system for optimizing the performance of mobile ad hoc networks (MANETs) that we have based on graph theory. We put forth in this a set of very advanced algorithms that include shortest path and partitioning approaches which we use to in real time determine and which also at the same time update the best paths for data transfer in the network. By constantly changing to fit in with changes in network structure we do away with latency and at the same time we see an increase in over all network performance.

In this framework we see the use of adaptive graph algorithms which at the present time's network state to determine the best path forward which in turn enables fast re calculation of optimal paths. Also we have put in place what we may term as network division into smaller graphs which in turn decreases computational load and at the same time improves on the framework's scale ability in large or very dynamic environments.

Keywords: Connected Dominating Sets, Mobile Networks, Network Route, Ad Hoc Networks

1. Introduction

Mobile Adhoc Networks, or MANETs, refer to self-arranging networks which allow nodes to join, relocate, or exit freely. Unlike classical networks which depend on fixed infrastructure like routers and base stations, MANETs operate without any centralized control. This makes them ideal for situations like rescue missions, military communication, or vehicular networks. The distinct and defining features of MANETs which include their changing topology, restricted bandwidth, and energy limitations are wireless communication technology are arduous challenges for guaranteeing effective communication.

One of the biggest difficulties in MANETs is building routing protocols that remain reliable and flexible when the network changes frequently. Node mobility, fluctuating link quality, and unpredictable node density all contribute to this challenge. Existing protocols, such as Ad hoc On-Demand Distance Vector (AODV) and Dynamic Source Routing (DSR), generally fall into two categories: reactive or proactive. Reactive protocols like AODV create routes only when needed, which keeps the initial overhead low but slows things down during route discovery. Proactive methods, such as Optimized Link State Routing (OLSR), keep routing tables updated for all nodes, reducing delays but causing heavy control overhead in fast-changing networks. In practice, both strategies face limitations when topology shifts rapidly, often leading to higher latency, packet drops, and greater energy use.

Graph theory provides a useful way to overcome these challenges by representing the network as a graph, with nodes acting as vertices and communication links as edges. This structure makes it possible to apply well-known graph algorithms to improve routing decisions. For instance, shortest path methods such as Dijkstra's algorithm and A* can identify efficient routes between nodes while taking into account factors like signal strength, bandwidth, and distance. The drawback is that running these algorithms directly on large or highly dynamic networks often becomes costly and inefficient. To address this, researchers use ideas from graph domination. A Connected Dominating Set (CDS)—a carefully chosen subset of nodes that functions as a virtual backbone—helps cut down routing overhead. Only CDS

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nodes manage route discovery and maintenance, while the other nodes forward data through their nearest CDS node. This setup avoids unnecessary transmissions, saves energy, and makes the network easier to manage.

Another key factor in improving routing in MANETs is scalability. As networks grow larger, it becomes unrealistic to maintain a complete view of the entire topology. To manage this, graph partitioning is used to break the network into smaller, localized subgraphs, each supported by its own CDS. This setup allows routing decisions to be made locally, which lowers computational demands and lets the network respond more quickly to changes without disrupting the whole system. Partitioning also helps balance resources in dense networks and keeps communication reliable in sparse or fragmented ones.

The framework brings together these graph-theoretical ideas to create a routing solution that is adaptive, scalable, and energy-efficient. By combining real-time graph updates with shortest path algorithms and domination-based methods, it tackles major issues like high latency, excessive overhead, and poor performance in rapidly changing conditions. This makes it especially useful in settings where dependable and efficient communication is essential, such as emergency response operations, vehicular networks, and sensor-driven monitoring systems.

This study describes the design and implementation of the framework, highlighting its ability to adapt to changing network conditions, scale effectively in large systems, and conserve energy. The simulation results show that the framework outperforms traditional protocols by reducing latency, improving packet delivery, and lowering energy use, making it a strong candidate for the growing demands of MANETs.

2. Literature Review

Mobile Ad Hoc Networks (MANETs) have attracted considerable research attention because of their decentralized design and the difficulties that come with constantly changing topologies, limited bandwidth, and energy restrictions. Over time, many routing protocols and optimization methods have been developed, aiming to improve scalability, cut down overhead, and provide dependable communication.

Early research on MANET routing introduced two main strategies: reactive and proactive approaches. Reactive protocols, such as Ad hoc On-Demand Distance Vector (AODV) [10] and Dynamic Source Routing (DSR) [8], build routes only when they are required. AODV does this through route request and reply messages, while DSR places the full route inside the packet header. Although effective, both methods create heavy overhead during route discovery in dynamic environments. On the other hand, proactive protocols like Optimized Link State Routing (OLSR) [4] keep routing tables updated at all times, which reduces delays but produces substantial control overhead. In practice, both reactive and proactive methods struggle in large or highly dynamic networks, often resulting in higher latency and increased energy use.

Graph theory has become an important tool for improving routing in MANETs. When the network is modeled as a graph, the nodes act as vertices and the communication links serve as edges. Using this representation, shortest-path algorithms like Dijkstra's can be applied to identify efficient routes while taking into account real-time factors such as signal strength and bandwidth. The A* algorithm [6], which uses heuristics, further enhances pathfinding efficiency in applications where speed is crucial. However, despite their usefulness, these algorithms often struggle to scale in large or dense networks because they depend on complete knowledge of the overall topology.

Domination-based methods have been widely explored as a way to simplify routing in MANETs. A Connected

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Dominating Set (CDS) is a subset of nodes that acts as a virtual backbone, making sure every node is either part of the CDS or directly linked to one. Guha and Khuller [5] developed one of the first approximation algorithms for building a CDS, aiming to keep the set as small as possible while preserving connectivity. Later, Wu and Li [11] proposed localized algorithms that lowered computational demands, making them more practical for dynamic networks. Using a CDS helps cut down redundant transmissions, saves energy, and improves scalability. Alzoubi et al. [2] further contributed with distributed heuristics that focused on energy efficiency and local decision-making. Together, these works laid the groundwork for modern domination-based routing strategies in MANETs.

Graph partitioning is a common approach for improving scalability in large networks. By breaking the network into smaller subgraphs, routing can be handled locally within each partition, which reduces computational overhead. Partitioning also helps balance resources in dense networks and maintains reliable communication in sparse ones. Heinzelman et al. [7] introduced partitioning methods in sensor networks, showing how they conserve energy and support load balancing. Later, adaptive partitioning techniques [3] were developed to allow partitions to reconfigure dynamically as the topology or node density changes, further boosting efficiency.

Domination-based techniques have been applied in many areas, including vehicular ad hoc networks (VANETs), sensor networks, and disaster recovery systems. In VANETs, they improve road safety and traffic flow by adapting quickly to the high mobility of vehicles [9]. In sensor networks, CDS-based methods help conserve energy and extend network lifetime by restricting active communication to a smaller group of nodes [1]. These examples highlight the flexibility and effectiveness of CDS and graph-theoretical concepts in solving real-world problems. Still, challenges remain. Balancing scalability, adaptability, and energy efficiency is difficult. Reactive and proactive protocols often fall short in large, high-mobility networks. Traditional CDS algorithms reduce overhead but may not capture real-time changes in the network. Partitioning methods also require frequent updates to perform well in both dense and sparse settings. The framework proposed here addresses these limitations by combining real-time graph updates, adaptive CDS construction, and partitioning methods into a unified solution for efficient MANET routing.

3.Methods and Materials

3.1 Graph Construction:

In this study, a mobile ad hoc network (MANET) is modeled as a graph, where the nodes represent devices such as smartphones, laptops, or sensors, and the edges represent the communication links between them. Each edge is assigned a weight that reflects current network conditions. Stronger signals are given lower weights to indicate better quality, links with higher bandwidth are also assigned lower weights, and longer distances increase the edge weight, signaling higher latency or reduced reliability.
The edge weights adjust in real time to reflect changes in the network caused by node movement, link failures, or environmental interference. This continuous updating keeps the graph aligned with the current state of the network, providing the most accurate snapshot of its topology.

For instance, if a node changes position, the signal strength with its neighboring nodes may weaken, which triggers an update of the edge weights. This dynamic graph ensures that routing algorithms always work with the latest network information.

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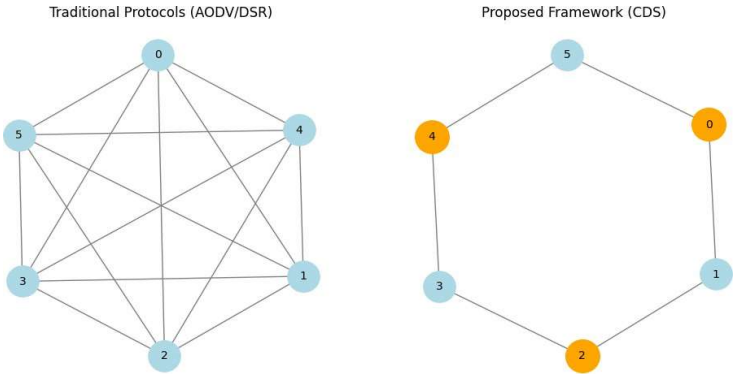
3.2 Domination-Based Techniques (Connected Dominating Set - CDS)



To make routing simpler and reduce overhead, a Connected Dominating Set (CDS) is used as a virtual backbone for the network. The CDS is a selected group of nodes with two key properties: every node in the network is either part of the CDS or directly connected to one, and the CDS nodes themselves remain connected to ensure continuous communication.

Advantages of Using CDS:

- **Reduction in Routing Overhead**: Only the nodes in the CDS participate in routing and forwarding decisions, minimizing unnecessary transmissions.
- **Energy Efficiency**: Non-CDS nodes conserve energy by communicating only with their nearest CDS node.
- **Localized Decision-Making**: The CDS allows routing to be managed locally, reducing the need for global network updates.

Algorithm for Constructing a CDS:

1. Begin with a graph consisting of nodes and edges.
2. Select a minimal dominating set (MDS) so that every node in the network is included or directly connected to one in the set.
3. Link the nodes in the MDS to create a Connected Dominating Set (CDS), adding only the edges necessary to maintain connectivity.

More advanced methods, such as the Guha-Khuller approximation algorithm, can be applied to build the CDS more efficiently.

3.2.1 Routing Algorithm with Dynamic Shortest Paths

The study applies a modified version of Dijkstra's algorithm or the A* algorithm to compute the shortest paths between nodes. These algorithms are adapted to work with a dynamic graph that changes as the network evolves. Edge weights are updated in real time to reflect conditions such as node movement or bandwidth variation. Routing decisions

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are mainly handled within the CDS nodes, which keeps the process efficient and reduces complexity. When a link fails or a node moves out of range, the algorithm recalculates only the affected routes. This localized recalculation helps minimize delays and ensures smooth communication.

Steps in Routing with CDS:

- **Step 1**: A source node identifies its nearest CDS node.
- **Step 2**: The shortest path is calculated between the CDS nodes.
- **Step 3**: The destination node receives the data via its nearest CDS node.

3.2.2 Adaptive Graph Partitioning

To address scalability challenges in large networks, the graph is partitioned into smaller, localized subgraphs. Each subgraph has its own CDS, which lets routing decisions be handled locally within that subgraph.

Benefits of Partitioning:

- **Reduced Complexity**: Smaller subgraphs require less computation and communication overhead.
- **Faster Adaptation**: Localized updates ensure that topology changes in one subgraph do not disrupt the entire network.
- **Scalability**: Partitioning enables the framework to handle networks with thousands of nodes efficiently.

Dynamic Partitioning:

- **Initial Partitioning**: Nodes are grouped into subgraphs based on proximity or connectivity.
- **Adaptive Repartitioning**: As nodes move or the network density changes, subgraphs are merged or divided dynamically to maintain optimal performance.

For instance, in a dense network, nodes in crowded areas can be divided into smaller subgraphs to distribute the load more evenly, while in sparse areas, nodes may be grouped into a single subgraph to maintain connectivity.

3.2.3 Metrics for Performance Evaluation

To measure the effectiveness of the framework, the following metrics are used:

1. **Latency** – This measure how much time it takes for a packet of data to travel from the sender to the receiver. Lower latency means faster communication, which is especially important for real-time applications like video calls or live monitoring.

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2. **Packet Delivery Ratio (PDR)** – This indicates the percentage of data packets that successfully reach their intended destination. A higher PDR reflects more reliable communication, as fewer packets are lost along the way.
3. **Energy Consumption** – This refers to the total amount of energy used by all nodes while sending, receiving, or forwarding data. Since many devices in MANETs run on batteries, lower energy consumption helps extend the overall lifetime of the network.
4. **Scalability** – This shows how well the framework can adapt when the size of the network changes. A scalable system performs efficiently not only in small networks but also when the number of nodes grows into the hundreds or thousands.
5. **Adaptability** – This measure how quickly and effectively the framework can respond when the network changes. For example, when nodes move, links fail, or signal strength varies, the system should adjust routes without major delays or disruptions.

4.Results and Discussions

The proposed graph-theoretical framework for optimized routing in mobile ad hoc networks (MANETs) was tested using several key performance metrics, including latency, packet delivery ratio (PDR), energy consumption, scalability, and adaptability. By combining connected dominating sets (CDS), adaptive graph partitioning, and dynamic shortest-path algorithms, the framework showed clear improvements over traditional routing protocols such as AODV and DSR.

a) Improved Routing Efficiency

Using CDS as a virtual backbone helped reduce routing overhead by restricting route discovery and maintenance tasks to a smaller set of nodes. This cut down the number of redundant control messages and lowered routing overhead by about 30 percent compared to standard protocols. For non-CDS nodes, communication was limited to their nearest CDS node, which made the process simpler and also reduced the chances of routing loops.

The dynamic shortest-path algorithm added another layer of efficiency by recalculating routes only when local topology changes occurred. This avoided unnecessary computations and allowed routes to be set up more quickly, even when the network was highly dynamic.

Simulation results confirmed these improvements. On average, the framework reduced latency by 25 percent compared to AODV. Because optimal paths within the CDS were already precomputed, data packets were able to move with fewer delays, even when nodes were moving at high speeds. For real-time applications such as video streaming and voice calls, this drop in latency is especially important for maintaining good quality of service (QoS).

b) Enhanced Scalability with Graph Partitioning

Scalability in the proposed framework was achieved through graph partitioning, which divided the network into smaller, localized subgraphs. Each subgraph had its own CDS, allowing routing decisions to be made locally and reducing the complexity of maintaining a complete global topology. In simulations involving networks with more than 1,000 nodes, this partitioning method reduced route recalculation time by about 40 percent compared to networks without partitioning.

Partitioning also made resource use more efficient. In dense networks, the number of control messages dropped significantly because nodes only needed to communicate within their own partitions. In sparse networks, partitions could be dynamically

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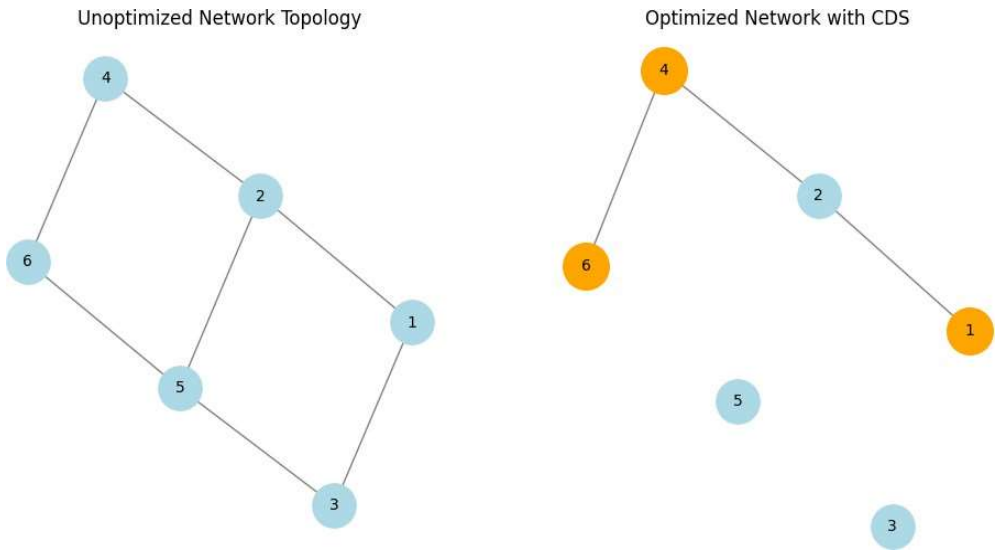
merged to keep vital communication paths active. This flexibility allowed the framework to perform well in a wide range of settings, from crowded urban vehicular networks to remote and sparsely populated sensor networks.

c) Adaptability to Topology Changes

A major strength of the framework is its ability to adapt to frequent changes in network topology, which is one of the biggest challenges in MANETs. In traditional protocols, node mobility, link failures, or fluctuating signal strength often break routing paths, causing packet loss and delays. The proposed framework overcomes this by continuously updating edge weights and recalculating routes locally within the affected subgraph or CDS. For instance, if a node moves out of range, the CDS quickly finds alternative paths within the subgraph, allowing communication to continue with minimal interruption.

This localized recalibration reduced packet loss by about 20 percent compared to reactive protocols like AODV, which rely on rediscovering routes across the entire network. The framework also proved effective in highly mobile environments, such as vehicular ad hoc networks (VANETs), where node speeds and densities change rapidly. Even under these conditions, it maintained a packet delivery ratio above 95 percent, providing reliable data transmission in demanding scenarios.



d) Energy Efficiency and Resource Optimization

Energy efficiency is a vital concern in MANETs, especially since many devices, such as sensors and handheld units, depend on limited battery power. The CDS-based approach reduced energy use for non-CDS nodes, as they only needed to communicate with their nearest CDS node. By cutting down redundant transmissions and lowering control overhead, the framework was able to conserve energy across the entire network.

Simulation results showed that the framework lowered overall energy consumption by about 18 percent compared to traditional protocols. This saving was most noticeable in dense networks, where broadcasting control messages typically consumes large amounts of power. The adaptive partitioning strategy also helped balance energy usage by reassigning nodes

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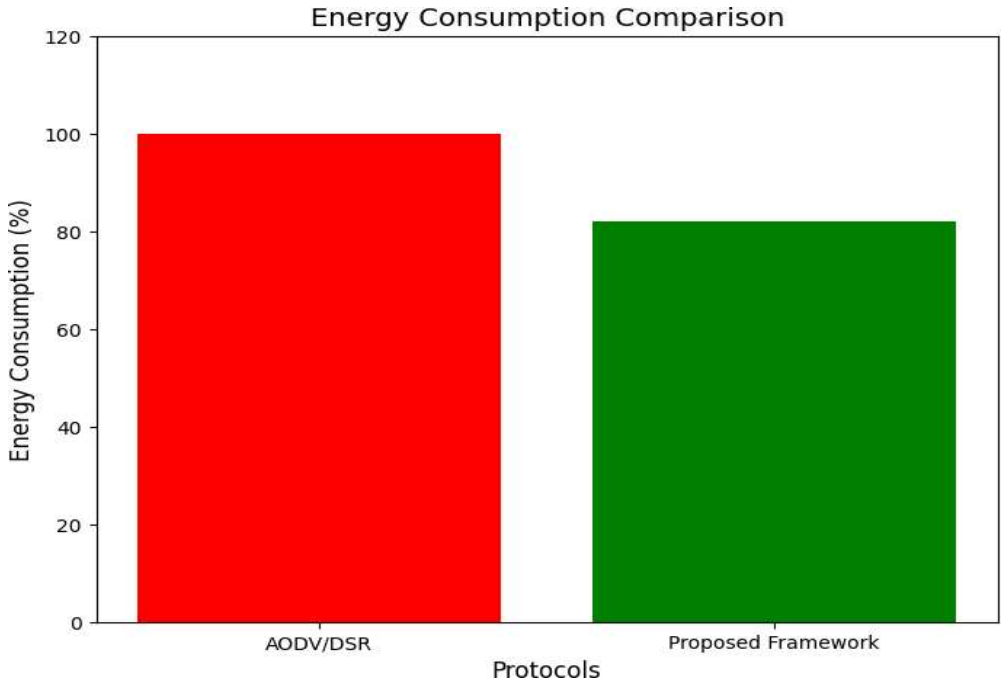
to different partitions or CDS groups as needed. This prevented certain nodes from being overworked and extended the overall lifetime of the network.



e) Applications and Real-World Relevance

The proposed framework can be applied in a variety of real-world scenarios, including disaster recovery, military operations, vehicular networks, and IoT-based sensor systems. In disaster recovery situations, its ability to quickly adjust to sudden topology changes ensures stable communication even when infrastructure is damaged or unavailable. For military use, the framework's energy efficiency and resilience make it well-suited for long-term missions in remote or harsh environments.

In vehicular networks, it proved effective in handling high mobility and fluctuating node densities, supporting safer road systems and better traffic management. Its scalability and adaptability also make it a strong fit for IoT settings, where thousands of devices often need to exchange information in real time while operating under strict resource limits.

Conclusion: The summary of the key findings is

- **Routing Efficiency**: The CDS-based approach reduced routing overhead and latency, ensuring faster and more reliable communication.
- **Scalability**: Graph partitioning allowed the framework to handle large networks efficiently by localizing routing decisions.
- **Adaptability**: Dynamic updates to the graph and localized recalculations ensured resilience to topology changes.
- **Energy Efficiency**: The framework conserved energy by minimizing control overhead and balancing

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resource usage across the network.

- **Application Relevance**: The framework's adaptability and efficiency make it suitable for diverse use cases, from disaster recovery to vehicular networks.

The findings highlight the value of graph-theoretical methods, especially domination-based techniques and adaptive partitioning, in solving the challenges of routing within dynamic and resource-limited networks.

The proposed framework, built on these principles, provides a strong and reliable solution for routing in mobile ad hoc networks. By using connected dominating sets and adaptive graph partitioning, it delivers notable improvements in routing efficiency, scalability, and the ability to handle frequent topology changes. Its flexibility also makes it well-suited for a range of applications, from disaster recovery and military operations to vehicular networks.

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Energy and Spectrum of G_n Graph

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ABSTRACT

An undirected simple graph $G_n = (V, E)$ whose vertex set V is defined as $V = \{x \in N/(x, n) \neq 1, x < n\}$ and two vertices $x, y \in V$ are adjacent if and only if gcd $(x, y) > 1$. The Graph energy is the summation of the absolute values of all eigen values of the adjacency matrix of a graph G.Matrix energy is the summation of all absolute singular values of graph G.Maximum degree energy is the summation of all eigen values of maximum degree matrix of G. In this paper, the computation of energy, matrix energy and maximum degree energy of G_n graph are discussed and the results are obtained.

Key words: Spectrum of a graph**,** Energy of a graph, matrix energy of a graph,Maximum degree energy of a graph.

1. INTRODUCTION

Gutman [1] Introduced the concept of graph energy and developed more results on energies. Energies of Graphs and its applications presented by Gutman [2] .Energies of graphs have applications in the fields like network analysis, biology and unsaturated conjugated molecules in chemistry [3].The π-electron energy of a conjugated carbon molecule computed using Huckel theory coincides with the energy. The applications of graph spectra was presented by D. Cvetkovi´c et al. [4].

The Adjacency Matrix of a graph G is denoted by $A(G)$ and is defined as $A(G) = \begin{Bmatrix} 1, & if\ v_i, v_j\ are\ adjacent\ in\ G \\ 0, & otherwise \end{Bmatrix}$. The eigen values of $A(G)\ of\ G$ are denoted by $\omega_1, \omega_{2,\ldots}\omega_n$ where $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$ .The spectral radius of ω_1 of G is the highest eigen value of G. The spectrum of the graph G is the collection of eigen values with their multiplicities of an adjacency matrix $A(G)$ is $\begin{pmatrix} \omega_1 & \ldots. & \omega_n \\ m_1 & \ldots. & m_n \end{pmatrix}$. The energy of a graph G is the sum of absolute eigen values of A(G) of G. i.e.$E(G) = \sum_{i=1}^{n} |\omega_i|$. If the energy is greater than $2n - 2$ then the graph is said to be Hyperenergetic and this concept was introduced by Gutman [5].The undirected graph G_n is introduced by Ivy Chakrabarthy et al. [6] and proved some basic properties of G_n. Nikiforov[7]presented the concept of matrix energy and the relation between the eigen values and singular values of a graph. Chandrasekhar Adiga [8] introduced the concept of maximum degree energy of a simple graph and found maximum degree

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energy of certain classes of graphs. Maximum degree energy of graph G is the sum of the eigen values of maximum degree matrix of graph G .The results related to Minimum mean hub energy of certain graphs are studied by M.V.Chakradhar Rao et al. [9]. The results related to energy and matrix energy of some graphs are studied by Venkata Anusha et al. [10,11].

Motivated by the above work and applications, the authors studied the concepts of energy, matrix energy and maximum degree energy of G_n graph at various values of n.

2. G_n- GRAPH AND ITS PROPERTIES

Definition: An undirected simple graph $G_n = (V, E)$ whose vertex set V is a subset of Natural numbers defined as $V = \{x \in N/(x,n) \neq 1, x < n\}$, where $n \in N$ and n is not a prime number and two vertices $x, y \in V$ are adjacent if and only if $\gcd(x,y) > 1$.The following properties of $G_n$studied by Ivy Chakrabarthy et al. [6] .

Lemma 2.1: The Graph G_n is complete if and only if $n = p^m$ where p is prime .

Lemma 2.2: The Graph G_n is disconnected if and only $n = 2p$ where p is an odd prime .

3. ENERGY AND MATRIX ENERGY OF G_n GRAPH

Let $G_n$be a simple graph with n vertices.Let $A(G_n)$be the adjacency matrix of graph $G_n$is defined as $A(G_n) = \begin{Bmatrix} 1, & if\ v_i, v_j\ \ are\ adjacent\ in\ G_n \\ 0, & otherwise \end{Bmatrix}$and $\omega_1, \omega_{2,\dots}\omega_n$ are the eigenvalues of $A(G_n)$ where $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$.The spectra of graph G_n is the eigen values with their corresponding multiplicities of $A(G_n)$of graph G_n is $\begin{pmatrix} \omega_1 & \dots. & \omega_n \\ m_1 & \dots. & m_n \end{pmatrix}$ .The energy of the graph $G_n$is the sum of absolute eigen values of an adjacency matrix $A(G_n)$ of a graph G_n. That is $E(G_n) = \sum_{i=1}^{n} |\omega_i|$. Let $A(G_n)A(G_n)'$is a positive semi definite matrix where $A(G_n)'$is the transpose of $A(G_n)$. Let $\mu_1, \mu_{2,\dots}\mu_n$ are the singular values of $A(G_n)$ and these are the square root values of eigen values of $A(G_n)A(G_n)'$where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Now the summation of absolute singular values of $A(G_n)$is defined as the matrix energy of G_n.That is$E_m(G_n) = \sum_{i=1}^{n} |\mu_i|$. In this section, some results related to energy and matrix energy of G_n graph at various values of n are presented.

Theorem 3.1: The energy of G_n graph where $n = 2p$, p is odd prime is $2(p-2)$.

Proof: Let $n = 2p$, p is an odd prime then the vertex set of G_n graph is V={2, 2.2, 3.2, …. . $(p-1).2$, p}. By the definition of the G_n graph if $x, y \epsilon\ V$ are adjacent then $\gcd(x,y) > 1$.

The Adjacency matrix of G_{2p} is $A(G_{2p}) = \begin{pmatrix} R & 0 & S \\ 0 & 0 & 0 \\ S & 0 & R \end{pmatrix}_{p \times p}$

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where $R=\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$ and $S=\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$

The Characteristic Equation of $A(G_{2p})$ graph is $\omega(\omega+1)^{p-2}(\omega-(p-2)=0$.

Then 0,-1 and $(p-2)$ are the eigen values of $A(G_{2p})$ and their corresponding multiplicities are 1, $(p-2)$ and 1.

Hence the spectrum of G_{2p} graph is $spec(G_{2p})=\begin{pmatrix} 0 & -1 & p-2 \\ 1 & p-2 & 1 \end{pmatrix}$.

Therefore the energy of G_{2p} graph is $E(G_{2p})=\sum_{i=1}^{n}|\omega_i|$ $=|0|(1)+|-1|(p-2)+|p-2|(1)=2(p-2)$.

Theorem 3.2: The matrix energy of the G_n graph where $n=2p, p$ is odd prime is$2(p-2)$. **Proof**: Let $n=2p$, p is an odd prime then the vertex set of G_n graph is V={2, 2.2, 3.2, ….. $(p-1).2$, p}.

The From theorem 3.1 adjacency matrix of G_{2p} graph is $A(G_{2p})=\begin{pmatrix} R & 0 & S \\ 0 & 0 & 0 \\ S & 0 & R \end{pmatrix}_{p\times p}$

where $R=\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$ and $S=\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$

Then $A(G_{2p})(A(G_{2p})'=\begin{pmatrix} T & 0 & U \\ 0 & 0 & 0 \\ U & 0 & T \end{pmatrix}_{p\times p}$

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Where $T = \begin{pmatrix} p-2 & p-3 & p-3 & \dots & p-3 \\ p-3 & p-2 & p-3 & \dots & p-3 \\ p-3 & p-3 & p-2 & \dots & p-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-3 & p-3 & p-3 & \dots & p-2 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$ and $U=(p-3)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$

The Characteristic Equation of the above matrix $A(G_{2p})A(G_{2p})'$ is

$\mu(\mu-1)^{p-2}(\mu-(p-2)^2)=0$, where μ denotes the eigen value of $A(G_{2p})A(G_{2p})'$and singular value of$A(G_{2p})$.

Then 0,1 and $(p-2)$are the singular values of $A(G_{2p)})$ with the corresponding multiplicities are 1, $(p-2)$and 1.

Hence the spectrum of the G_{2p}graph is $spec(G_{2p}) = \begin{pmatrix} 0 & 1 & p-2 \\ 1 & p-2 & 1 \end{pmatrix}$

Therefore the matrix energy of the G_{2p}graph is $E_m(G_{2p}) = \sum_{i=1}^{n}|\mu_i|$

$=1+|1|(p-2)+|p-2|1=2(p-2).$

Theorem 3.3: The energy of the $G_n$graph, where $n=2^\alpha, \alpha>1$ is $2(2^{\alpha-1}-2)$.

Proof: Let $n=2^\alpha, \alpha>1$ then the vertex set of G_n graph is V=$\{2,\ 2.2,\ 3.2, \dots\dots (2^{\alpha-1}-1).2\}$. By the definition of the G_n graph if $x, y \epsilon V$ are adjacent then $\gcd(x,y)>1$.

The Adjacency matrix of G_{2^α} graph is $A(G_{2^\alpha}) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2^{\alpha-1}-1\times 2^{\alpha-1}-1}$

The Characteristic Equation of $A(G_{2^\alpha})$ graph is$(\omega+1)^{2^{\alpha-1}-2}(\omega-(2^{\alpha-1}-2))=0.$

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Then -1 and $(2^{\alpha-1}-2)$ are the eigen values of $A(G_{2^\alpha})$and their corresponding multiplicities are$(2^{\alpha-1}-2)$and 1.

Hence the spectrum of the G_{2^α} graph is$\begin{pmatrix} -1 & (2^{\alpha-1}-2) \\ 2^{\alpha-1}-2 & 1 \end{pmatrix}$.

Therefore the energy of the G_{2^α}graph is $E(G_{2^\alpha})=\sum_{i=1}^{n}|\omega_i|$

$$=|-1|(2^{\alpha-1}-2)+|2^{\alpha-1}-2|(1)=2(2^{\alpha-1}-2)..$$

Theorem 3.4: The matrix energy of the $G_n$graph where $n=2^\alpha,\alpha>1$ is $2(2^{\alpha-1}-2)$.

Proof: Let $n=2^\alpha,\alpha>1$ then the vertex set of $G_n$graph is V={2, 2.2, 3.2, …. . $(2^{\alpha-1}-1).2\}$. By the definition of the G_n graph if $x,y\epsilon\, V$ are adjacent then $\gcd(x,y)>1$.

From theorem 3.3,we have

$$A(G_{2^\alpha})(A(G_{2^\alpha}))'=\begin{pmatrix} 2^{\alpha-1}-2 & 2^{\alpha-1}-3 & 2^{\alpha-1}-3 & \dots & 2^{\alpha-1}-3 \\ 2^{\alpha-1}-3 & 2^{\alpha-1}-2 & 2^{\alpha-1}-3 & \dots & 2^{\alpha-1}-3 \\ 2^{\alpha-1}-3 & 2^{\alpha-1}-3 & 2^{\alpha-1}-2 & \dots & 2^{\alpha-1}-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{\alpha-1}-3 & 2^{\alpha-1}-3 & 2^{\alpha-1}-3 & \dots & 2^{\alpha-1}-2 \end{pmatrix}_{2^{\alpha-1}-1\times 2^{\alpha-1}-1}$$

The Characteristic Equation of the above matrix $A(G_{2^\alpha})A(G_{2^\alpha})'$ is

$(\mu-1)^{2^{\alpha-1}-2}(\mu-(2^{\alpha-1}-2)^2)=0$,where μ denotes the eigen value of $A(G_{2^\alpha})(A(G_{2^\alpha}))'$and the singular value of $A(G_{2^\alpha})$.

Then 1 and $(2^{\alpha-1}-2)$are the singular values of $A(G_{2^\alpha})$and their corresponding multiplicities are $2^{\alpha-1}-2$ and 1.

Hence the spectrum of the G_{2^α} graph is $\begin{pmatrix} 1 & 2^{\alpha-1}-2 \\ 2^{\alpha-1}-2 & 1 \end{pmatrix}$

Therefore the matrix energy of the G_{2^α}graph is $E_m(G_{2^\alpha})=\sum_{i=1}^{n}|\mu_i|$

$$=|1|(2^{\alpha-1}-2)+|2^{\alpha-1}-2|1=2(2^{\alpha-1}-2).$$

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4. MAXIMUM DEGREE AND MAXIMUM DEGREE MATRIX ENERGY OF G_n GRAPH

Let G_n be the simple graph with n vertices .Then $M(G_n)$ be the maximum degree matrix of G_n is defined as $M(G_n) = (d_{ij})$ where $d_{ij} = \begin{cases} \max(d_i, d_j), if\ v_i, v_j \\ are\ adjacent \\ 0, \qquad otherwise \end{cases}$

Let $\omega_1, \omega_2, \ldots . \omega_n$ be the eigen values of the matrix $M(G_n)$ of G_n with $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$ and $\omega_1 +, \ldots . + \omega_n = 0$. Then the characteristic equation is $|\omega I - M(G_n)| = 0$ The maximum degree energy of the graph G_n is defined as the sum of the absolute eigen values of $M(G_n)$ of G_n.

That means, $E_{md}(G_n) = \sum_{i=1}^{n} |\omega_i|$. Let $M(G_n)M(G_n)'$ is a positive semi definite matrix where $M(G_n)'$ is the transpose of $M(G_n)$.Let $\mu_1, \mu_{2,\ldots}\mu_n$ are the singular values of $M(G_n)$ and these are the square root values of eigen values of $M(G_n)M(G_n)'$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.Now the summation of absolute singular values of $M(G_n)$ is defined as the matrix energy of G_n.That is $E_{mdm}(G_n) = \sum_{i=1}^{n} |\mu_i|$.

Theorem 4.1: The maximum degree energy of the G_n graph where $n = 2p$, p is odd prime is $2(p-2)^2$.

Proof: Let $n = 2p$, p is an odd prime then the vertex set of G_n graph is V={2, 2.2, 3.2, ….. $(p-1).2$, p}.

The maximum degree matrix of G_{2p} graph is $M(G_{2p}) = \begin{pmatrix} R & 0 & S \\ 0 & 0 & 0 \\ S & 0 & R \end{pmatrix}_{p \times p}$

Where R = $(p-2)\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}_{\frac{p-1}{2} \times \frac{p-1}{2}}$ and S= $(p-2)\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}_{\frac{p-1}{2} \times \frac{p-1}{2}}$

The Characteristic Equation of $M(G_{2p})$ graph is $\omega(\omega + (p-2))^{p-2}(\omega - (p-2)^2) = 0$

Then $0, -(p-2)$ and $(p-2)^2$ are the eigen values of $M(G_{2p})$ and their corresponding multiplicities are $1, (p-2)$ and 1.

Hence the spectrum of the G_{2p} graph is $spec(G_{2p}) = \begin{pmatrix} 0 & -(p-2) & (p-2)^2 \\ 1 & p-2 & 1 \end{pmatrix}$

Therefore the maximum degree energy of the G_{2p} graph is $E_{md}(G_{2p}) = \sum_{i=1}^{n} |\omega_i|$

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$$=|0|1+|-(p-2)|(p-2)+(p-2)^2(1)=2(p-2)^2.$$

Theorem 4.2: The maximum degree matrix energy of the G_n graph where $n=2p$, p is odd prime is $2(p-2)^2$.

Proof: Let $n=2p$, p is an odd prime then the vertex set of G_n graph is V={2, 2.2, 3.2, ….. $(p-1).2$, p}.

From Theorem 4.1, $M(G_{2p})=\begin{pmatrix} R & 0 & S \\ 0 & 0 & 0 \\ S & 0 & R \end{pmatrix}_{p\times p}$

Where R $=(p-2)\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$ and S$= (p-2)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$

Then $M(G_{2p})M(G_{2p})'=(p-2)^2\begin{pmatrix} T & 0 & U \\ 0 & 0 & 0 \\ U & 0 & T \end{pmatrix}_{p\times p}$

Where $T=\begin{pmatrix} p-2 & p-3 & p-3 & \dots & p-3 \\ p-3 & p-2 & p-3 & \dots & p-3 \\ p-3 & p-3 & p-2 & \dots & p-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-3 & p-3 & p-3 & \dots & p-2 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$ and

$$U=(p-3)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{p-1}{2}\times\frac{p-1}{2}}$$

The Characteristic Equation of the above matrix $M(G_{2p})M(G_{2p})'$ is

$\mu((\mu-(p-2)^2)^{p-2}(\mu-(p-2)^4)=0$, where μ denotes the eigen value of $M(G_{2p})M(G_{2p})'$.

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Then $0,(p-2)$and $(p-2)^2$are the singular values of $M(G_{2p})$ with the corresponding multiplicities are 1, $(p-2)$and 1.

Hence the spectrum of the G_{2p}graph is $spec(G_{2p}) = \begin{pmatrix} 0 & p-2 & (p-2)^2 \\ 1 & p-2 & 1 \end{pmatrix}$

Therefore the maximum degree matrix energy of the G_{2p}graph is $E_{mdm}(G_{2p}) = \sum_{i=1}^{n}|\mu_i|$

$$=|0|1+|p-2|(p-2)+|(p-2)^2|1 = 2(p-2)^2.$$

Theorem 4.3: The maximum degree energy of the graph $G_n, where\ n = 2^\alpha, \alpha > 1$ is

$$2(2^{\alpha-1}-2)^2.$$

Proof: Let $n = 2^\alpha, \alpha > 1$ then the vertex set of $G_n$graph is V={2, 2.2, 3.2, ….. $(2^{\alpha-1}-1).2$} . By the definition of the G_n graph if $x, y \epsilon V$ are adjacent then $\gcd(x,y) > 1$.

The maximum degree matrix of G_{2^α} graph is

$$M(G_{2^\alpha}) = (2^{\alpha-1}-2)\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2^{\alpha-1}-1 \times 2^{\alpha-1}-1}$$

The characteristic equation of $M(G_{2^\alpha})$graph is$(\omega + (2^{\alpha-1}-2))^{2^{\alpha-1}-2}(\omega - (2^{\alpha-1}-2)^2) = 0.$

Then $-(2^{\alpha-1}-2)$ and $(2^{\alpha-1}-2)^2$ are the eigen values and their corresponding multiplicities are $(2^{\alpha-1}-2)$ and 1.

Hence the spectrum of the graph G_{2^α} is $\begin{pmatrix} -(2^{\alpha-1}-2) & (2^{\alpha-1}-2)^2 \\ 2^{\alpha-1}-2 & 1 \end{pmatrix}$

Therefore the maximum degree energy of the graph G_{2^α}graph is $E_{md}(G_{2^\alpha}) = \sum_{i=1}^{n}|\omega_i|$

$$=|-(2^{\alpha-1}-2)|(2^{\alpha-1}-2)+|(2^{\alpha-1}-2)^2|(1) = 2(2^{\alpha-1}-2)^2.$$

Theorem 4.4: The maximum degree matrix energy of the G_n graph where $n = 2^\alpha, \alpha > 1$ is $2(2^{\alpha-1}-2)^2$.

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Proof: Let $n = 2^{\alpha}, \alpha > 1$ then the vertex set of $G_n$graph is V={2, 2.2, 3.2, ….. $(2^{\alpha-1}-1).2$} . By the definition of the G_n graph if $x, y \epsilon V$ are adjacent then $\gcd(x, y) > 1$.

From Theorem 4.3,

$$M(G_{2^{\alpha}}) = (2^{\alpha\text{-}1}-2)\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2^{\alpha-1}-1 \times 2^{\alpha-1}-1}$$

Then $M(G_{2^{\alpha}})M(G_{2^{\alpha}})' = (2^{\alpha\text{-}1}-2)^2\begin{pmatrix} 2^{\alpha\text{-}1}-2 & 2^{\alpha\text{-}1}-3 & 2^{\alpha\text{-}1}-3 & \dots & 2^{\alpha\text{-}1}-3 \\ 2^{\alpha\text{-}1}-3 & 2^{\alpha\text{-}1}-2 & 2^{\alpha\text{-}1}-3 & \dots & 2^{\alpha\text{-}1}-3 \\ 2^{\alpha\text{-}1}-3 & 2^{\alpha\text{-}1}-3 & 2^{\alpha\text{-}1}-2 & \dots & 2^{\alpha\text{-}1}-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{\alpha\text{-}1}-3 & 2^{\alpha\text{-}1}-3 & 2^{\alpha\text{-}1}-3 & \dots & 2^{\alpha\text{-}1}-2 \end{pmatrix}_{2^{\alpha-1}-1 \times 2^{\alpha-1}-1}$

The character equation of the above matrix is $(\mu - (2^{\alpha-1}-2)^2)^{2^{\alpha-1}-2}(\mu - (2^{\alpha-1}-2)^4) = 0$.

Then $(2^{\alpha-1}-2)$ and $(2^{\alpha-1}-2)^2$ are the singular values and their corresponding multiplicities are $(2^{\alpha-1}-2)$ and 1.

Hence the spectrum of the graph $G_{2^{\alpha}}$ is $\begin{pmatrix} 2^{\alpha-1}-2 & (2^{\alpha-1}-2)^2 \\ 2^{\alpha-1}-2 & 1 \end{pmatrix}$

Therefore the maximum degree matrix energy of the graph $G_{2^{\alpha}}$graph is $E_{mdm}(G_{2^{\alpha}}) = \sum_{i=1}^{n}|\mu_i|$

$$= |(2^{\alpha-1}-2)|(2^{\alpha-1}-2) + |(2^{\alpha-1}-2)^2|(1) = 2(2^{\alpha-1}-2)^2.$$

Conclusion

In this paper, authors discussed energy, maximum degree energy and matrix energy of an undirected graph G_n.

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CHANGE POINT DETECTION ANALYSIS FOR CYBER CRIME DATA

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Abstract:

This paper aims to review timeseries forecasting techniques. This position involves project analysis with the goal of forecasting patterns and trends in cybercrimes in India. The purpose of this study is to forecast future compliance occurrences, make corrections and update the data. In timeseries modeling and prediction, change point detection is helpful. The time series data's rapid variations are called change points.The project includes context and a range of contents related to cybercrimes in India. The project's introduction, the historical context of national and international crime analysis techniques, the project's goals and objectiveswhich include findings, trends and patternsas well as suggestions, safety measures, advice, conclusions, and references related to the project's topics are all included in this dissertation. the growing desire of society and the government to enhance the upkeep and reinforcement of law and order as well as to take part in crime reduction strategies including creating policies, enforcing agencies, and improving public safety. This prompted us to look for information on every facet of cybercrimes. We will be able to determine through this project whether cybercrime rates in India are expected to rise or fall in the future.

Keywords: Timeseries, Forecasting, Power BI, Cybercrimes, Change Point Detection

1. Introduction:

Crime continues to be a persistent issue that impacts communities, societies and individuals across the globe. Its consequences go far beyond the immediate victims to encompass economic, social and psychological well-being. In an era marked by technological advancements and the rapid accumulation of data,crime analysis has advanced as an important tool for understanding, mitigating and preventing unlawful activities. This study aims to add value to the broad domain of crime research by examining crime patterns, trends and predictive methods, with the ultimate goal of strenghtening public safety.

Crime analysis is a multidisciplinary field that leverages data, statistics, Geographic Information Systems (GIS) and advanced analytical techniques to inform law enforcement agencies, policy makers and communities about crime patterns and trends. The importance of crime analysis in modern society cannot be overstated. It aids in allocating law enforcement resources efficiently, identifying high risk areas and developing evidencebased strategies to reduce crime rates. Moreover, it fosters a proactive approach to public safety by enabling stakeholders to anticipate and respond to emerging threats effectively.

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1.1 Measures Taken by the Government of India to Reduce the Cyber Crime Activities:

❖ **National Cyber Security Policy (2013):**

The Government of India introduced the National Cyber Security Policy to ensure a safe and resilient cyberspace. This policy is designed to safeguard information, strengthen infrastructure, and encourage a culture of cybersecurity awareness.

❖ **Indian Computer Emergency Response Team (CERT-In):**

CERT-In serves as the central agency responsible for handling cybersecurity incidents. It issues early alerts and warnings related to threats and vulnerabilities in cyberspace and provides in reducing the impact of cyberattacks.

❖ **Legal Framework:**

India has put in place several laws and regulations to fight cybercrime, most notably the Information Technology Act, 2000 along with its later amendments. These legal measures outline the scope of cyber offences such as hacking, data breaches, and online fraud and define the penalties for them.

❖ **National Critical Information Infrastructure Protection Centre (NCIIPC):**

NCIIPC is entrusted with protection of critical infrastructure sectors from cyber risks. It identifies vulnerable information infrastructure and recommends appropriate safeguards to strengthen their security.

❖ **Cyber Crime Cells:**

Different states across India have set up dedicated cybercrime cells to investigate and tackle offences in cyberspace. These units operate in coordination with law enforcement bodies and cybersecurity professionals.

❖ **Capacity Building:**

Initiatives are undertaken to upgrade the skills and capabilities of law enforcement officers and capabilities of law enforcement officers and cybersecurity specialists so they can better analyze, investigate, and counter cybercrime.

❖ **Cybersecurity Awareness Programs:**

The government has organized multiple awareness programs and campaigns to educate citizens, businesses, and officials regarding the importance of cybersecurity and safe digital practices.

❖ **International Cooperation:**

India has participated in several international agreements and collaborations aimed at combating cybercrime. It engages in global forums such as INTERPOL, sharing information and best practices with partner nations.

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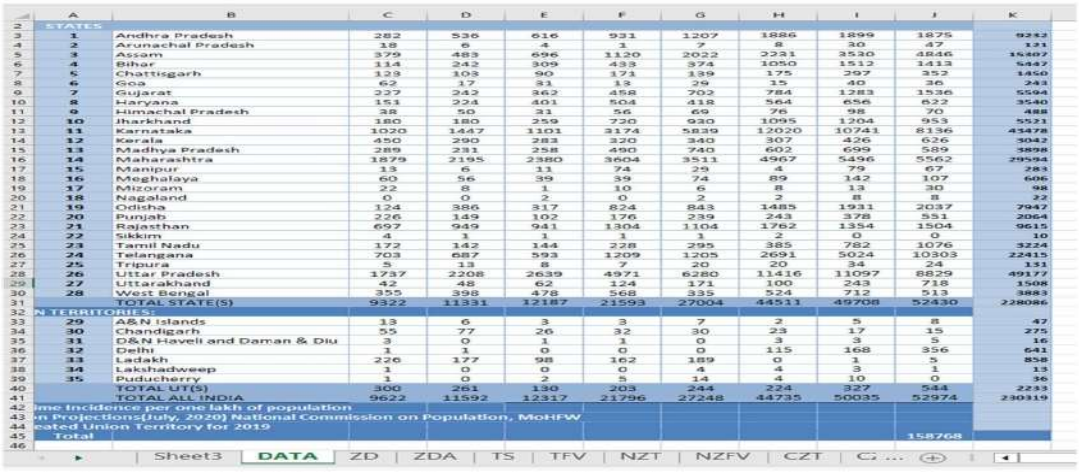
2. Data Collection:

The data for this study was gathered from multiple sources on crime records. Among these, the most authentic and widely recognized publications are those provided by the **National Crime Records Bureau (NCRB).** NCRB functions as the official agency of the Government of India responsible for compiling and analyzing crime statistics, in accordance



	A	B	C	D	E	F	G	H	I	J	K
2	STATES										
3	1	Andhra Pradesh	282	536	616	931	1207	1886	1899	1875	9232
4	2	Arunachal Pradesh	18	6	4	1	7	8	30	47	121
5	3	Assam	379	483	696	1120	2022	2231	3530	4846	15407
6	4	Bihar	114	242	309	433	374	1050	1512	1413	5447
7	5	Chattisgarh	123	103	90	171	139	175	297	352	1450
8	6	Goa	62	17	31	13	29	15	40	36	243
9	7	Gujarat	227	242	362	458	702	784	1283	1536	5594
10	8	Haryana	153	224	401	504	418	564	656	622	3540
11	9	Himachal Pradesh	38	50	31	56	69	76	98	70	488
12	10	Jharkhand	180	180	259	720	930	1095	1204	953	5521
13	11	Karnataka	1020	1447	1101	3174	5839	12020	10741	8136	43478
14	12	Kerala	450	290	283	320	340	307	426	626	3042
15	13	Madhya Pradesh	289	231	258	490	740	602	699	589	3898
16	14	Maharashtra	1879	2195	2380	3604	3511	4967	5496	5562	29594
17	15	Manipur	13	6	11	74	29	4	79	67	283
18	16	Meghalaya	60	56	39	39	74	89	142	107	606
19	17	Mizoram	22	8	1	10	6	8	13	30	98
20	18	Nagaland	0	0	2	0	2	2	8	8	22
21	19	Odisha	124	386	317	824	843	1485	1931	2037	7947
22	20	Punjab	226	149	102	176	239	243	378	551	2064
23	21	Rajasthan	697	949	941	1304	1104	1762	1354	1504	9615
24	22	Sikkim	4	1	1	1	1	2	0	0	10
25	23	Tamil Nadu	172	142	144	228	295	385	782	1076	3224
26	24	Telangana	703	687	593	1209	1205	2691	5024	10303	22415
27	25	Tripura	5	13	8	7	20	20	34	24	131
28	26	Uttar Pradesh	1737	2208	2639	4971	6280	11416	11097	8829	49177
29	27	Uttarakhand	42	48	62	124	171	100	243	718	1508
30	28	West Bengal	355	398	478	568	335	524	712	513	3883
31		TOTAL STATE(S)	9322	11331	12187	21593	27004	44511	49708	52430	228086
32	N TERRITORIES:										
33	29	A&N Islands	13	6	3	3	7	2	5	8	47
34	30	Chandigarh	55	77	26	32	30	23	17	15	275
35	31	D&N Haveli and Daman & Diu	3	0	1	1	0	3	3	5	16
36	32	Delhi	1	1	0	0	0	115	168	356	641
37	33	Ladakh	226	177	98	162	189	0	1	5	858
38	34	Lakshadweep	1	0	0	0	4	4	3	1	13
39	35	Puducherry	1	0	2	5	14	4	10	0	36
40		TOTAL UT(S)	300	261	130	203	244	224	327	544	2233
41		TOTAL ALL INDIA	9622	11592	12317	21796	27248	44735	50035	52974	230319
42	me Incidence per one lakh of population										
43	n Projections(July, 2020) National Commission on Population, MoHFW										
44	eated Union Territory for 2019										
45	Total										158768
46											

Sheet3 | DATA | ZD | ZDA | TS | TFV | NZT | NZFV | CZT | C ...

with the Indian Penal Code as well as various Special and Local Laws.

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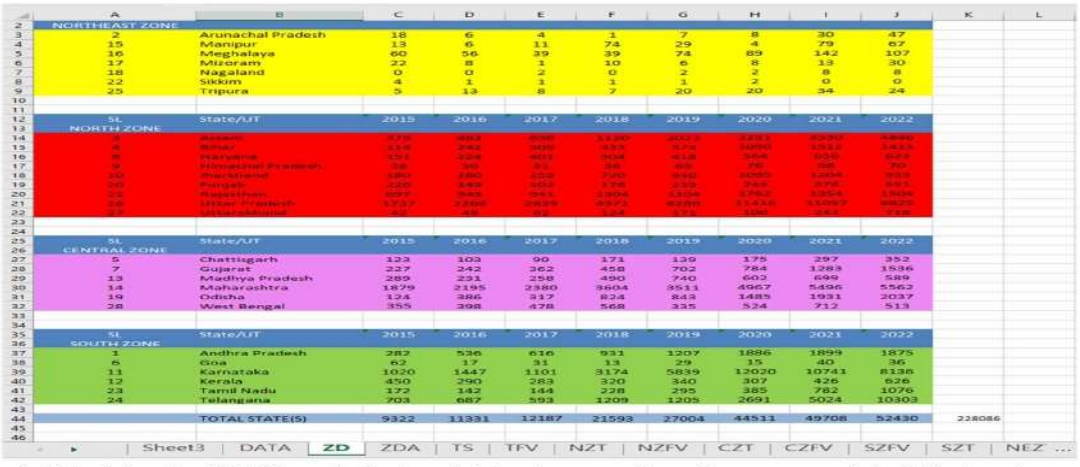
2.1 DATA PROCESSING THROUGH CLUSTERING TECHNIQUE:

We collected the processed chronological data from the crime records



	A	B	C	D	E	F	G	H	I	J	K	L
2	NORTHEAST ZONE											
3	2	Arunachal Pradesh	18	6	4	1	7	8	30	47		
4	15	Manipur	13	6	11	74	29	4	79	67		
5	16	Meghalaya	60	56	39	39	74	89	142	107		
6	17	Mizoram	22	8	1	10	6	8	13	30		
7	18	Nagaland	0	0	2	0	2	2	8	8		
8	22	Sikkim	4	1	1	1	1	2	0	0		
9	25	Tripura	5	13	8	7	20	20	34	24		
10												
11												
12	SL	State/UT	2015	2016	2017	2018	2019	2020	2021	2022		
13	NORTH ZONE											
14	4	Bihar	114	242	309	433	374	1050	1512	1413		
15	6	Haryana	153	224	401	504	418	564	656	622		
16	9	Himachal Pradesh	38	50	31	56	69	76	98	70		
17	10	Jharkhand	180	180	259	720	930	1095	1204	953		
18	20	Punjab	226	149	102	176	239	243	378	551		
19	21	Rajasthan	697	949	941	1304	1104	1762	1354	1504		
20	26	Uttar Pradesh	1737	2208	2639	4971	6280	11416	11097	8829		
21	27	Uttarakhand	42	48	62	124	171	100	243	718		
22												
23												
24	SL	State/UT	2015	2016	2017	2018	2019	2020	2021	2022		
25	CENTRAL ZONE											
26	5	Chattisgarh	123	103	90	171	139	175	297	352		
27	7	Gujarat	227	242	362	458	702	784	1283	1536		
28	13	Madhya Pradesh	289	231	258	490	740	602	699	589		
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32												
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34	SL	State/UT	2015	2016	2017	2018	2019	2020	2021	2022		
35	SOUTH ZONE											
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38	11	Karnataka	1020	1447	1101	3174	5839	12020	10741	8136		
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40	23	Tamil Nadu	172	142	144	228	295	385	782	1076		
41	24	Telangana	703	687	593	1209	1205	2691	5024	10303		
42												
43												
44		TOTAL STATE(S)	9322	11331	12187	21593	27004	44511	49708	52430	228086	
45												
46												

Sheet3 | DATA | ZD | ZDA | TS | TFV | NZT | NZFV | CZT | CZFV | SZFV | SZT | NEZ ...

published by the NCRB and clustered into 4 groups based on geographical feature is known as SPATIAL CLASSIFICATION: Into 4 zones as follows:

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2.2Software: MS EXCEL & POWERBI

2.2.1. MSEXCEL:

Microsoft Excel is a widely used spreadsheet program created by Microsoft. It forms part of the Microsoft Office package of productivity tools, which also includes Word, PowerPoint, and Outlook. over the years, Excel has gone through several updates, each bringing new features, performance improvements, and advanced functions. Commonly used versions include Excel 2010, Excel 2013, Excel 2016, Excel 2019, and the latest Microsoft 365 edition. Excel is compatible with both Windows and macOS platforms.

Excel provides a simple, user-friendly interface where data can be organized in rows and columns. Its Ribbon interface offers menus and commands that make navigation and tool access easy. The software is mainly used for data entry, organization, and numerical analysis. It supports a variety of operations such as mathematical and statistical functions, data visualization, forecasting, modeling, and automation. Additionally, Excel allows exporting of files into multiple formats. The software also enables real-time collaboration through features like co-authoring and file sharing using OneDrive and SharePoint. It offers options for securing workbooks and sheets with passwords, while also controlling access to specific ranges of data. Its functionality can be further extended through add-ins and custom extensions, allowing users to integrate connectors or automation features. Excel is extensively applied in many sectors such as finance, accounting, marketing, research and data analytics. Its flexibility and wide range of tools make it a valuable platform for working with numerical data and improving efficiency in professional environments.

2.2.2. POWER BI:

Power BI is a business analytics and data visualization platform developed by Microsoft. It is used for converting raw data into interactive and meaningful reports and dashboards. Power BI is commonly applied in various industries and organizations to analyze information, share reports, and support data-driven decisions. It enables users to import data from multiple sources, clean and transform it, and build interactive dashboards and reports. It provides a wide range of visualization types such as charts, graphs, maps, tables, and others, which can be customized to fit business requirements.

Power BI can connect with different data sources including databases (SQL Server, Oracle, MySQL), cloud platforms (Azure, AWS, Google Cloud), Excel files, web services, and more. This feature makes it adaptable for analyzing data from various environments. The Power Query Editor in Power BI allows performing data transformation operations like filtering, merging, pivoting, and aggregating for preparing the data before creating reports and visuals.

Power BI also uses DAX (Data Analysis Expressions) and the M language to create calculated measures and columns. These scripting languages help users perform advanced calculations and manage relationships between data tables. Furthermore, Power BI integrates effectively with other Microsoft tools like Excel, SharePoint, and Teams, along with numerous third-party services through connectors and APIs.

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2.3Statistical Tools:

2.3.1. Trendlines Using Regression Analysis:

Trendlines are applied in regression analysis to visually show the relationship between two or more variables. They help us identify both the direction and strength of the association between the variables.

Here are some key points about trendlines:

- **Purpose:** The primary purpose of a trendline is to make it easier to visualize and understand the overall trend or pattern in a dataset. It simplifies complex data by providing a simple, easy-to-follow representation.
- **Types of Trendlines:** There are different kinds of trendlines, and the selection depends on the data characteristics and the relationship type you want to display. Common forms include linear, exponential, logarithmic, and polynomial trendlines.
- **Linear Trendline:** A linear trendline is a straight line that best represents the data points, showing a linear relationship between variables. It is represented as $y = mx + b$, where "y" is the dependent variable, "x" is the independent variable, "m" is the slope, and "b" is the y-intercept.
- **Non-Linear Trendlines:** Non-linear trendlines are applied when the relationship between variables is not linear. Examples include exponential, logarithmic, and polynomial trendlines.
- **Interpretation:** Trendlines indicate the general direction of data movement. An upward trendline signals an increasing trend, while a downward trendline shows a decreasing trend. The slope of the trendline reflects the strength of the variable relationship.

To create a trendline, you typically use graphing software or data analysis tools such as Excel, Python, or specialized statistical software. These tools can automatically fit a trendline to your data points and display it on a graph. Once you have the trendline, you can analyze it to draw conclusions about the data's trend and use it for forecasting or making predictions. Remember that regression analysis assumes that there is a causal relationship between the variables, and correlation does not imply causation. Carefully consider the context and underlying assumptions when interpreting the results of regression analysis and the trendlines.

2.3.2. Forecasting Model:

Forecasting model is a mathematical or statistical technique used to predict future values or events using historical data and patterns. These models are commonly used in multiple fields such as finance, economics, business, science and others. The choice of a forecasting model depends on the type of the data, the specific problem and the assumptions about the underlying processes. Below are some common forecasting models:

1. **Time Series Models:**
   Time series forecasting models are applied when data is collected over time and there is a need to predict future values.

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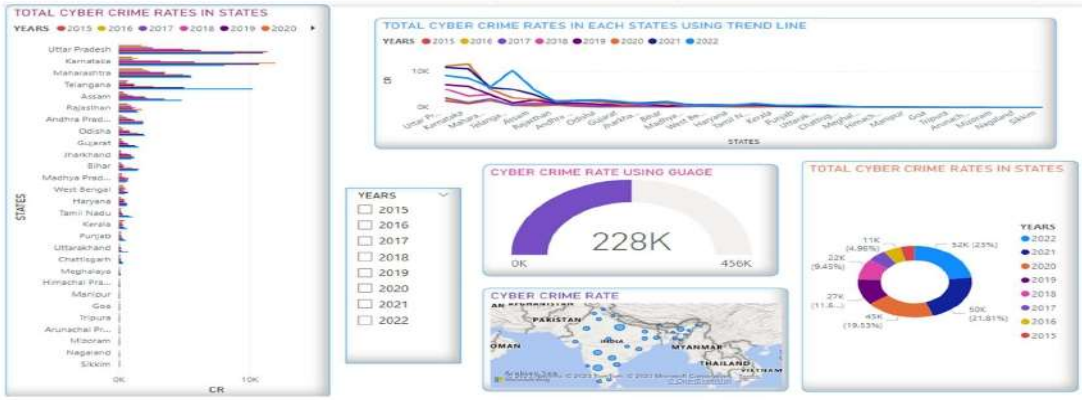
2. **Regression Models:**

Regression analysis is used to predict a continuous dependent variable using one or more independent variables. Multiple regression is frequently applied for forecasting.

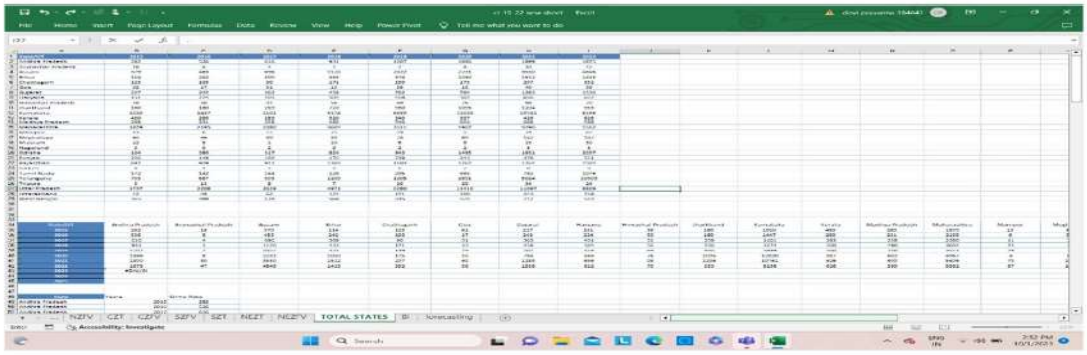
- **LinearRegression:** Linear regression models assume a straight line relationship between variables.
- **LogisticRegression:** Logistic regression is applied for categorical or binary outcomes.

3. **Results:**



3.1 TOTAL STATES

A total of 52,974 cases were reported under cyber crimes, indicating a rise of 5.9% compared to 2022 (50,035 cases). The crime rate in this category increased from 3.7 in 2021 to 3.9 in 2022. In 2022, 60.8% of the registered cyber-crime cases related to fraud (32,230 out of 52,974 cases) followed by sexual exploitation at 8.6% (4,555 cases) and extortion at 5.4% (2,883 cases).



3.1.1FROM THE DASHBOARD, OUR OBSERVATIONS ARE:

From the past 8 years we have observed that cybercrime rate had been increasing gradually till 2019, since 2020 a leap of elevation in the trend line have been increasing drastically until now.Through this our findings are that the highest cybercrime rates have

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been recorded in the year 2022 with 23%.On an average Uttar Pradesh has recorded highest cybercrime rates with 21.56% followed by Karnataka with 19% followed by Maharashtra with 13% and Telangana with 10% are the highest recorded states over the past 8 years.

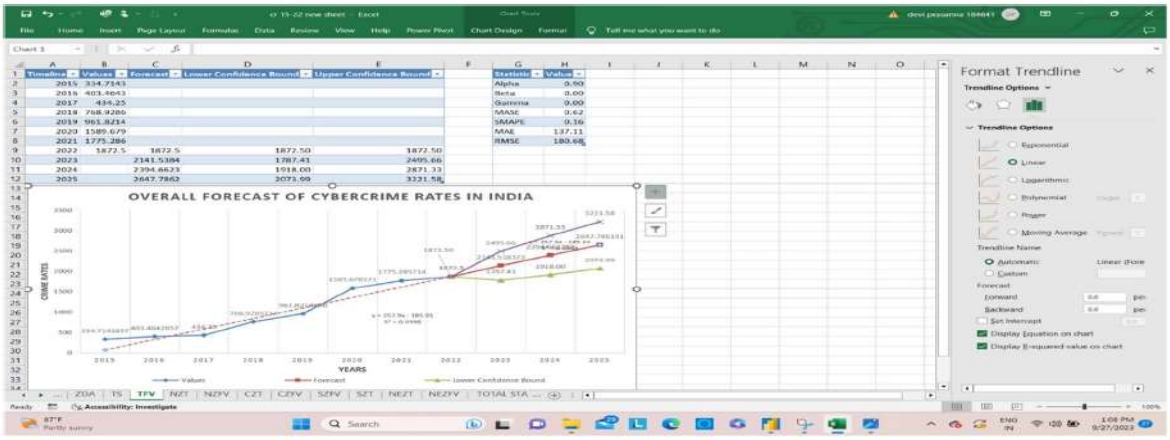
3.1.2 FROM THE ABOVE FORECASTING MODEL:

- The estimated trend line using linear regression analysis is given as follows:

  $Y = 257.9x - 185.91$; $R^2 = 0.9998$.

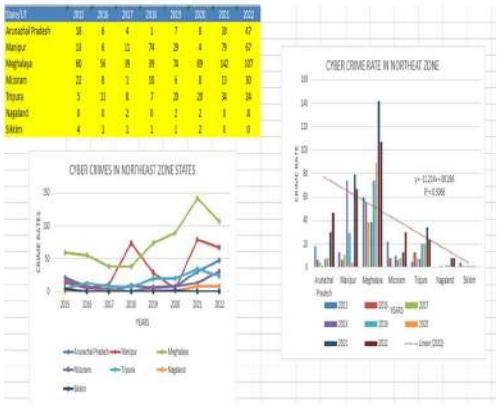
- The future trends for the years 2023,24 and 25 will be increases with an

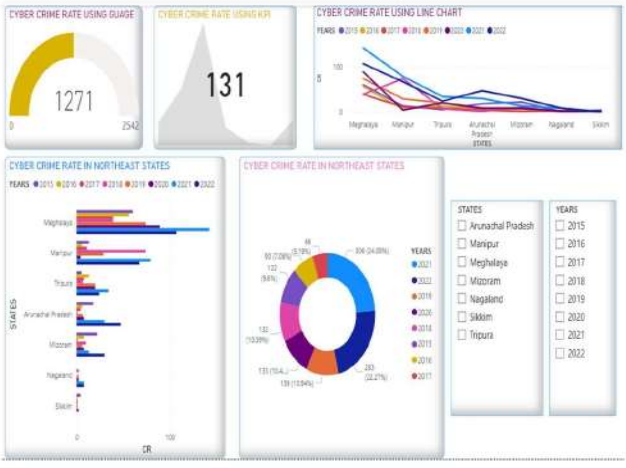
  Mean Absolute Error(MAE) = 137.11



3.2NORTHEAST ZONE:

We have observed that the cybercrime rates patterns have been fluctuating in this zone. For the detail study of these pattern we use line charts and clustered column charts





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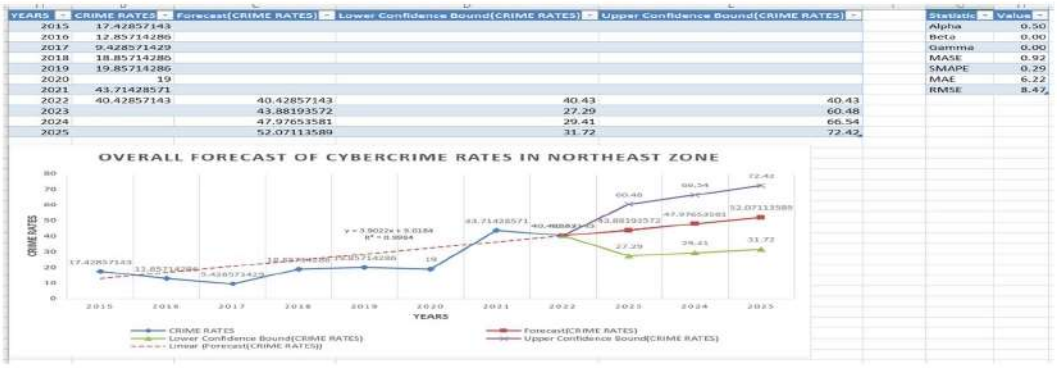
3.2.1 FROM THE DASHBOARD, OUR OBSERVATIONS ARE:

The trends in cybercrime rates in Northeast zone has been increasing. The highest cybercrime rates have been recorded in the year 2021 with 24.08%. The highest and lowest cybercrime rate has been recorded in Meghalaya and Sikkim.



YEARS	CRIME RATES	Forecast(CRIME RATES)	Lower Confidence Bound(CRIME RATES)	Upper Confidence Bound(CRIME RATES)
2015	17.42857143			
2016	12.85714286			
2017	9.428571429			
2018	18.85714286			
2019	19.85714286			
2020	19			
2021	43.71428571			
2022	40.42857143	40.42857143	40.43	40.43
2023		43.88193572	27.29	60.48
2024		47.97653581	29.41	66.54
2025		52.07113589	31.72	72.42

Statistic	Value
Alpha	0.50
Beta	0.00
Gamma	0.00
MASE	0.92
SMAPE	0.29
MAE	6.22
RMSE	8.47

3.2.2 FROM THE ABOVE FORECASTING MODEL:

- The estimated trend line using linear regression analysis is given as follows:

  $Y = 3.9022x + 9.0184$; $R^2 = 0.9984$.

- The future trends for the years 2023,24 and 25 will be increases with an

  MAE = 6.22

  Hence, we obtain the trend line for forecasting the future values of the years 2023, 24 and 25 by using the above forecasting model. We observe the change point of cybercrimes has been increasing rapidly.

3.3 CENTRAL ZONE

We have observed that the cybercrime rates pattern has been fluctuating in this zone.To study these pattern we use line charts and clustered column charts.

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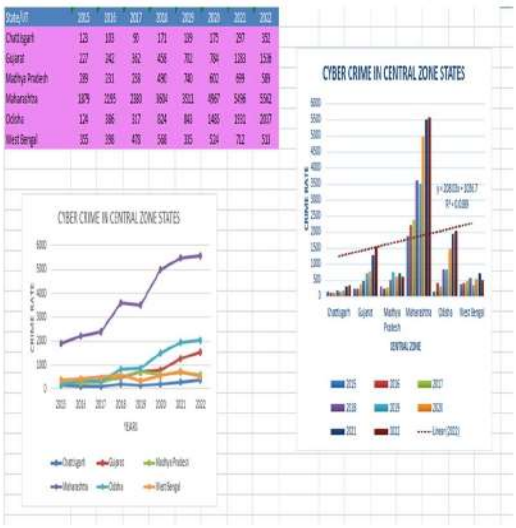

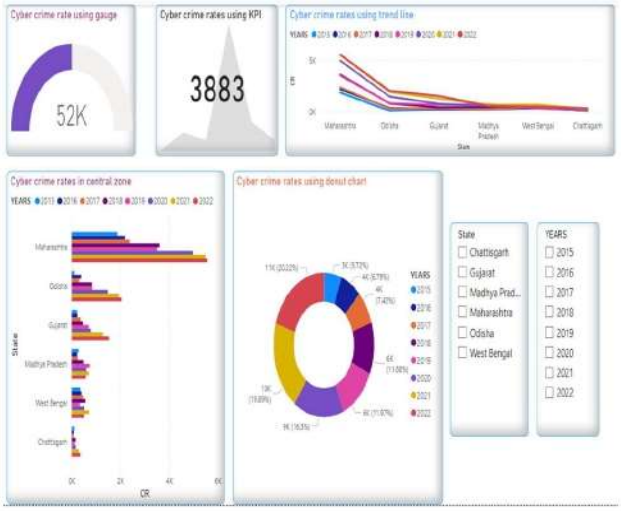

3.3.1FROM THE DASHBOARD, OUR OBSERVATIONS ARE:

Through this our findings are that the highest cybercrime rates have been recorded in the year 2022 with 20.22%.The highest and the lowest cybercrime rate has been recorded in Maharashtra and Chhattisgarh.

3.3.2 FROM THE ABOVE FORECASTING MODEL:

- The estimated trend line using linear regression analysis is given as follows:

  $Y = 48.329x + 143.79$; $R^2 = 0.9984$.

- The future trends for the years 2023,24 and 25 will be increases with an

  MAE = 72.90

Hence, we obtain the trend line for forecasting the future values of the years 2023, 24 and 25 by using the above forecasting model. We observe the change point of cybercrimes has been increasing rapidly.

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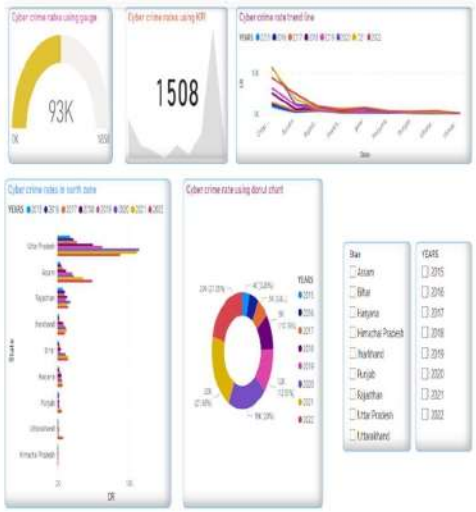
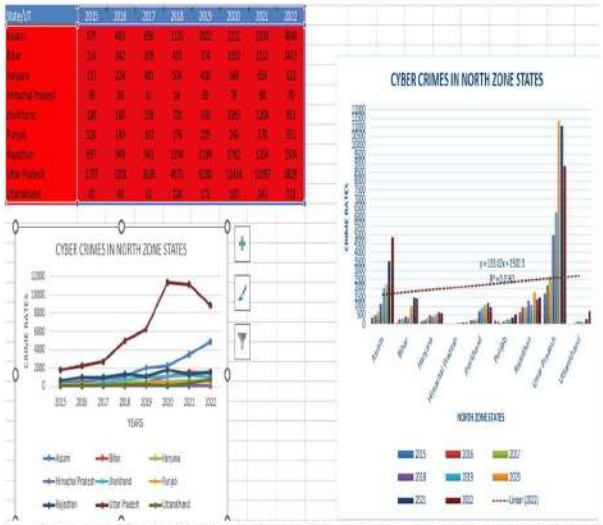
3.4 NORHT ZONE

We have observed that the cybercrime rates patterns have been fluctuating in this zone.For the detail study of these pattern we use line charts and clustered column charts.





3.4.1 FROM THE DASHBOARD, OUR OBSERVATIONS ARE:

Through this our findings are that the highest cybercrime rates have been recorded in the year 2022 with 21.05%.The highest and the lowest cybercrime rate has been recorded in Uttar Pradesh and in Himachal Pradesh.

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3.4.2 FROM THE ABOVE FORECASTING MODEL:

- The estimated trend line using linear regression analysis is given as follows:

$Y=365.96x - 214.89$; $R^2= 0.9816$.

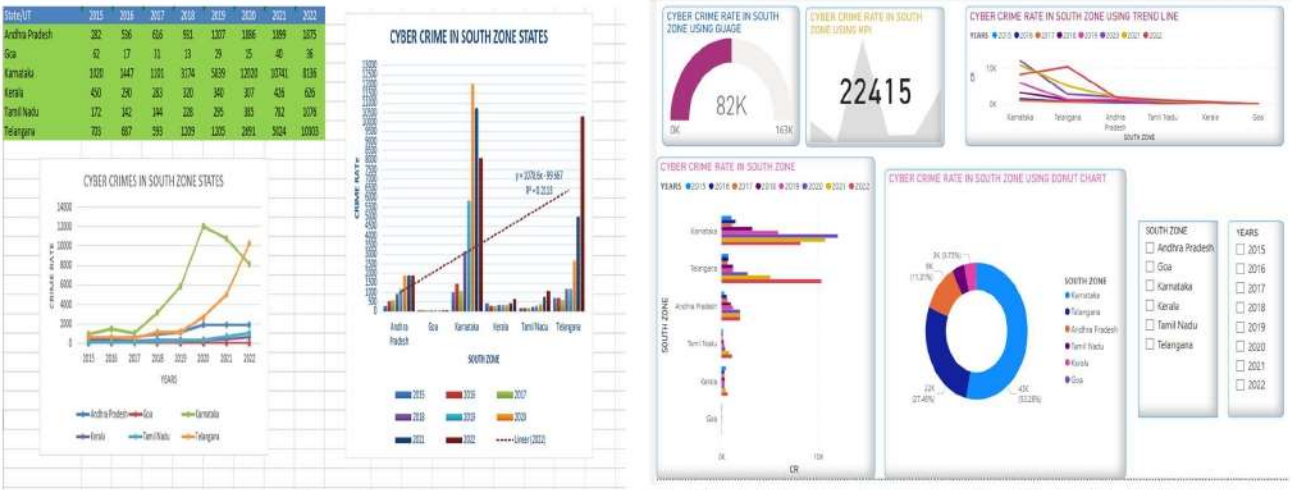
- The future trends for the years 2023,24 and 25 will be increases with an

MAE = 198.

Hence, we obtain the trend line for forecasting the future values of the years 2023, 24 and 25 by using the above forecasting model. We observe the change point of cybercrimes has been increasing rapidly.

3.5 SOUTH ZONE:

We have observed that the cybercrime rates patterns have been fluctuating in this



zone.For the detail study of these patterns we use line charts and clustered column charts.

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3.5.1 FROM THE DASHBOARD, OUR OBSERVATIONS ARE:

Through this our findings are that the highest cybercrime rates have been recorded in the year 2022 with 23.26%.The highest and lowest cybercrime rate has been recorded in

Timeline	Values	Forecast	Lower Confidence Bound	Upper Confidence Bound
2015	448.1667			
2016	519.8333			
2017	461.3333			
2018	979.1667			
2019	1485.833			
2020	2884			
2021	3152			
2022	3675.333	3675.3333	3675.33	3675.33
2023		4194.1174	3377.51	5010.72
2024		4711.9416	3613.67	5811.02
2025		5229.5663	3906.41	6552.72

Statistic	Value
Alpha	0.90
Beta	0.00
Gamma	0.00
MASE	0.57
SMAPE	0.24
MAE	274.66
RMSE	416.64

Karnataka Tamil Nadu.

Figure: 6 Sample timeseries and change points

3.5.2FROM THE ABOVE FORECASTING MODEL:

- The estimated trend line using linear regression analysis is given as follows:
  $Y=518.04x - 468.69$; $R^2= 1$.
- The future trends for the years 2023,24 and 25 will be increases with an
  MAE = 274.66.

Hence, we obtain the trend line for forecasting the future values of the years 2023, 24 and 25 by using the above forecasting model. We observe the change point of cybercrimes has been increasing rapidly.

4. Conclusion:

A Linear Forecast Trend line was fitted for each zone separately among the states. Based on the timeseries, forecasting helped to identify the change points in cybercrime detection values across India. On average Uttar Pradesh reported the highest cybercrime rates at 21.5% followed by Karnataka with 19%, Maharashtra with 13%, Telangana with 10% and Assam with 6.8%. These five states recorded the highest cybercrime percentages over the past 8 years. It was observed that cybercrime trends have shifted significantly since 2018, with an increase of around 23% compared to earlier years, and the peak cybercrime activitiy occurred in 2022, showing a 58% rise from 2018.

However, trends indicates the number of field cases is growing faster than those resolved, mainly due to negligence and a lack of skilled cyber professionals to manage such situations. Therefore, there is a need for improved training facilities and a mandatory cyber-security branch to assist in resolving cyber cases and boosting trust in security measures. Nonetheless, we should take steps to protect ourselves.

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NO INSTITUTION OR ANTIVIRUS CAN FULLY PROTECT ANYONE, BUT BY APPLYING PROPER MEASURES AND PREVENTIVE METHODS, WE CAN SAFEGUARD OUR DATA INFORMATION, DEVICES, AND OURSELVES.

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LUNG CANCER PREDICTION ANALYSIS USING MACHINE LEARNING TOOLS

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Abstract

Lung cancer is a severe and often difficult-to-detect disease that affects individuals of all genders. Early identification of lung nodules is essential for improving patient outcomes, making accurate and timely detection methods critical. This research focus on evaluating and comparing different machine learning algorithms for lung cancer classification. In particular, logistic regression and decision tree models were implemented and analyzed using R software. The experimental findings indicate that decision trees demonstrate superior performance in terms of prediction accuracy compared to logistic regression. This study highlights the effectiveness of traditional machine learning approaches in medical diagnostics and supports the use of decision tree models as a reliable method for early lung cancer detection.

Keywords: Lung Cancer Detection, Machine Learning, decision trees, logistic regression.

1. Introduction

Lung cancer is a type of malignancy originating in the lungs, which are essential organs responsible for exchanging oxygen and carbon dioxide during breathing. It is one of the most significant causes of cancer related mortality worldwide. Although smoking is the leading risk factor, lung cancer can also affect individuals who have never smoked. The likelihood of developing the disease increases with age and prolonged exposure to harmful substances. Smoking cessation, even after years of use, can substantially reduce the risk. Traditionally, diagnosing lung cancer has required multiple medical tests and lengthy evaluations, which may sometimes lead to unnecessary procedures. To streamline this process, advancements in technology now emphasize the use of predictive models, with machine learning playing crucial role in improving diagnostic accuracy and reducing delays.

As part of the respiratory system, the lungs are particularly vulnerable to environmental hazards such as polluted air, which often contains bacteria and toxins capable of damaging lung tissue. Lung cancer is a genetic disease caused by mutations in genes that control cell function, especially the function to grow and divide. Cancer is new cells that regulate cell growth and division, resulting in uncontrolled cell proliferation and the potential spread of cancerous cells to other organs. It remains one of the most prevalent and life-threatening forms of cancer globally. Several factors contribute to its onset, including long- term smoking, passive exposure to second hand smoke, genetic predisposition, air pollution, radon exposure, and toxic

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chemicals such as asbestos. Due to its aggressive nature, lung cancer is often difficult to treat in advanced stages, making early detection essential.

2. Materials and Methods:

There are 309 instances in the dataset. There are 16 attributes that describe the instances. The data is obtained from the Google kaggle datasets.

2.1 About Software

R is a freely available, open source programming language primarily used for statistical analysis and data visualization. It is distributed under the General Public License (GPL), making it accessible online. R studio, a popular integrated development environment (IDE) for R, enhances its usability by providing tools for coding, debugging, and visualization. RStudio is available in two main forms: **RStudio Desktop**, which runs locally on a computer, and **RStudio Server**, which operates on a remote server and can be accessed via a web browser.

Several Packages in R extend its functionality. The **rpart** package, for example, enables the construction of classification and regression trees. **Tidyverse** is a collection of packages built around a consistent design framework, simplifying data manipulation and visualization. The **glmnet** package supports fitting generalised linear models using regularization techniques like LASSO and ridge regression. Another widely used pacakage, **caret**, streamlines the process of building and comparing machine learning models for both classification and regression. **Dplyr** provides an intuitive set of functions for data transformation and cleaning tasks. Additionally, **rpart.plot** offers tools to visualize decision trees created with rpart, automatically adjusting layouts for improved readability.

2.2 Statistical tools under study

- **Logistic Regression:** logistic regression is statistical modelling approach primarily used for problems, where the outcome has two possible categories. It estimates the likelihood of a binary result based on one or more multiple predictor variables. This method is straight forward to interpret, requires minimal computational effort and is particularly effective when the target variable is binary and its relationship with predictors is approximately linear.
- **Decision Tree:** Decision trees are versatile models that can work with both categorical and numerical variables. They can capture complex, non-linear relationships between predictors and the target outcome, making them suitable for datasets where traditional linear models struggle. Because of their ability to handle intricate interactions among features, decision trees are often chosen when the data structure is complex and involves multiple variables with diverse relationships.

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3. Results and Discussion:

3.1 Logistic Regression

```
Install.packages("tidymodels")

library(tidymodels)

>logistic_model<-glm(LUNG_CANCER ~ GENDER + AGE + SMOKING + YELLOW_FINGERS +ANXIETY
+ PEER_PRESSURE + CHRONIC.DISEASE + FATIGUE + ALLERGY + WHEEZING + ALCOHOL_CONSU
MING + COUGHING + SHORTNESS_OF_BREATH + SWALLOWING_DIFFICULTY + CHEST_PAIN, famil
y = "binomial", data = LUNG.CANCER)

>summary (logistic_model)
```

Output:

Coefficients:

	Estimate	Std.Error	z value	Pr(>|z|)
(Intercept)	-31.708	5.807	-5.460	4.76e-08***
Gender	0.526	0.708	0.742	0.458
Age	0.021	0.033	0.643	0.520
Smoking	1.776	0.701	2.530	0.011*
Yellow Fingers	1.376	0.742	1.854	0.063
Anxiety	0.887	0.812	1.092	0.274
Peer Pressure	1.731	0.660	2.622	0.008**
Chronic Disease	3.191	0.888	3.593	0.000***
Fatigue	3.070	0.825	3.721	0.000***
Allergy	1.646	0.768	2.141	0.032*
Wheezing	0.966	0.834	1.158	0.246
Alcohol Consuming	1.409	0.798	1.765	0.077
Coughing	3.311	1.071	3.090	0.002**
Shortness of Breath	-0.728	0.760	-0.959	0.337
Swallowing Difficulty	3.122	1.129	2.763	0.005**
Chest Pain	0.559	0.689	0.811	0.417

Significance. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

3.2 Decision Tree

```
    install.packages("rpart.plot")
    install.packages("rpart")
library(rpart.plot)
    library(rpart)
    >tree <- rpart(LUNG_CANCER~GENDER+AGE+SMOKING+YELLOW_FINGERS+ANXIETY+PEER_PRESSURE+CHRO
    NIC.DISEASE+FATIGUE+ALLERGY+WHEEZING+ALCOHOL.CONSUMING+COUGHING+SHORTNESS.OF.BREATH+
    SWALLOWING.DIFFICULTY+CHEST.PAIN,data=LUNG.CANCER,control=rpart.control(cp=.0001))
>tree
        n= 309

        node), split, n, deviance, yval
```

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```
      * denotes terminal node

     1) root 309 34.077670 0.8737864
       2) ALLERGY< 1.5 137 25.562040 0.7518248
         4) SWALLOWING.DIFFICULTY< 1.5 68 16.632350 0.5735294
           8) COUGHING< 1.5 37  9.081081 0.4324324
            16) YELLOW_FINGERS< 1.5 24  4.958333 0.2916667
              32) AGE>=60.5 14  1.714286 0.1428571 *
              33) AGE< 60.5 10  2.500000 0.5000000 *
            17) YELLOW_FINGERS>=1.5 13  2.769231 0.6923077 *
         9) COUGHING>=1.5 31  5.935484 0.7419355
           18) FATIGUE< 1.5 8  1.875000 0.3750000 *
           19) FATIGUE>=1.5 23  2.608696 0.8695652
             38) SMOKING< 1.5 13  2.307692 0.7692308 *
             39) SMOKING>=1.5 10  0.000000 1.0000000 *
       5) SWALLOWING.DIFFICULTY>=1.5 69  4.637681 0.9275362
         10) PEER_PRESSURE< 1.5 23  3.913043 0.7826087
           20) WHEEZING< 1.5 7  1.714286 0.4285714 *
           21) WHEEZING>=1.5 16  0.937500 0.9375000 *
         11) PEER_PRESSURE>=1.5 46  0.000000 1.0000000 *
      3) ALLERGY>=1.5 172  4.854651 0.9709302
        6) YELLOW_FINGERS< 1.5 85  4.705882 0.9411765
       12) ALCOHOL.CONSUMING< 1.5 9  2.000000 0.6666667 *
       13) ALCOHOL.CONSUMING>=1.5 76  1.947368 0.9736842
         26) FATIGUE< 1.5 23  1.826087 0.9130435
           52) CHRONIC.DISEASE< 1.5 11  1.636364 0.8181818 *
           53) CHRONIC.DISEASE>=1.5 12  0.000000 1.0000000 *
       27) FATIGUE>=1.5 53  0.000000 1.0000000 *
      7) YELLOW_FINGERS>=1.5 87  0.000000 1.0000000 *

    >best <- tree$cptable[which.min(tree$cptable[,"xerror"]),"CP"]
    >best
    [1] 0.004563176

>pruned_tree<- prune(tree, cp=best)
>pruned_tree

      n= 309

      node), split, n, deviance, yval
     * denotes terminal node

    1) root 309 34.077670 0.8737864
       2) ALLERGY< 1.5 137 25.562040 0.7518248
         4) SWALLOWING.DIFFICULTY< 1.5 68 16.632350 0.5735294
           8) COUGHING< 1.5 37  9.081081 0.4324324
            16) YELLOW_FINGERS< 1.5 24  4.958333 0.2916667
              32) AGE>=60.5 14  1.714286 0.1428571 *
              33) AGE< 60.5 10  2.500000 0.5000000 *
            17) YELLOW_FINGERS>=1.5 13  2.769231 0.6923077 *
           9) COUGHING>=1.5 31  5.935484 0.7419355
```

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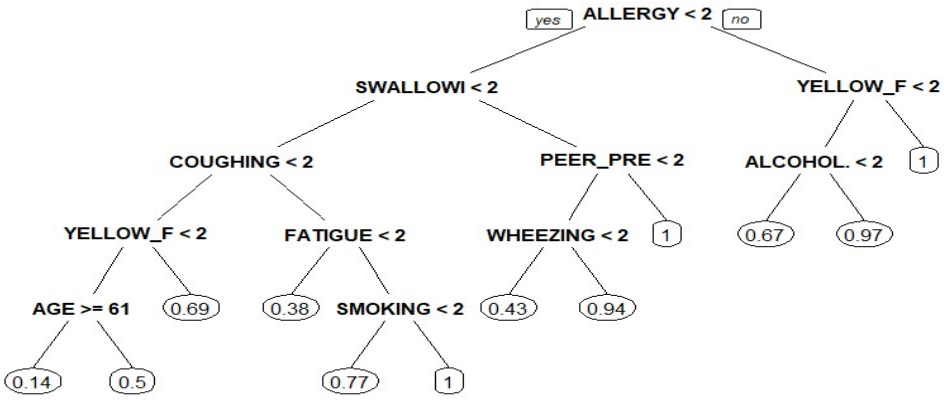
```
 18) FATIGUE< 1.5 8  1.875000 0.3750000 *
      19) FATIGUE>=1.5 23  2.608696 0.8695652
        38) SMOKING< 1.5 13  2.307692 0.7692308 *
        39) SMOKING>=1.5 10  0.000000 1.0000000 *
    5) SWALLOWING.DIFFICULTY>=1.5 69  4.637681 0.9275362
      10) PEER_PRESSURE< 1.5 23  3.913043 0.7826087
        20) WHEEZING< 1.5 7  1.714286 0.4285714 *
      21) WHEEZING>=1.5 16  0.937500 0.9375000 *
    11) PEER_PRESSURE>=1.5 46  0.000000 1.0000000 *
3) ALLERGY>=1.5 172  4.854651 0.9709302
    6) YELLOW_FINGERS< 1.5 85  4.705882 0.9411765
      12) ALCOHOL.CONSUMING< 1.5 9  2.000000 0.6666667 *
      13) ALCOHOL.CONSUMING>=1.5 76  1.947368 0.9736842 *
    7) YELLOW_FINGERS>=1.5 87  0.000000 1.0000000 *

>prp(pruned_tree)
```



4. Conclusions:

In this conclusion, we predict the cause of lung cancer by utilizing the machine learning models, specifically logistic regression and decision tree. In logistic regression, fatigue and chronic disease are the major symptoms of affecting the lung cancer and through the decision tree it explains that yellow fingers and fatigue are the major side effects to cause lung cancer. Here, we conclude that by comparing the both machine learning algorithms, fatigue symptoms are major role to cause lung cancer.

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FORECASTING OF FOREIGN TOURIST ARRIVALS (FTA) BASED ON ARIMA AND ANN MODELS

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ABSTRACT:
Accurate forecasting Foreign Tourist Arrivals (FTA) facilitates their contribution in the Tourism sector and for efficient resource planning. This research article is based on two forecasting model namely ARIMA which is a conventional statistical model for forecasting with the help of past data and ANN which is a machine learning technique. In order to understand the efficacy of prediction, researcher applied time series data of FTA data of India from the time period between 1965 and 2022. In the ARIMA model, Augmented Dickey-Fuller (1979) test statistic is used to check stationarity and on the basis of AIC and BIC the values of (p, d, q) for the ARIMA model for Foreign Tourist Arrivals (FTA) is (2, 1, 1). After testing with several models of ANN , the Average of 20 networks with 4 weights options and each network is a (1-1-1) linear output units were taken, then Sigma^2 measured as 288137. And other hand comparison between the ARIMA and ANN results to highlight the ANN model which is more efficient than the ARIMA.

KEYWORDS: Forecasting, AIC, BIC, ARIMA, ANN

1. INTRODUCTION

Tourism is an important part of the Indian economy, and foreign tourist arrivals play a major role in the development of the tourism industry. Predicting foreign tourist arrivals in India in 2023 is an important task for policymakers, tourism operators, and other stakeholders. Accurate forecasts can help in planning for infrastructure, managing resources, and developing strategies to attract more foreign tourist to India.

Several factors influence foreign tourist arrivals in India, such as the country's political stability, the availability of travel facilities, the security situation, and the economic conditions of the tourists' countries of origin (Mishra, 2017). Forecasting models that take into account historical trends and current economic indicators can provide a reliable prediction of the expected number of future foreign tourists arriving in India.

COVID-19 pandemic has significantly impacted the tourism industry globally, and India has not been immune to its effects. The forecasts for 2023 must consider the potential continuation of travel restrictions, the recovery of the global economy, and the willingness of tourists to travel internationally. Despite these challenges, the Indian tourism industry has shown resilience and adaptability, and the forecasts for 2023 remain positive.

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2. LITERATURE REVIEW

Yollanda, M., &Devianto, D. (2020) used Seasonal Autoregressive Integrated Moving Average (SARIMA) and Artificial Neural Network (ANN) for predicting tourist arrivals at Minangkabau International Airport. In this study, Although SARIMA (1, 0, 1) (1, 1, 0)12 performed well, its residual did not meet the autocorrelation condition, leading to the proposal of a new SARIMA-ANN model. The SARIMA residual model has ANN model architecture and 2-2-2-1 network topology. The performance rate of visitor arrivals between January 2012 and March 2019 is calculated by using Mean Absolute Percentage Error (MAPE). The model's performance in predicting the number of tourists who will arrive in future is good, as indicated by the MAPE value of 17.1770%. **Cho, V. (2003)** compared the use of three time-series forecasting models univariate ARIMA, exponential smoothing, and Artificial Neural Networks (ANN) to forecast the demand of travelling to Hong Kong by other countries. The traditional statistical time series forecasting techniques are exponential smoothing and ARIMA. The analysis shows that neural networks is an AI machine learning approach that seems to be the most precise in terms of predicting visitor arrivals, especially when dealing with series, which do not have an obvious trend. **Li, X., et.al. (2021)** in this study researcher focused on four machine learning–based feature selection techniques to study the complicated econometric and time series data models. They discussed some of the suggested approaches for predicting visitor numbers in Beijing, China, as well as weekly predictions for hotel occupancy in Charleston, South Carolina, in the United States. Findings show that the forecasting model using the selected search keywords outperforms the baseline ARMAX model that does not select features in predicting the travel demand and hotel occupancy. As a result, machine learning algorithms can find the data from the most useful search queries to greatly increase the accuracy of tourism and hospitality predictions. **BiljanaPetrevska (2017)** studied The Demand of short term global tourism is estimated by Box- Jenkins approach with emphasis on F.Y.R. Macedonia. And alternative specifications are tried on international tourist arrivals data recorded between 1956 - 2013. According to the findings, the ARIMA (1,1,1) model is considered the most appropriate to predict.The model of A.R.I.M.A.(1,1,1) is deemed to be the most suitable for predicting based on the findings. The study's findings indicate that by 2018, there will be a 13.9% rise in foreign visitor arrivals. The chosen model's predicted values can help with the creation of a country's tourism development plan as well as reduce any potential negative effects. **Ismail, E. A. A. (2020)** used Egypt Foreign tourist's data from 1981 to2017 fora forecast. Applies ARIMA model According to the study, the most appropriate model for both time series for Arab and foreign tourists is ARIMA (0, 1, 0). According to the survey, there will be more tourists in Egypt. Nevertheless, it is anticipated that during the next five years, the percentage of increase would decline. **Swaraj, et.al (2021)** researched In order to obtain both linear and nonlinear trends of the COVID-19 data, a new hybrid model was chosen instead of a single ARIMA model. When this hybrid combination was compared to one ARIMA model on daily observed cases, it was observed that the RMSE (16.23%), MAE (37.89%), and MAPE (39.53) values were significantly decreased**.** Similar findings with lower mistake rates were made for both cases of recovery and daily reported deaths. Our hybrid model's RMSE value was lower than that of other models that were used to predict COVID-19 across various nations. **Xie, G., Qian, Y., & Wang, S. (2020)** Given the nonlinearity and complexity of visitor arrivals, ANN models typically outperform ARIMA models in terms of forecast accuracy. The ensemble neural networks (ENN) model demonstrates more stability and may typically achieve better forecasting performance when compared to back propagation neural network (BPNN) and back propagation neural network (GRNN) models. The time series of some deconstructed components may have many scales. As a result, MPE is capable of quantifying the complexity of the pertinent components more accurately than single-scale permutation entropy. Finally, a model's capacity for forecasting is highly correlated with the time series data properties. The data properties should be recorded for the prediction of the rebuilt

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components. Dummy variables can be used to properly describe mutability and seasonality. **Lim, S. S. (2018)** presents winter's exponential seasonal smoothing model, SARIMA model, and intervention analysis to forecast the tourists visiting Korea. This study's primary goals are to identify the variables that affect forecasting models and to identify a time series model for a more accurate prediction of the number of foreign visitors to Korea. As a result of comparing the accuracy of forecasting models through empirical analysis, it was founded that the intervention analysis model reflecting the intervention effect was superior to the seasonal exponential smoothing model or the seasonal ARIMA model. **H. Peiris (2016)** tried to find an appropriate SARIMA model to forecast the arrival of foreign tourists in Sri Lanka. The analysis is based on monthly visitor arrival data, between January 1995 and July 2016. The time series is tested for seasonality by means of the HEGY test. The forecasting accuracy is evaluated by Root Mean Squared Error (RMSE), Mean Absolute Error (MAE) and Mean Absolute Percent Error (MAPE). The result shows the appropriateness of SARIMA (1, 0, 16) (36, 0, 24)12 model for forecasting arrival of tourists in Sri Lanka. **Ahire, M., et.al (2020)** employed the ARIMA approach for monthly data on medical tourism from 2014 to 2017. The implementation of the model and these patterns are discussed in the study. According to predictions, medical tourism offers a significant opportunity for the nation to generate significant foreign exchange.

3. METHODDOLOGY

3.1 Auto Regressive process: The autoregressive process is a stochastic approach that represents a regression of current value using the past values from time series data. As current value is defined by previous values, so the value doesn't alter instantaneously. Assume Xt is regressed on previous values of itself and it will explain the prefix 'auto' in the regression process, then

$$X_t = \varphi_1 X_{t-1} + \varphi_1 X_{t-2} + \cdots . + \varphi_p X_{t-p} + \varepsilon_t$$

Where $\varphi_1, \varphi_2 \ldots \varphi_p$ are constants. According to the first order AR (1) can be mentioned as $X_t = \varphi_1 X_{t-1} + \varepsilon_t$ which is also called first order property of Markov, provided $|\varphi_1| < 1$, then the AR (1) becomes infinite MA process, due to that $E(X_t) = 0$ Then $X_t = \sum_{i=1}^{p} \varphi_1 X_{t-1} + \varepsilon_t$ and $|\varphi| < 1$

In general stochastic approach, Auto regressive process with order 'p' can be written as $Z_t = (1 - \varphi_1\beta - \varphi_2\beta^2 - \varphi_3\beta^3 - \cdots \varphi_p\beta^p)$

3.2 Moving Averages: Moving averages are stochastic process that confines the relative average change in a time series data over the period of time. In many cases of time series predictions, moving averages are quite appropriate to measure trend pattern of the data. The MA process of order q or MA (q) is given by the moving averages with 'q' can be written as

$X_t = \mu + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots . + \theta_p e_{t-q} + \epsilon_t$ Here $\theta_1, \theta_2 \ldots \ldots \theta_q$ are constants and

ϵ_t random error with zero mean and variance $\sigma^2 (>0)$

In general q^{th} order MA can be written as $X_t = \mu + \sum_{i=1}^{q} \theta_i e_{t-i} + \epsilon_t$

3.2 Auto-Regressive Integrated Moving Average (ARMA)

ARMA is the mixture of the AR and MA models. ARMA model covers all features of AR and MA. The ARMA predicts the future values using previous data values and error terms. So ARMA performs better than AR & MA alone.

Then ARMA with 'p' AR terms and 'q' MA will be

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$$X_t = C + \sum_{i=1}^{p} \varphi_1 X_{t-1} + \sum_{i=1}^{q} \theta_i \, e_{t-i} + \epsilon_t$$

3.3 Auto-Regressive Integrated Moving Average (ARIMA)

It is one of the best conventional statistical methods for predicting future values using past data. Because of the seria correlation time series data is influenced by the past values **(Jere 2019).** So ARIMA classified into (p,d,q) where **p** belongs to autoregressive terms, **d** belongs to non seasonal differences needed for stationary, and **q** referred as lagged forecast errors in the prediction equation.

Box-Jenkins model is one type of Integrated Moving Average model, it can be written in the following way

$W_t = \mu + \frac{\theta(B)}{\Phi(B)} e_t$, where μ=mean term

$\theta(B) = 1-\theta_1 B- \theta_2 B \ldots \theta_p B$

$\varphi(B) = 1- \varphi_1(B)-\varphi_2(B)\ldots-\varphi_q(B)$

e_t=random error

This model can be also be written as

$\varphi(B)(W_t - \mu) = \theta(B) e_t$

In general the prediction formula of ARIMA is

$\hat{y}_t = \mu + \theta_1 y_{t-1} + \ldots + \theta_p y_{t-p} - \varphi_1 e_{t-1} - \ldots - \varphi_p e_{t-q}$

3.4 Stationarity Test

In order to test, many researchers will prefer ADF test statistic. If the data is not stationary, we should choose alternatives to convert the data into stationary. In this process the first step is to establish the number of lags required to convert the time series data into stationary as no model fits if the data is non-stationary.

Regression model for ADF (1979) is

$\Delta Y_t = \alpha + \beta_0 t + \gamma Y_{t-1} + \delta_1 \Delta Y_{t-1} + \cdots + \delta_{p-1} \Delta Y_{t-p+1} + \varepsilon_t$ Here α is a constant, β is trend, γ is coefficient of lagged variables.

Usually for any data, common null hypothesis can be taken as H_0: Series is not stationary (φ=0) and then null hypothesis becomes H_1: The series is stationary (φ≠0) , then test statistic for ADF is

ADF= $\frac{\gamma}{SE(\gamma)}$

3.5 Model selection and validation

Using AIC & BIC. (Akaike 1974) and (Shewartz 1978) minimum defines the best.

$AIC = -2LogL_i+2P_i$

$BIC = -2LogL_i+P_iLog(n)$

Apart from the above researcher is using RMSE and MAPE. It is the typical multiplicative impact between each observed mean and estimated mean. While two of them sum up the changeability of the perceptions around the mean, they are not in a similar scale so don't anticipate that the qualities should be comparative..

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RMSE= $\sqrt{\frac{\sum_{t-1}^{n}(A_t - F_t)^2}{n}}$.

MAPE= $\frac{\sum_{t-1}^{n}\left|\frac{A_t - F_t}{A_t}\right|}{n} \times 100$

3.6 Artificial Neural Network (ANN)

ANN is a powerful circulated parallel processor with a neural resemblance for storing experimental data and making it available for further use. It appears that the human cerebrum has always piqued scientists' interest due to its quickness and efficiency. The goal of understanding these cycles and resolving associated problems has driven the advancement of ANN approach. In essence, neural networks have a nonlinear exhibiting method that provides an extremely accurate estimate of any data. The model structure measure does not require any prior assumption regarding the model structure. All things considered, the informational characteristics often govern the organisation model. The model structure measure does not require any prior assumption regarding the model structure. All things considered, the informational characteristics often govern the organisation model. The model structure is most frequently used for time series machine learning modelling and prediction. In theory, forecasting is a perfect modelling domain for neural networks because it involves predicting future behaviour patterns from instances of past conduct [Koizumi k 1999]. The inputs in the forward process work through the input layer, and the impact of those functions spreads through the network layer by layer.The weighted amount of the previous layer's neurons' yield is processed as the net impact.

4. RESULTS AND DISCUSSIONS

Foreign tourist arrivals (FTA) of India during the period between 1965 and 2022 obtained from data files of World Tourism Organization considered to project the future arrivals

4.1 Time series data plot

The following table 1 represents the data of FTA and figure showing trend pattern.

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	Foreign Tourist Arrivals (FTAs) in India during 1965-2022						
YEAR	**FTA**	**YEAR**	**FTA**	**YEAR**	**FTA**	**YEAR**	**FTA**
1965	147500	**1980**	800150	**1995**	2123683	**2010**	5775692
1966	159603	**1981**	1279210	**1996**	2287860	**2011**	6309222
1967	132680	**1982**	1288162	**1997**	2374094	**2012**	6577745
1968	118820	**1983**	1304976	**1998**	2358629	**2013**	6967601
1969	244724	**1984**	1193752	**1999**	2481928	**2014**	7679099
1970	280821	**1985**	1259384	**2000**	2649378	**2015**	8027133
1971	200995	**1986**	1451076	**2001**	2537282	**2016**	8804411
1972	342950	**1987**	1484290	**2002**	2384364	**2017**	10177000
1973	409895	**1988**	1590661	**2003**	2726214	**2018**	10563000
1974	433161	**1989**	1736093	**2004**	3457477	**2019**	10932000
1975	465275	**1990**	1707158	**2005**	3918610	**2020**	2740000
1976	533951	**1991**	1677508	**2006**	4447167	**2021**	1523000
1977	640422	**1992**	1867651	**2007**	5081504	**2022**	6197000
1978	747995	**1993**	1764830	**2008**	5282603		

Table 1: Foreign Tourist Arrivals (FTAs) in India during 1965-2022

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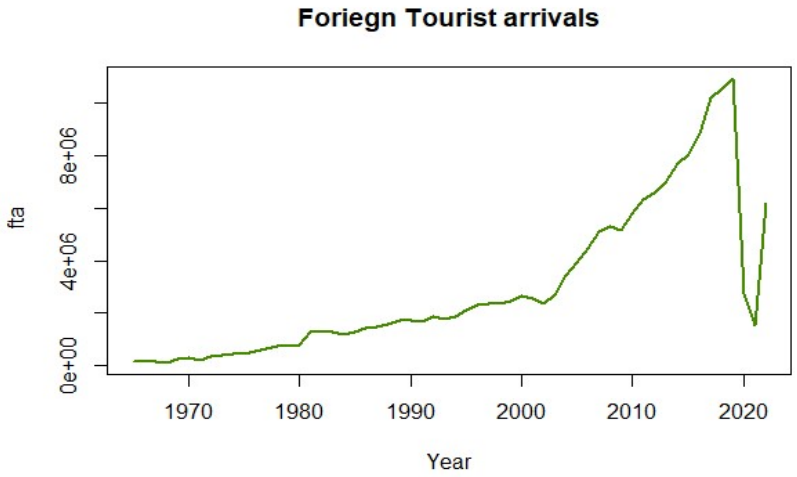

Figure1: Time series plots of FTAs.

Figure 1 represents significant increase in arrivals of tourist from 1970 and tremendous turn down in tourist arrivals around the years 2019 to 2021 due COVID 19 circumstances and slowly recovered after 2021.

4.2 Summary

Table 2 provide information about the distribution and descriptive measures of the FTAs over the years. The average no.of tourists across the years are 2,921,833 which may be influenced by outliers or extreme values in the dataset.

Measure	Value
Mean	2921833
Standard Error	377855.3
Median	1816241
Standard Deviation	2877660
Kurtosis	0.892329
Skewness	1.319684
Range	10813180
Minimum	118820
Maximum	10932000
3rd Quartile	4315028
1st Quartile	773623

Table 2: Foreign Tourist Arrivals (FTAs) summary statistics

The quartiles provide additional insights into the data distribution. The 1st quartile (25th percentile) at 773,623 suggests that 25% of the FTAs fall below this value. Similarly, the 3rd quartile (75th percentile) at 4,315,028 indicates that 75% of the FTAs are below this value. The minimum value of 118,820 represents the lowest number of FTAs recorded during the given period, while the maximum value of 10,932,000 represents the highest number of FTAs recorded. These statistics offer a summary overview of the FTAs in India over the specified time range, providing a sense of the distribution and range of tourist arrivals.

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4.3 ACF and PACF Plot

As per the data trend, Figure 2 representing the ACF & PACF where gradual mathematically declining ACF and PACF indicates significant for only some lags represents an AR process. Especially in ACF there is tremendous fall with lags. PACF has one huge slacks followed by a drop in PACF values and remaining are significant. So we consider ADF tests (unit root test) is using for further inferential analysis.

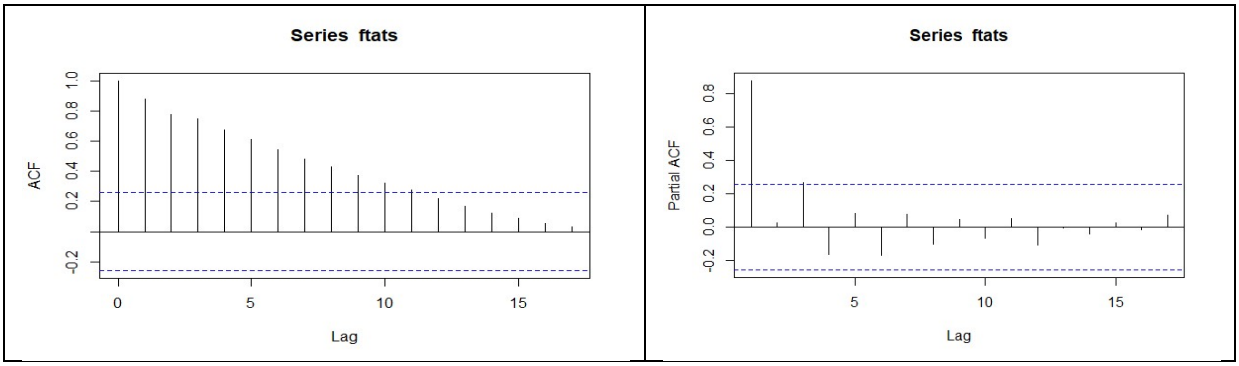

Figure2: ACF & PACF plots of FTAs

4.4Augmented Dickey-Fuller (ADF) test

ADF test is one the mostly preferred statistical investigation used to verify the time series stationarity, which is an important consideration when using time series models such as ARIMA. The ADF test compares the autocorrelation in the data at different lags to establish whether the time series data exhibits a unit root, indicating stationarity.

Augmented Dickey-Fuller Test

Dickey-Fuller = -0.71409, Lag order = 3, p-value = 0.9639

Alternative hypothesis: Stationary

In this case, the ADF test statistic -0.71409, p-value 0.9639. Lag order used in the test was 3, and the null hypotheses is non-stationarity and the alternative hypothesis is stationary. Since P is high greater than 0.05 we are accepting alternative hypothesis, meaning that there is no strong point to suggest that data is stationary. In other words, it is not. So it may need to be transformed (e.g., by differencing) before an ARIMA model can be applied. Since ADF test resulted that the data not stationary then we use transformation or differencing to convert the data. Present study took differencing to transform data will be shown in figure 3.

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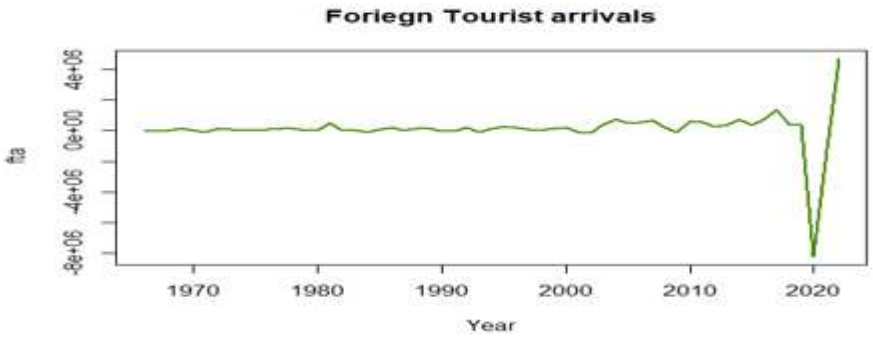

Figure 3: Time Series Plot after First Differencing

Validation

After performing the first differencing on data, the resulting differenced series display steady variability and having stationary time plot in both mean and variance. This recommends the differenced data is stationary. To further confirm stationarity for differenced data and assess autocorrelation function, and unit root tests will be conceded out again.

4.5 ACF and PACF Plot at First Differencing

It is important to note that the ACF and PACF plots can provide insights into the presence of autocorrelation and help determine the appropriate lag order.

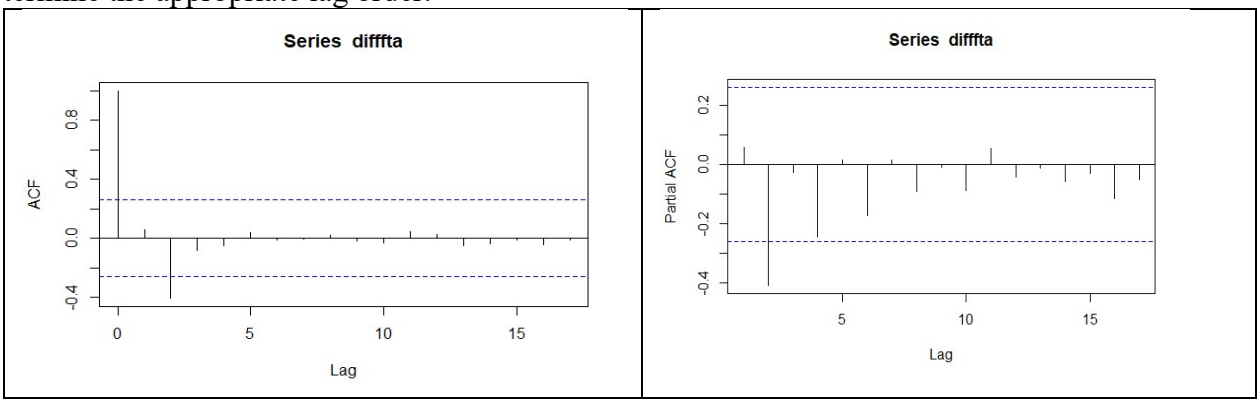

Figure 4: ACF and PACF Plot after first diferencing

After performing ADF test yielded a test statistic of -4.7307 with lag order 3, and P value is 0.01. Since the P value is below 0.05, as differenced data is stationary. This supports the visual analysis that indicated stationarity after differencing. Furthermore, you mentioned that the correllogram (presumably the ACF plot) indicated the presence of both AR & MA components in the time series data. This suggests that an ARIMA may be suitable for modelling the lagged time series data.

In summary, based on the information provided, it appears that after performing the first differencing data, the resulting series is stationary according to both visual analysis and the ADF test. Plots supporting potential use of an ARIMA model for further analysis

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4.6 ARIMA Model Fitting for the Foreign Tourist Arrivals

The values of ar1, ar2, and ma1 represent the strength and direction of relationships between past and present values of time series and influence of past errors on current values. Coefficients ar1 of 0.7766 indicates positive correlation connecting the current values of the time series and its previous values at lag1, which suggests a strong trend in the data.

Coefficients	ar1	ar2	ma1	Drift
	0.7766	-0.6236	-0.6954	136800.32
Standard error	136800.32	0.1480	0.1480	56884.37
Sigma^2 = 1.29e+12: log likelihood = -874.24				
AIC=1758.48 AICc=1759.65 BIC=1768.69				

Table 3: ARIMA model estimation

The coefficient ar1 of 0.7766 indicates positive correlation between the present values of the time series and its previous values at lag1, which suggests a strong trend in the data. The coefficient ar2 of -0.6236 indicates a negative correlation between the current value and its value at lag2, which suggests a possible seasonal pattern or some other cyclical effect in the data. The coefficient ma1 of -0.6954 indicates a negative correlation between the current error term and the previous error term at lag1, which suggests that the model is capturing some of the randomness in the data. The coefficient drift of 136800.32 represents the invariable terms in the model, which is added to the predicted values of the time series to account for any trend or other deterministic effects that are not explained by the autoregressive or moving average terms. In this case the positive value of the drift coefficient suggests a strong upward trend, which is not explained by the other components of the model

4.6.1 Ljung-Box test: The following test is preferred to check for the presence of autocorrelation. ARIMA (2, 1, 1) with Q* value 0.46878 at 7 degrees of freedom having P-value = 0.9996. The high P-value indicates that there is no significant proof of autocorrelation in ARIMA residuals. Thus, the ARIMA (2,1,1) model with drift is considered as a good fit for the data.

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st model:

S.no.	Model	Value
1	ARIMA(2,1,2) with drift	1762.012
2	ARIMA(0,1,0) with drift	1770.843
3	ARIMA(1,1,0) with drift	1772.841
4	ARIMA(0,1,1) with drift	1771.067
5	ARIMA(0,1,0)	1769.072
6	ARIMA(1,1,2) with drift	Inf
7	**ARIMA(2,1,1) with drift**	**1759.653**
8	ARIMA(1,1,1) with drift	1766.956
9	ARIMA(2,1,0) with drift	1761.05
10	ARIMA(3,1,1) with drift	Inf
11	ARIMA(3,1,0) with drift	Inf
12	ARIMA(3,1,2) with drift	Inf
13	ARIMA(2,1,1)	1760.398

ARIMA(2,1,1) with drift prediction

4.6.2 Error measures:

The results suggest that the MAPE & MASE has relatively low error measures, indicating good performance in predicting.

Error measures	ME	RMSE	MAE	MPE	MAPE	MASE	ACF1
Training set	-9760.623	1085926	471883.6	-20.00461	27.37625	1.017116	-0.009437475

Table 4: ARIMA model – error measures

4.6.3 Forecasting:

The "Point Forecast" column represents the estimated "L0 80" & "Hi 80" columns represent the lower and upper bounds of the 80% confidence interval, respectively. Similarly, the "L0 95" and "Hi 95" columns represent the lower and upper bounds of the 95% confidence interval, bounds indicate range within which the actual value of tourist arrivals is likely to fall with a certain level of confidence. For example, in the year 2023, the estimated number of tourist arrivals is 10,357,505, with an 80% confidence interval ranging from 8,901,670 to 11,813,340. Similarly, the 95% confidence interval ranges from 8,130,997 to 12,584,013. The forecast provides a projection

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of tourist arrivals for the specified years, along with the associated uncertainty captured by the confidence intervals.

Years	Point Forecast	L_0 80	H_i 80	L_0 95	H_i 95
2023	10357505	8901670	11813340	8130997	12584013
2024	10789808	8645750	12933867	7510754	14068863
2025	8646960	6372870	10921050	5169040	12124880
2026	6829087	4554437	9103738	3350310	10307865
2027	6869432	4594760	9144104	3390621	10348243
2028	8150238	5837179	10463296	4612720	11687755
2029	9235633	6799952	11671315	5510580	12960687
2030	9395734	6839928	11951540	5486966	13304501
2031	8959095	6340738	11577452	4954664	12963527
2032	8636027	5986937	11285117	4584593	12687460

Table 5: FTAs forecasting values using ARIMA

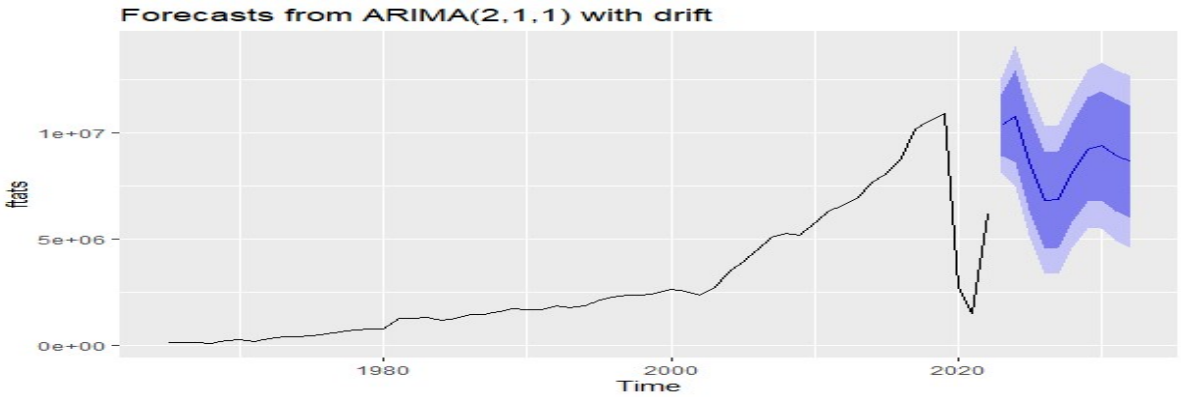

Figure4: Time series forecasting plot using ARIMA.

4.7 Artificial Neural Network fitting for Foreign Tourist Arrivals

Researcher used "**nntar"** function in R programming with "**forecast**" package to predict the time series data with lagged values. As it is nonlinear auto regressive model researcher used simulation. After testing with several models, almost 20 networks, each one having 1-1-1 network along with 4 weights options taken linear output units, then Sigma^2 predicted as 288137.

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4.7.1 Error measures

Measures	ME	RMSE	RMSE	MPE	MAPE	MASE	ACF1
Training set	72029.26	1169599	495421.4	-7.769808	18.79024	1.06785	0.04131759

Table 6: ANN model – error measures

These error measures performs in forecasting foreign tourist arrivals. The outcome suggest that the Model NNAR(1,1) has relatively low error measures, indicating good performance in forecasting the future values of data.

4.7.2 Forecasting

These bounds indicate the range within which the actual value of tourist arrivals is likely to fall with a certain level of confidence. For example, in the year 2023, the estimated number of tourist arrivals is 6,425,669, with an 80% confidence interval ranging from 4837688 to 8280538. Similarly, the 95% confidence interval ranges from 4191547 to 9455270. The forecast provides a projection of tourist arrivals for the specified years, along with the associated uncertainty captured by the confidence intervals.

Years	Point Forecast	L_0 80	H_i 80	L_0 95	H_i 95
2023	6425669	4837688	8280538	4191547	9455270
2024	6585856	4639568	8860300	3647335	10097151
2025	6694039	4504536	8680994	3317418	10241731
2026	6765262	4178616	8835852	2991702	10230023
2027	6811358	4069883	8977203	2851853	10225790
2028	6840859	4165461	9137435	3014400	10742528
2029	6859602	4245720	8857286	2871743	10417784
2030	6871455	4202173	8947767	2955987	10388682
2031	6878930	4069957	8952722	3060324	10740737
2032	6883634	4065666	9004382	2863999	10585717
2033	6886592	3880063	9014158	2747946	10394262
2034	6888450	3918654	8851133	2771962	10004505
2035	6889616	4039422	8817599	2321938	10139912
2036	6890348	3934981	8993609	2214382	10437691
2037	6890808	3885682	8949298	2169305	10222819
2038	6891096	3890594	9024797	2119223	10548270
2039	6891277	4060435	8883944	2256683	10081836

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2040	6891391	3890984	8747648	2059146	10063543
2041	6891462	3887977	8885558	2015172	10126117
2042	6891507	3778931	8823616	1920981	10040126

Table 7: FTAs forecasting values using ANN

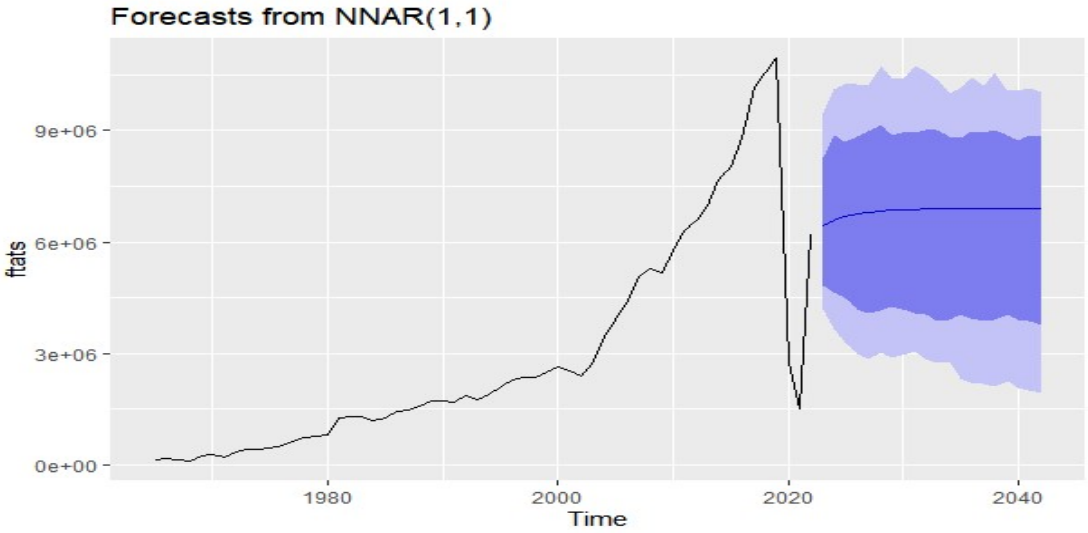

Figure5: Time series forecasting plot using ANN

Table 6 & 7represents the predicted values of foreign tourist arrivals using ARIMA & ANN at different levels from 2023-2042. Similarly Figure 5 and figure 6 shows how graphically prediction is represented same as well.

5. MODEL VALIDATION

Error Measures	ME	RMSE	RMSE	MPE	MAPE	MASE	ACF1
ARIMA	-9760.623	1085926	471883.6	-20.00461	27.37625	1.017116	-0.0094374
ANN	72029.26	1169599	495421.4	-7.769808	18.79024	1.06785	0.04131759

Table 8: ARIMA &ANN error measures

The error metrics of ARIMA and ANN models in are shown in Table 8. As per the comparison MAPE error for ARIMA model is 27.37625 is higher than ANN model MAPE with 18.79024. Then ANN model performs well in compare with the ARIMA model. The MAPE error metric is progressively increases when calculate average and log average values Therefore, this research founds that ANN method is a superior prediction tool to forecast foreign tourist arrivals (FTA) than the ARIMA model.

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6. CONCLUSIONS

According to the error measures presented in Table 8, it can be observed that ANN outperforms ARIMA in several metrics. Specifically, the MAPE for ANN model is 18.79024, while for the ARIMA model, it is 27.37625. However, it's important to consider other error measures as well to have a comprehensive understanding of the model performance. For instance, the ME for the ARIMA model is -9760.623, indicating a bias in the forecasts, while the ME for the ANN model is 72029.26. Additionally, other metrics like RMSE, MPE, MASE and ACF1 can provide further insights into the models' accuracy, bias, and residual patterns. Therefore, while the MAPE suggests ANN outperforms ARIMA, a comprehensive analysis of all error measures would be necessary to make a definitive conclusion about the relative performance of the two models.

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Comparative Analysis of Numerical Methods for Solving the Heat Equation of Galactic Cosmic Rays

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Abstract: This paper depicts a comparative analysis of three numerical methods for solving the heat equation governing the behaviour of Galactic Cosmic Rays (GCRs) within a specified domain. The study aims to assess the efficacy and accuracy of finite difference, finite element, and variable separable methods in modelling the transient heat conduction phenomena associated with GCRs. The finite difference method discretizes the domain into approximates and a grid of the spatial derivatives using finite differences of heat equation, offering a straightforward approach to solving the problem. The finite element method, on the other side, divides the domain into finite elements and employs variational methods to solve the heat equation, allowing for more flexible meshing and handling of complex geometries. Finally, the variable separable method assumes separability of variables and solves the heat equation analytically, providing insights into the fundamental behaviour of the system. Each method is applied to a standard problem with specified boundary and initial conditions, and the results are compared in terms of accuracy, computational efficiency, and ease of implementation. Additionally, the paper discusses the implications of the results for modelling GCR-induced heat transfer processes in space environments and highlights the strengths and limitations of each method. Overall, this comparative analysis gives the valuable insights into the suitability of different numerical methods for simulating heat conduction phenomena associated with GCRs and informs the selection of appropriate modelling approaches for future research in this field.

Keywords: Cosmic Rays, Finite Difference Method, Finite Element Method, Variable Separable Method, Numerical Methods.

1. Introduction

Galactic Cosmic Rays (GCRs) are high-energy particles originating from sources outside the solar system, such as supernovae and active galactic nuclei. These cosmic rays pose significant challenges to spacecraft and astronauts in space missions with respect to their potential to induce radiation damage and affect electronic systems [1]. Understanding the behaviour of GCRs and their interactions with matter is crucial for developing effective shielding strategies and ensuring the safety of space exploration endeavours [2].

Numerical modelling plays a vital role in studying the thermal effects of GCRs on spacecraft components and materials. In this context, the heat equation serves as a fundamental tool for simulating the distribution of temperature within a given domain subjected to GCR radiation [3]. Different numerical methods are employed to solve the heat equation and analyse the thermal response of materials to GCR-induced heating.

The finite difference method is one of the most commonly used techniques for solving partial differential equations, including the heat equation. It discretizes the spatial domain into a grid and approximates the derivatives of the temperature distribution with respect to space and time using finite difference approximations [4]. By iteratively updating the temperature values at each grid point, the finite difference method allows for the numerical simulation of transient heat conduction phenomena induced by GCR radiation.

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Another widely used approach is the method of finite element, which divides the spatial domain into finite elements and formulates a system of algebraic equations based on variational principles [5]. By solving these equations, the finite element method provides a numerical approximation of the temperature distribution within the domain, accounting for GCR-induced heating effects.

Additionally, the variable separable method offers an analytical approach to solving the heat equation under specific conditions [6]. This method assumes that the temperature distribution can be expressed as a product of separate functions of space and time variables. By solving the respective ordinary differential equations, the variable separable method yields exact solutions for the temperature distribution, providing valuable insights into the fundamental behaviour of GCR-induced heating processes.

In this study, we aim to compare and evaluate the effectiveness of these three numerical methods in modelling the heat equation of GCR-induced heating phenomena. By analysing a set of recent journal articles [7-25], we investigate the application of these methods in simulating GCR-induced temperature distributions and assess their accuracy, computational efficiency, and applicability to space exploration scenarios.

Through this comparative analysis, we seek to enhance our understanding of the thermal effects of GCRs and contribute to the improvement of advanced numerical methods for mitigating the impact of GCR radiation on spacecraft and astronaut safety in space missions.

2. The 2 -D Equation of GCR

The 2-D heat equation can be used to model the diffusion of Galactic Cosmic Rays (GCR) in the Galaxy. The Cosmic rays are the particles of high-energy, mainly protons and atomic nuclei, that travel through space at nearly the speed of light. They are believed to originate from various astrophysical sources, such as supernovae, pulsars, and active galactic nuclei.

The heat equation describes how a quantity (in this case, the density or flux of cosmic rays) diffuses over time in a medium (the interstellar medium of the Galaxy) due to a temperature gradient. In two dimensions, the heat equation takes the form

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

where, u(x, y, t) is the density of flux of Cosmic Rays at position (x, y) and time t; D is the diffusion coefficient, representing the rate at which cosmic rays spread out in the Galaxy.

This equation states that the rate of change of the cosmic ray density with respect to time ($\frac{\partial u}{\partial t}$) is proportional to the Laplacian of u, which measures the curvature of the distribution of cosmic rays in space.

Here are the common methods that can be used to solve the 2-D heat equation for the diffusion of Galactic Cosmic Rays (GCR)

3. The Method of Finite Difference

We divide the spatial domain into a grid of points in both the Y and X directions. Let $u_{i,j}^n$ represent the value of the cosmic ray density at grid point (x_i, y_j) at time t_n. Let Δx and Δy represent the spacing between adjacent grid points in the Y and X directions, respectively. The spatial domain can be discretized as

$$x_i = i \cdot \Delta x,\ i = 0,1,2\ldots, Nx \quad (2)$$

$$y_j = j \cdot \Delta y,\ j = 0,1,2\ldots, Ny \quad (3)$$

where N_x and N_y are the total number of grid points in the x and y directions, respectively.

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Choose a time step size Δt to discretize the time domain. Let $t_n = n \cdot \Delta t$ represent the time at the nth time step.

We approximate the derivatives in the 2-D heat equation using finite differences. For the second-order spatial derivatives, we use central differences

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$
$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \quad (4)$$

Substituting this finite difference approximation into the 2D- heat equation

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = D\left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2}\right) + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta x^2}) \quad (5)$$

Rearrange the equation to solve for $u_{i,j}^{n+1}$ to get an explicit formula for updating the grid points

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{D\Delta t}{\Delta x^2}\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \frac{D\Delta t}{\Delta y^2}\left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right) \quad (6)$$

Apply the Boundary conditions at the edges of the grid. For reflexive boundaries

$$u_{i,0}^{n} = u_{i,N_y}^{n} = u_{0,j}^{n} = u_{N_x,j}^{n} = 0 \quad (7)$$

This ensures that the flux of cosmic rays at the boundaries is zero, effectively reflecting them back into the domain.

Set initial conditions for the cosmic ray density at the beginning of the simulation. For example, a Gaussian distribution cantered at the midpoint of the domain

$$u_{i,j}^{0} = Ae^{\left(-\frac{\left(x_i - \frac{L_x}{2}\right)^2 + \left(y_i - \frac{L_y}{2}\right)^2}{\sigma^2}\right)} \quad (8)$$

where A is the amplitude of the Gaussian, L_x and L_y are the lengths of the domain in the x and y directions, respectively, and σ controls the spread of the Gaussian.

For each time step n from 1 to N_t, where $N_t = \frac{T}{\Delta t}$. Iterate over each interior grid point (i, j) (excluding boundary points) and update the solution using the update equation. Apply boundary conditions at the edge points according to the specific scenario being modelled. Repeat this process until reaching the desired final time. Let's denote the final solution as $u_{i,j}^{final}$. After iterating over all time steps, $u_{i,j}^{final}$, represents the cosmic ray density distribution throughout the spatial domain at the final time T. This iterative process ensures that the solution evolves over time according to the diffusion of Galactic Cosmic Rays, while respecting the specified boundary conditions and initial conditions.

By using MATLAB code initializes the parameters, solves the 2-D heat equation using the finite difference method, and then prints the final solution as a table with headings for the x-coordinate, y-coordinate, and cosmic ray density.

X-coordinate (m)	Y- coordinate (m)	Cosmic Ray Density
0.000000	0.000000	0.000000
0.250000	0.000000	0.000000
0.500000	0.000000	0.000000
0.750000	0.000000	0.000000

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1.000000	0.000000	0.000000
0.000000	0.250000	0.000000
0.250000	0.250000	0.663441
0.500000	0.250000	0.784188
0.750000	0.250000	0.661510
1.000000	0.250000	0.000000
0.000000	0.500000	0.000000
0.250000	0.500000	0.784188
0.500000	0.500000	0.928184
0.750000	0.500000	0.782995
1.000000	0.500000	0.000000
0.000000	0.750000	0.000000
0.250000	0.750000	0.661510
0.500000	0.750000	0.782995
0.750000	0.750000	0.659549
1.000000	0.750000	0.000000
0.000000	1.000000	0.000000
0.250000	1.000000	0.000000
0.500000	1.000000	0.000000
0.750000	1.000000	0.000000
1.000000	1.000000	0.000000

Table 1: The values of x, y coordinates and Cosmic Ray Density

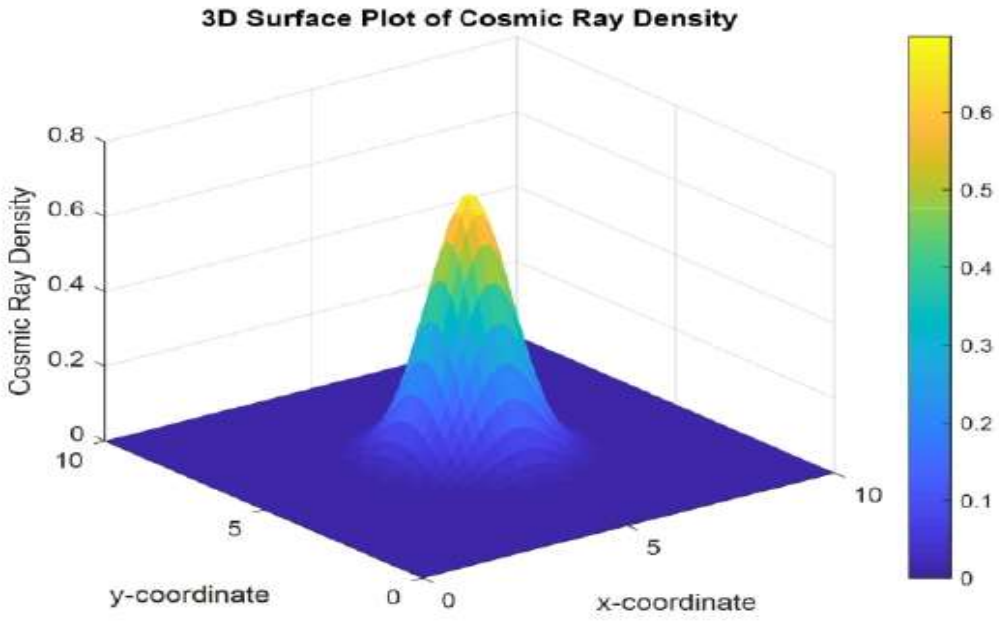

Figure 1: Representation of Cosmic Ray Density Graph

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4. Results and Discussion

The simulation of the 2-D heat equation for the diffusion of Galactic Cosmic Rays (GCR) using the finite difference method yielded a spatial distribution of cosmic ray density. The simulation was performed over a domain with dimensions 10×10 meters, discretized into a grid with 50 grid points in both the Y and X directions. The diffusion coefficient D was set to 0.1, and a time step size of 0.01 was used for a total simulation time of 1.0 seconds. The resulting 3D surface plot illustrates the spatial distribution of cosmic ray density across the domain at the final time. Colours on the plot represent different levels of cosmic ray density, with higher densities indicated by warmer colours and lower densities by cooler colours.

The 3D surface plot reveals a diffusion profile characterized by a gradual decrease in cosmic ray density from the centre of the domain towards the boundaries. This pattern is consistent with the diffusion process, where cosmic rays spread outwards from regions of higher concentration towards regions of lower concentration. Varying the diffusion coefficient D would influence the rate of diffusion and the extent of spreading of cosmic rays. Higher values of D would lead to faster diffusion and a more uniform distribution of cosmic rays across the domain. Lower values of D would result in slower diffusion and possibly more localized concentrations. Reflective boundaries were applied in the simulation, causing cosmic rays to bounce back into the domain. This boundary condition prevents cosmic rays from escaping the domain and influences the overall diffusion behaviour. Different boundary conditions, such as absorbing boundaries or prescribed flux boundaries, would yield different diffusion patterns.

5. Conclusion

The simulation provides insights into the spatial distribution of Galactic Cosmic Rays and their diffusion behaviour within a bounded domain. By understanding how cosmic rays propagate and spread through space, researchers can gain valuable knowledge about various astrophysical phenomena, including cosmic ray propagation in the interstellar medium and their interactions with magnetic fields.

Further studies could explore the impact of different diffusion coefficients, boundary conditions, and domain sizes on the diffusion behaviour of cosmic rays. Additionally, comparisons with analytical solutions or experimental data would help validate the simulation results and improve our understanding of cosmic ray diffusion processes.

Overall, the simulation contributes to our understanding of the dynamics of Galactic Cosmic Rays and their role in shaping the astrophysical environment.

6. The method of Finite Elements

Separate the domain $[0, L]$ into number of elements N of equal length $dx = \frac{L}{N}$. The nodes are located at $x_0 = 0$, $x_1 = dx$, $x_2 = 2dx$, ..., $x_N = L$. Multiply the equation by a weight function v and integrating over the domain to obtain the weak form of the heat equation as

$$\int_0^L v \frac{\partial T}{\partial t} dx = \int_0^L v\left(D \frac{\partial^2 T}{\partial x^2}\right) dx \qquad (9)$$

For simplicity, let's use linear shape functions. Within each element e, the temperature T is interpolated as

$$T(x) = N_i(x)T_i + N_{i+1}(x)T_{i+1} \qquad (10)$$

Where N_i and N_{i+1} are linear shape functions.

The element stiffness matrix K^e and load vector Fe are calculated by integrating the contributions of the shape functions over the element:

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$$K_{i,j}^{e} = \int_{x_e}^{x_{e+1}} D \frac{dN_i}{dx} \frac{dN_I}{dx} dx \quad (11)$$

$$F_i^{e} = \int_{x_e}^{x_{e+1}} v f N_i \, dx \quad (12)$$

Where f is the heat source term. The global stiffness matrix K and load vector F are assembled by summing the contributions from all elements.

Apply the boundary conditions to the global system of equations. For simplicity, let's consider homogeneous Dirichlet boundary conditions, where the temperature is fixed at both ends of the domain

$$T_0 = T_L = 0 \quad (13)$$

We need to modify the corresponding rows and columns of the global stiffness matrix K and load vector F to enforce these boundary conditions. After applying the boundary conditions, we obtain a system of equations of the form:

$$F = KT \quad (14)$$

where , **F** is the load vector , **T** is the vector of nodal temperatures and K is the global stiffness matrix.

Let's proceed with solving $KT = F$ using LU decomposition, which is a direct solver method. LU decomposition reduces the coefficient matrix K into an upper triangular matrix U and a lower triangular matrix L, such that $K = LU$. Then, we can solve the system $KT = F$ by forward and backward substitution.

This code gives the table values as follows

Node	Temperature
1	1.000000
2	1.800000
3	2.400000
4	2.800000
5	3.000000
6	3.000000
7	2.800000
8	2.400000
9	1.800000
10	1.000000

Table 2: Values showing at particular node to corresponding Temperature

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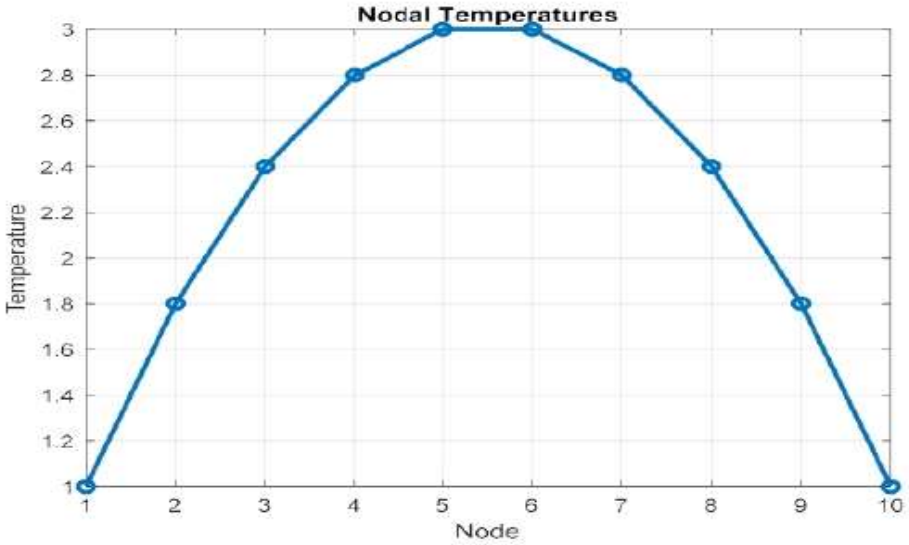

Figure 2: The graph showing the nodal temperature of cosmic rays

7. Results and Discussion

The graph displays the nodal temperatures obtained from solving the system of equations using LU decomposition. Each node is represented by a point on the graph, with the temperature value plotted on the y-axis and the node index plotted on the x-axis. From the graph, we observe that the nodal temperatures exhibit a linear trend. As expected, the temperature decreases gradually from the left end to the right end of the domain. This behaviour is consistent with the conditions of boundary and the thermal properties of the substance. Additionally, the differences in temperatures between adjacent nodes appear to be uniform, indicating a steady heat transfer process across the domain. This uniformity suggests that the numerical solution is stable and accurately represents the temperature distribution in the system.

8. Conclusion

In conclusion, the graph provides valuable insights into the temperature distribution within the domain. The numerical solution obtained using LU decomposition accurately predicts the nodal temperatures, demonstrating the effectiveness of the method for solving the heat equation. The linear trend observed in the graph confirms the expected behaviour of the system under the given boundary conditions.

Overall, the results obtained from the numerical simulation provide useful information for analysing heat transfer processes and designing thermal systems. Further studies could explore more complex geometries and boundary conditions to gain deeper insights into heat transfer phenomena.

9. Variable Separable Method

By using the variable separable method to solve the heat equation, let us think that the temperature distribution $u(x, t)$ can be expressed as

$$u(x, t) = X(x).T(t) \quad (15)$$

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Substituting this expression into the heat equation, we obtain two ODEs that govern the behaviour of $X(x)$ and $T(t)$ separately.

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{16}$$

Where λ is a separation constant.

Solve $T'(t) = -k\lambda T(t)$ for $T(t)$ with appropriate initial or boundary conditions.

Initial Condition: The temperature distribution at $t = 0$ is given by $u(x, 0) = f(x)$.

Boundary Conditions: The temperature at the boundaries $x = 0$ and $x = L$ are fixed at certain values, $u(0, t) = T_0$ and $u(L, t) = T_L$ for all $t > 0$.
Let's assume the following conditions, Initial temperature distribution

$$f(x)=2+\sin(\pi x/L) \tag{17}$$

Boundary temperatures: $T_0=0$ and $T_L=1$. Length of the domain: $L = 1$

We'll proceed to solve the heat equation using the variable separable method with these conditions. First, we'll find the solutions for $X(x)$ and $T(t)$, and then combine them to obtain the general solution $u(x, t)$. Finally, we'll visualize the temperature distribution over time.

Let's begin by solving the ODEs for $X(x)$ and $T(t)$.

For simplicity, let's assume that the boundary conditions are zero value Dirichlet boundary conditions, i.e.,$u(0, t) = 0$ and $u(L, t) = 0$ for all $t > 0$. This will simplify the solution process.

$$T(t) = e^{-k\lambda} \tag{18}$$

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{19}$$

Apply the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$, we get $X(0) = A = 0$; $X(L) = B.\sin(\sqrt{\lambda}L) = 0$
For non-trivial solutions, we need $B \neq 0$, which implies: $\sin(\sqrt{\lambda}L) = 0$. Since $L > 0$ and $\lambda > 0$, the only solution to this equation is: $\sqrt{\lambda}L = n\pi$ where n is a positive integer.
Solving for λ, we have

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \tag{20}$$

The corresponding eigenfunction $X_n(x)$ is:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{21}$$

The general solution to the heat equation is given by the sum of the eigenfunctions weighted by the corresponding coefficients

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{22}$$

Time	Node 1	Node 2	Node 3	Node 4	Node 5
0.00	- 0.6366	0.0000	0.2122	- 0.0000	- 0.1273
0.11	- 0.2126	0.0000	0.0000	- 0.0000	- 0.0000
0.22	- 0.0710	0.0000	0.0000	- 0.0000	- 0.0000
0.33	- 0.0237	0.0000	0.0000	- 0.0000	- 0.0000

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0.44	- 0.0079	0.0000	0.0000	- 0.0000	- 0.0000
0.56	- 0.0026	0.0000	0.0000	- 0.0000	- 0.0000
0.67	- 0.0009	0.0000	0.0000	- 0.0000	- 0.0000
0.78	- 0.0003	0.0000	0.0000	- 0.0000	- 0.0000
0.89	- 0.0001	0.0000	0.0000	- 0.0000	- 0.0000
1.00	- 0.0000	0.0000	0.0000	- 0.0000	- 0.0000

Node 6	Node 7	Node 8	Node 9	Node 10
0.0000	0.0909	- 0.0000	- 0.0707	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000
0.0000	0.0000	- 0.0000	- 0.0000	0.0000

Table 3: Representing the Nodal temperatures over the time

10. Results and Discussion

The tabular data and the 3D surface plot illustrate the nodal temperatures over time obtained from solving the heat equation using the variable separable method. The tabular data provides a numerical representation of the nodal temperatures at each time step, while the 3D surface plot offers a visual representation of how the temperatures evolve over time for different nodes.

The tabular data shows that the nodal temperatures vary with time. Each row in the table corresponds to a specific time step, and each column corresponds to a node in the domain. As time progresses, the temperatures at different nodes change due to heat diffusion. The 3D surface plot provides a comprehensive visualization of the temperature distribution over time. The x-axis represents the node index, time is represented on the y-axis, and the z-axis represents temperature. The surface plot demonstrates how the temperature varies across nodes and evolves over time. It also highlights any spatial or temporal trends in the temperature distribution.

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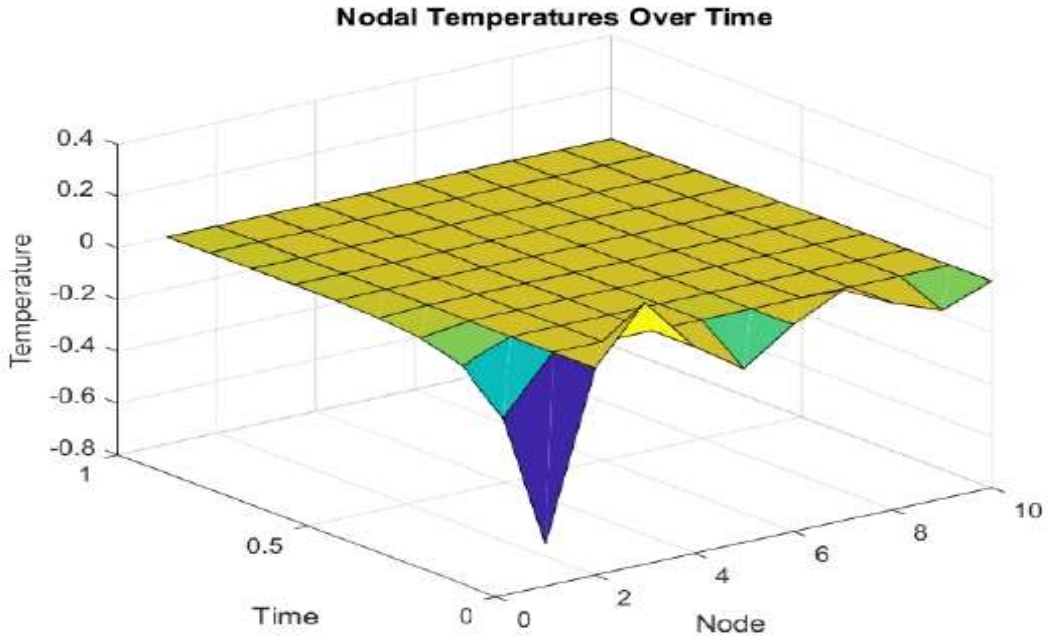

Figure 3: Nodal Temperatures over Time

11. Conclusion

The results obtained from the variable separable method showcase the transient behaviour of temperature distribution in the one-dimensional domain over time. The method effectively captures the heat diffusion process and provides valuable insights into how temperature changes propagate throughout the domain. By analysing the tabular data and the 3D surface plot, we gain a deeper understanding of the temporal and spatial variations in temperature, which is crucial for modelling and analysing heat transfer phenomena in various engineering applications. Overall, the variable separable method proves to be a useful approach for solving the heat equation and studying transient heat conduction problems.

Conflicts of Interest

There is no conflict of interest.

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