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A NOTE ON Ga SECOND OVERPARTITIONS OF *n*

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Abstract:

The Mathematicians Corteel and Lovejoy [2] derived *overpartitions* of n. Hanuma Reddy [3] proposed a formula for the sum of i^{th} greatest parts of *overpartitions* of positive integer n. Sagar G.V.R.K[5] derived *Ga partitions* of positive integer n. In this chapter we define $Ga\,2^{nd}$ *overpartitions* of n and $r-Ga\,2^{nd}$ *overpartitions* of n and derive the generating functions for the number of the smallest parts and the sum of smallest parts of $Ga\,2^{nd}$ *overpartitions* of positive integer n by utilizing $r-Ga2^{nd}$ *overpartitions* of n.

Keywords: Partition, r-partition, overpartition, *Ga* overpartition, *Ga* second overpartition Smallest part of the *Ga* second overpartition.

Subject classification: 11P81 Elementary theory of Partitions.

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1. Introduction:

We define $Ga2^{nd}$ *overpartition* of n. It is a non-increasing sequence of natural numbers whose sum is n and smallest parts are of the form $a^{k-1}, k \in N$ in which first (equivalently, the final) occurrence of a number may be two times over lined. We denote the $Ga2^{nd}$ *overpartition* of λ, denoted by $\overline{\overline{Ga\,\lambda}}$ and the number of $Ga2^{nd}$ *overpartition* of n by $\overline{\overline{Ga\;p(n)}}$. Since the over lined parts form a *Ga partition* into distinct parts.

For example, the number of 2^{nd} *overpartition* of 4 is 27 and are as follows:

$$4, \bar{4}, \bar{\bar{4}}, 3+1, \bar{3}+1, \bar{\bar{3}}+1, 3+\bar{1}, \bar{3}+\bar{1}, \bar{\bar{3}}+\bar{1}, 3+\bar{\bar{1}}, \bar{3}+\bar{\bar{1}}, \bar{\bar{3}}+\bar{\bar{1}}, 2+2, \bar{2}+2$$
$$\bar{\bar{2}}+2, 2+1+1, \bar{2}+1+1, \bar{\bar{2}}+1+1, 2+\bar{1}+1, \bar{2}+\bar{1}+1, \bar{\bar{2}}+\bar{1}+1, 2+\bar{\bar{1}}+1,$$
$$\bar{2}+\bar{\bar{1}}+1, \bar{\bar{2}}+\bar{\bar{1}}+1, 1+1+1+1, \bar{1}+1+1+1, \bar{\bar{1}}+1+1+1.$$

Let $\overline{\overline{Ga\,\xi(n)}}$ denote the set of all $Ga2^{nd}$ *overpartition* of n and $\overline{\overline{Ga\;p(n)}}$ the cardinality of $\overline{\overline{Ga\,\xi(n)}}$ for $n \in N$ and $\overline{\overline{Ga\,p(0)}} = 1$. If $1 \le r \le n$ write $\overline{\overline{Ga\;p_r(n)}}$ for the number of $Ga2^{nd}$ *overpartition* of n each consisting of exactly r parts, i.e $r-Ga2^{nd}$ *overpartition* of

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n. If $r \le 0$ or $r \ge n$ we write $\overline{\overline{Ga\, p_r(n)}} = 0$. Let $\overline{\overline{Ga\, p(k,n)}}$ represent the number of $Ga2^{nd}$ *overpartition* of n using natural numbers at least as large as k only.

Let $\overline{\overline{Ga\, spt(n)}}$ denote the number of smallest parts including repetitions in all $Ga2^{nd}$ *overpartition* of n.

1.1 We derive $\overline{\overline{p_r(n)}} = \frac{q^r(-2,q)_r}{(q)_r}$ **as follows:**

$$\overline{\overline{p_1(n)}} = (2+1)\,p_1(n) = \frac{(2+1)q}{(1-q)} = \frac{q(-2,q)_1}{(q)_1}$$

$$\overline{\overline{p_2(n)}} = (1+2)^2\, p_2(n) - (1+2)2p_1\left(\left[\frac{n}{2}\right]\right)$$

$$= \frac{(1+2)^2 q^2}{(1-q)(1-q^2)} - \frac{(1+2)2q^2}{(1-q^2)}$$

$$= \frac{(1+2)q^2}{(1-q^2)}\left\{\frac{(1+2)}{(1-q)} - 2\right\} = \frac{q^2(1+2)(1+2q)}{(1-q)(1-q^2)} = \frac{q^2(-2,q)_2}{(q)_2}$$

By induction, we get

$$\overline{\overline{p_r(n)}} = \frac{q^r(1+2)(1+2q)(1+2q^2)\ldots(1+2q^{r-1})}{(1-q)(1-q^2)(1-q^3)\ldots(1-q^r)} = \frac{q^r(-2,q)_r}{(q)_r}$$

and $$\overline{\overline{p_r(n-a)}} = \frac{q^{r+a}(-2,q)_r}{(q)_r} \qquad (1.1)$$

Notations that are employed in this chapter are given here under.

1.2 Some more notations:

i) $\overline{\overline{Ga\, f}}(a^{k-1},n)$: number of $Ga2^{nd}$ *overpartition* of n with least part a^{k-1}.

ii) $\overline{\overline{Ga\, p_r}}(a^{k-1},n)$: number of $r - Ga2^{nd}$ *overpartition* of n with least part greater than or equal to a^{k-1}.

iii) $\overline{\overline{Ga\, f}}_r(a^{k-1},n)$: number of $r - Ga2^{nd}$ *overpartition* of n with least part a^{k-1}.

iv) $Ga\, n_s\left(\overline{\overline{\lambda}}\right)$: number of the smallest parts including repetitions in $\overline{\overline{Ga\,\lambda}}$.

v) $\overline{\overline{\lambda}}$: 2^{nd} *overpartitions* of λ.

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vi) $\overline{\overline{\xi}}(n)$: set of all 2^{nd} *overpartitions* of n.

vii) $\overline{\overline{p}}(n)$: number of 2^{nd} *overpartitions* of n .

viii) $\overline{\overline{\xi_r}}(n)$: set of $r-2^{nd}$ *overpartitions* of n.

ix) $\overline{\overline{p_r}}(n)$: number of $r-2^{nd}$ *overpartitions* of n .

x) $\overline{\overline{p}}(a^{k-1},n)$: number of 2^{nd} *overpartitions* of n with least part greater than or equal to a^{k-1}.

xi) $\overline{\overline{f}}(a^{k-1},n)$: number of 2^{nd} *overpartitions* of n with least part a^{k-1}.

xii) $\overline{\overline{p_r}}(a^{k-1},n)$: number of $r-2^{nd}$ *overpartitions* of n with least part greater than or equal to a^{k-1}.

xiii) $\overline{\overline{f_r}}(a^{k-1},n)$: number of $r-2^{nd}$ *overpartitions* of n with least part a^{k-1}.

xiv) $\overline{\overline{spt}}(n)$: number of the smallest parts including repetitions in all 2^{nd} *overpartitions* of n.

xv) $n_s\left(\overline{\overline{\lambda}}\right)$: number of the smallest parts including repetitions in $\overline{\lambda}$.

The Mathematicians Corteel and lovejoy [2] derived *overpartitions* of n. By utilizing $r-2^{nd}$ *overpartitions* of n, we propose a formula for finding the number of smallest parts of n.

2. Generating function for $\overline{\overline{Ga\,spt(n)}}$.

2.1 Theorem:

$$\overline{\overline{Ga\,spt(n)}} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{\overline{p\left(a^{k-1},n-ta^{k-1}\right)}} + 2\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{\overline{p\left(a^{k-1}+1,n-ta^{k-1}\right)}} + 3\left\{d(n)\mid d(n)\text{ is divisors of form } a^{k-1}\right\}$$

Proof: Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r)=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},\left(a^{k-1}\right)^{\alpha_l}\right)$ be any $r-partition$ of n with l distinct parts. For corresponding to it there are 3^l times $r-Ga\,2^{nd}$ *overpartitions* of n.

(2.1)

Case 1: Let $r>\alpha_l=t$ which implies $\lambda_{r-t}>a^{k-1}$

Subtract all a^{k-1}'s , we get $n-ta^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}}\right)$

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Hence $n-ta^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t)-Ga\ partition$ of $n-ta^{k-1}$ with $l-1$ distinct parts and each part is greater than or equal to $a^{k-1}+1$. For corresponding to it they are 3^{l-1} times $(r-t)-Ga\,2^{nd}\ overpartitions$ of $n-ta^{k-1}$. From (2.1), we know that the total number of $r-Ga\,2^{nd}\ overpartitions$ are 3^l.

Now we get, 3 times the number $\overline{\overline{Ga\,p_{r-t}\left(a^{k-1}+1,n-ta^{k-1}\right)}}$ of $r-Ga\ j^{th} overpartitions$ from $r-Ga\ partitions$ of n with exactly t smallest elements as a^{k-1}.

Case 2: Let $r>\alpha_l>t$ which implies $\lambda_{r-t}=a^{k-1}$

Omit a^{k-1}'s from last t places, we get $n-ta^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}},\left(a^{k-1}\right)^{\alpha_l-t}\right)$

Hence $n-ta^{k-1}=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\dots,\mu_{l-1}^{\alpha_{l-1}},\left(a^{k-1}\right)^{\alpha_l-t}\right)$ is a $(r-t)-partition$ of $n-ta^{k-1}$ with l distinct parts and the least part is a^{k-1}. For corresponding to it there are 3^l times of $r-2^{nd}\ overpartitions$ of $n-ta^{k-1}$ with least part a^{k-1}

Now we get the number of $r-Ga\,2^{nd}\ overpartitions$ with smallest part a^{k-1} that occurs more than t times among all $r-Ga\,2^{nd}\ overpartitions$ of n is $\overline{\overline{f_{r-t}\left(a^{k-1},n-ta^{k-1}\right)}}$.

Case 3: Let $r=\alpha_l=t$ which implies all parts in the $Ga\ partition$ are equal and each part is of the form a^{k-1}. For each $r-Ga\ partition$ with equal parts have 3 times of $r-Ga\,2^{nd}\ overpartitions$ of n.

The number of $Ga\ partitions$ of n with equal parts and each part is of the form a^{k-1} is equal to the number of divisors of n which are in the form a^{k-1}. Since the number of such divisors of n is $\{d(n)\mid d(n)\text{ is divisors of form } a^{k-1}\}$ the number of $Ga\,2^{nd}\ overpartitions$ of n with all parts are equal is $3\{d(n)\mid d(n)\text{ is divisors of form } a^{k-1}\}$.

From cases (1), (2) and (3) we get $r-Ga\,2^{nd}\ overpartitions$ of n with smallest part a^{k-1} that occurs t times is

$$\overline{\overline{f_{r-t}\left(a^{k-1},n-ta^{k-1}\right)}}+3.\overline{\overline{p_{r-t}\left(a^{k-1}+1,n-ta^{k-1}\right)}}+3.\beta \quad \text{where} \quad \beta=\begin{cases}1 & \text{if } \dfrac{n}{r}=a^{k-1}\\ 0 & \text{otherwise}\end{cases}$$


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$$= \overline{\overline{f_{r-t}\left(a^{k-1}, n-ta^{k-1}\right)}} + \overline{\overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}}$$
$$+ 2.\overline{\overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}} + 3.\beta$$

$$= \overline{\overline{p_{r-t}\left(a^{k-1}, n-ta^{k-1}\right)}} + 2.\overline{\overline{p_{r-t}\left(a^{k-1}+1, n-ta^{k-1}\right)}} + 3.\beta \qquad (2.2)$$

From [3], the number of smallest parts in $Ga\ 2^{nd}\ overpartitions$ of n is

$$\overline{\overline{Ga\,spt(n)}} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{\overline{p\left(a^{k-1}, n-ta^{k-1}\right)}} + 2\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{\overline{p\left(a^{k-1}+1, n-ta^{k-1}\right)}}$$
$$+3\left\{d(n) \mid d(n) \text{ is divisors of form } a^{k-1}\right\}.$$

2.2. Theorem: $\overline{\overline{p_r\left(a^{k-1}+1, n\right)}} = \overline{\overline{p_r\left(n-a^{k-1}r\right)}}$ (2.3)

Proof: Let $n = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_i > a^{k-1}\ \forall i$ be any $r-overpartition$ of n.

Subtracting a^{k-1} from each part, we get

$$n - a^{k-1}r = \left(\lambda_1 - a^{k-1}, \lambda_2 - a^{k-1}, \ldots, \lambda_r - a^{k-1}\right)$$

Hence $n - a^{k-1}r = \left(\lambda_1 - a^{k-1}, \lambda_2 - a^{k-1}, \ldots, \lambda_r - a^{k-1}\right)$ is a $r-overpartition$ of $n - a^{k-1}r$.

Therefore the number of $r-overpartitions$ of n with parts greater than or equal to $a^{k-1}+1$ is $\overline{\overline{p_r\left(n-a^{k-1}r\right)}}$.

2.3. Theorem: $\sum_{n=0}^{\infty}\overline{\overline{Ga\,spt(n)}}q^n = \frac{(-2,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{3q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-2,q)_{a^{n-1}+1}}$

Proof: From theorem (2.1) we have

$$\overline{\overline{Ga\,spt(n)}} = \sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\overline{\overline{p\left(a^{k-1}, n-ta^{k-1}\right)}} + 2\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\overline{\overline{p\left(a^{k-1}+1, n-ta^{k-1}\right)}}$$
$$+3\left\{d(n) \mid d(n) \text{ is divisors of form } a^{k-1}\right\}$$

Replace $a^{k-1}+1$ by a^{k-1}, n by $n-ta^{k-1}$ for first part and n by $n-ta^{k-1}$ for second part in (2.3)

$$\overline{\overline{Ga\,spt(n)}} = \sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\overline{\overline{p_r\left(n-ta^{k-1}-r\left(a^{k-1}-1\right)\right)}} + 2\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\overline{\overline{p_r\left(n-ta^{k-1}k-ra^{k-1}\right)}}$$
$$+3\left\{d(n) \mid d(n) \text{ is divisors of form } a^{k-1}\right\}$$

where $d(n)$ is the number of positive divisors of n.

From(1.1)

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$$\overline{\overline{Ga\,spt(n)}} = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+ta^{k-1}+r(a^{k-1}-1)}(-2,q)_r}{(q)_r} + 2\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+ta^{k-1}+ra^{k-1}}(-2,q)_r}{(q)_r} + \sum_{k=1}^{\infty}\frac{3q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{ta^{k-1}+ra^{k-1}}(-2,q)_r}{(q)_r} + 2\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+ta^{k-1}+ra^{k-1}}(-2,q)_r}{(q)_r} + \sum_{k=1}^{\infty}\frac{3q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{ta^{k-1}}\left[\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r(-2,q)_r}{(q)_r}\right] + 2\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{ta^{k-1}}\left[\sum_{r=1}^{\infty}\frac{q^r\left(q^{a^{k-1}}\right)^r(-2,q)_r}{(q)_r}\right] + \sum_{k=1}^{\infty}\frac{3q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$= \sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r(-2,q)_r}{(q)_r}\right)-1\right]$$

$$+2\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left[\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}+1}\right)^r(-2,q)_r}{(q)_r}\right)-1\right] + \sum_{r=1}^{\infty}\frac{3q^{a^{k-1}}}{1-q^{a^{k-1}}}$$

$$= \sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}}\right)^r(-2,q)_r}{(q)_r}\right) + 2\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^{a^{k-1}+1}\right)^r(-2,q)_r}{(q)_r}\right)$$

$$= \sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+2q^rq^{a^{k-1}}}{1-q^rq^{a^{k-1}}}\right) + 2\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+2q^rq^{a^{k-1}+1}}{1-q^rq^{a^{k-1}+1}}\right) \qquad \text{from [1]}$$

$$= \sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+2q^{r+a^{k-1}}}{1-q^{r+a^{k-1}}}\right) + 2\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\prod_{r=0}^{\infty}\left(\frac{1+2q^{r+a^{k-1}+1}}{1-q^{r+a^{k-1}+1}}\right)$$

$$= \frac{(-2,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}-1}}{(-2,q)_{a^{k-1}}} + \frac{(-2,q)_\infty .2}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}}}{(-2,q)_{a^{k-1}+1}}$$

$$= \frac{(-2,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}-1}}{(-2,q)_{a^{k-1}}}\left[1+\frac{2\left(1-q^{a^{k-1}}\right)}{\left(1+2q^{a^{k-1}}\right)}\right]$$

$$= \frac{(-2,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{3q^{a^{k-1}}}{\left(1-q^{a^{k-1}}\right)}\frac{(q)_{a^{k-1}-1}}{(-2,q)_{a^{k-1}+1}}$$

$$= \frac{(-2,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{3q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-2,q)_{a^{n-1}+1}}$$

2.4 Corollary: The generating function for $\overline{\overline{Ga\,A_c(n)}}$, the number of smallest parts of the $Ga2^{nd}$ *overpartition* of n which are multiples of c is


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$$\sum_{n=0}^{\infty}\overline{\overline{Ga\,A_c(n)}}q^n = \frac{(-2,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{3q^{ca^{n-1}}}{\left(1-q^{ca^{n-1}}\right)}\frac{(q)_{ca^{n-1}-1}}{(-2,q)_{ca^{n-1}+1}}$$

2.5 Theorem: The generating function for the sum of smallest parts of $Ga2^{nd}$ *overpartition* of n is

$$\sum_{n=0}^{\infty}\overline{\overline{sum\,Ga\,spt(n)}}q^n = \frac{(-2,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{3a^{n-1}q^{a^{n-1}}}{\left(1-q^{a^{n-1}}\right)}\frac{(q)_{a^{n-1}-1}}{(-2,q)_{a^{n-1}+1}}$$

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