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MATHEMATICAL EDUCATION, THEIR MEANINGS AND UNDERSTANDING – A PERSPECTIVE

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Abstract

Mathematics has always been, and will always be an indivisible and essential part of human culture. According to many scientists, it is a vital part of human nature to get to know and understand one's natural surroundings, and also the social milieu in which one lives and works. This is the basis of scientific, technical and technological progress. Lately, a new quality has been added to mathematics: mathematics education is now considered to be an important component in the development of an individual. Mathematics education is a treasure which every man has the right to, and it is the responsibility of the society (the national and international organizations) to provide an opportunity to every person to use that treasure. Some key elements for developing a theory for understanding mathematical concepts are outlined. We shall argue for the need to complement the psychological facets of understanding - 'as a mental experience', and 'connection of internal representations in information networks'- with the sociocultural approach, that is, understanding as 'correspondence between personal and institutional meanings'. The role of situation-problems and semiotic instruments is also emphasized in both personal and institutional dimensions of understanding processes. This research article to be discussed about the Mathematical Education their meanings and understanding.

Keywords: Mathematical Education, Structure of Mathematics, Knowledge of Mathematic, Epistemological facts, Methods of Understanding, Theoretical Models.

Introduction

Statement of the Problem

“Mathematics is the most beautiful and most powerful creation of the human spirit”

Stefan Banach

Polish Mathematician

A familiar mathematician in the world

The first school in which the concept of mathematics education was formed was founded more than 1200 years ago (in 795, to be precise) in Aachen, during the reign of Charles the Great. It is known that, at his personal request, a monk called Alcuin of York was brought to Britain because mathematics was taught at that school. It seems that the monk wrote the first textbook in medieval Europe, titled “Problems to Sharpen the Yong” (in English) or “Propositiones and Acuendos Juvenes” (in Latin). The 18th assignment in the book was a well-known problem called “on wolf, goat and cabbage”. Since then, that assignment has been used in many mathematics text books. ‘The sharpening of the memory’ is the aim of mathematics education at any level, especially in higher education in the areas of technical and humanistic sciences.

It is worth noting, in connection with the history of the first mathematics textbook, that on the 8th of August, 1900, David Hilbert read his famous paper “Mathematical problems” at the Second International Congress of Mathematicians, held in Paris. The paper began with the words: “Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?” At this point, Hilbert had already formulated a sufficiently general view of the state of mathematics at the time, and set a number of unsolved mathematical problems that formed an outline of the considerable efforts being made by mathematicians across the world.

In the 20th century, no mathematician has repeated David Hilbert's attempt and nothing of a similar nature has been written since the beginning of the 21st century. At best, a few syntheses that referred to separate mathematical fields have been made. For example, International Meetings of Mathematicians, including the one held in Berlin in 1998, are testament to that fact. No mathematician has yet presented a general mathematical view at any of these meetings. This may be because of the complexity and clumsiness of mathematical disciplines. According to Victor Antonovich Sadovnichii, who presented a paper at the mathematical conference “Mathematics Education: The Present and Future”, which was held in 2000 in Russia, the reasons lie elsewhere



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Mathematics is an international science. Gogol once said that Pushkin's name immediately sparks thoughts about his role as the national poet of Russia. A similar statement regarding any mathematician would make no sense. In contrast to national languages, mathematical language in an international language and mathematical truths have no national borders.

Mathematics has always been, and will always be an indivisible and essential part of human culture. According to many scientists, it is a vital part of human nature to get to know and understand one's natural surroundings, and also the social milieu in which one lives and works. This is the basis of scientific, technical and technological progress. Lately, a new quality has been added to mathematics: mathematics education is now considered to be an important component in the development of an individual.

Mathematics education is a treasure which every man has the right to, and it is the responsibility of the society (the national and international organizations) to provide an opportunity to every person to use that treasure.

Understanding in Mathematics Education

As Hiebert and Carpenter asserted, one of the most widely accepted ideas in mathematics education is that students should understand mathematics. Sierpiska starts her book on understanding in mathematics with similar words: "how to teach so that students understand? What exactly don't they understand? What do they understand and how?" mention the interest towards teaching and learning mathematics with understanding, which is shown in recent curricular reforms in many countries. This interest is also reflected in conference proceedings and research articles in psychology and artificial intelligence.

The importance of the idea of understanding for mathematics education is emphasized in recent works by Sierpiska, Pirie and Kieren, Koyama, amongst others. Nevertheless, characterizing understanding "in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems"

The use of the term 'understanding' (or 'comprehension') is varied, depending on institutional contexts, although the dominant psychological approach emphasizes the mental facet of understanding, which is strongly challenged by Wittgenstein. The cognitive revolution supported by Vygotsky -who claims the analytical and genetic priority of socio-cultural factors when attempt to understand individual psychological processes-, Bruner (1990) -with his proposal of a cultural psychology, or Chevallard (1992) -who speaks of cognitive and didactical anthropology- requires a reconceptualization of mathematical knowledge and its understanding.

The book by Sierpiska (1994) represents an important step forward, when discerning between understanding acts and processes and when relating "good understanding" of a mathematical situation (concept, theory, problem) to the sequence of acts of overcoming obstacles specific to this situation. Through the historico-empirical approach it is possible to identify meaningful acts for understanding a concept. Nevertheless, we think that taking the notion of object as primitive and deriving meaning from understanding cause some difficulties in analyzing the processes of assessing students' understanding.

From our point of view, a theory of conceptual understanding useful for mathematics education should not be limited to saying, for example, that understanding the concept of function is a person's mental experience assigning some object to the term 'function'. A cultural entity, a very complex, not ostensive object is designated with the term 'function'. Therefore, in order to define what is understanding the concept of function, we need to clarify a previous question: What object must the student assign to the term 'function' so that the teacher may say that he/she understands the object function?

The problem of understanding is, consequently, closely linked to how the nature of mathematical knowledge is conceived. Mathematical terms and expressions denote abstract entities whose nature and origin should be researched for elaborating a useful and effective theory for what it is to understand such objects. This research requires answering questions such as: What is the structure of the object to be understood? What forms or ways of understanding exist for each concept? What are the possible and desirable aspects or components of mathematical concepts for students to learn at a given time and under certain circumstances? How are these components developed?

If, for example, we consider mathematical knowledge as internally represented information, understanding occurs when the representations achieved are connected by progressively more structured and cohesive networks (Hiebert and Carpenter, 1992). However, we consider it excessively reductionist to view mathematical activity as information processing. From our point of view, the theories of understanding derived from this conception do not adequately describe the teaching and learning processes of mathematics, especially the social and cultural aspects involved in these processes.



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We think that a theory of understanding mathematical abstractions must be supported by a previous theory concerning the nature of such objects. In this research report, we first present a summary of our previous theoretical articles on the nature of the mathematical objects and its meanings. After this synthesis, we shall provide some elements deduced from this theory, to be taken into account for the development of a theory of understanding in the teaching and learning of mathematics.

Pragmatic and relativist for mathematics

Our theory is based on the following epistemological and cognitive assumptions about mathematics, which take into account some recent tendencies in the philosophy of mathematics (Tymoczko, 1986, Ernest, 1991): Mathematics is a human activity involving the solution of problematic situations. In finding the responses or solutions to these external and internal problems, mathematical objects progressively emerge and evolve. According to Piagetian constructivist theories, people's acts must be considered the genetic source of mathematical conceptualization.

- a) Mathematical problems and their solutions are shared in specific institutions or collectives involved in studying such problems. Thus, mathematical objects are socially shared cultural entities.
- b) Mathematics is a symbolic language in which problem-situations and the solutions found are expressed. The systems of mathematical symbols have a communicative function and an instrumental role.
- c) Mathematics is a logically organized conceptual system. Once a mathematical object has been accepted as a part of this system, it can also be considered as a textual reality and a component of the global structure. It may be handled as a whole to create new mathematical objects, widening the range of mathematical tools and, at the same time, introducing new restrictions in mathematical work and language.

To build our model, we take the notion of problem -situation as a primitive idea. Problems do not appear in isolation; the systematic variation of the variables intervening in problem-situations yields different fields of problems, sharing similar representations, solutions, etc.

The subject performs different types of practices, or actions intended to solve a mathematical problem, to communicate the solution to other people or to validate or generalize that solution to other settings and problems. The genesis of a subject's knowledge arises as a consequence of that subject's interaction with the field of problems, which is mediated by institutional contexts.

Two primary units of analysis to study cognitive and didactic processes are the meaningful practices, and the meaning of an object, for which we postulate two interdependent dimensions: personal and institutional. A practice is meaningful for a person (resp. institution) if it fulfills a function for solving the problem, or for communicating, validating, or extending the solution. This notion is used to conceptualize mathematical objects, both in their psychological and epistemological facets (personal and institutional objects). Mathematical objects -abstractions or empirico and operative generalizations (Dörfler, 1991)- are considered as emergents from the systems of personal (institutional) practices made by a person (or within an institution) when involved with some problem-situations.

The system of meaningful prototype practices, i.e., the system of efficient practices to reach the goal aimed at is defined as the personal (institutional) meaning of the object. It is considered to be the genetic (epistemological) origin of personal objects (institutional objects). It is linked to the field of problems from which this object emerges at a given time and it is a compound entity. Its nature is opposed to the intentional character of the object, and it allows us to focus, from another point of view, on the issues for designing teaching situations and assessing subjects' knowledge.

To sum up, we postulate a relativity of the emergent objects, intrinsic to the different institutions involved in the field of problems, and also depending on the available expressive forms. This assumption could be useful to explain the adaptations (or transpositions) and mutual influences that mathematical objects undergo when transmitted between people and institutions.

The systemic complexity we postulate for the meaning of mathematical concepts -understood from a pragmatic perspective - is well illustrated by the problem list that Sierpinska outlines on the meaning of the term 'function', expressed by its use:

How can a function be? Or, what adjectives can we use with the noun 'function' (Defined/ non-defined, defined in a point/ in an interval/ everywhere; increasing, decreasing, invertible, continuous in a point/ in an interval; smooth; differentiable in a point/ in an interval; integrable ..., etc). The socio-epistemic relativism for the meaning of mathematical abstractions that we postulate in our



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theoretical model is justified, for example, when we study the diversity of conceptions of the function concept identified by Ruíz-Higueras, not only from a historical perspective, but also in didactic transpositions at the teaching institutions (curricula, text books, mathematics classroom).

Elements for a model of understanding in Mathematics Education

From our theoretical positions summarized in the previous Section, we identify the following consequences that should be considered in order to elaborate a theory on understanding mathematics.

According to our pragmatic and relativist conception of mathematics, a theory of mathematical understanding, which is to be useful and effective to explain teaching and learning processes, should recognize the dialectical duality between the personal and institutional facets of knowledge and its understanding.

The definition of understanding by Sierpinska as the 'mental experience of a subject by which he/she relates an object (sign) to another object (meaning)' emphasizes one of the senses in which the term 'understanding' is used, well adapted for studying the psychological processes involved. Nevertheless, in mathematics teaching the term 'understanding' is also used in the processes for assessing students' learning. School institutions expect subjects to appropriate some culturally fixed objects, and assign the teacher with the task of helping the students to establish the agreed relationships between terms, mathematics expressions, abstractions, and techniques. In this case, understanding is not merely a mental activity, but it is converted into a social process. As an example, we may consider that a pupil sufficiently "understands" the function concept in secondary teaching and that he/she does not understand it, if the judgment is made by a university institution.

Furthermore, from a subjective sense, understanding cannot merely be reduced to a mental experience but it involves the person's whole world. As Johnson states, our understanding "is the way we are meaningfully situated in our world through our bodily interactions, our cultural institutions, our linguistic tradition, and our historical context".

Systemic nature

Since, in our theoretical model, we start out from the notions of object and meaning, personal understanding of a concept is "grasping or acquiring the meaning of the object". Therefore, the construct 'meaning of an object' is not conceived as an absolute and unitary entity, but rather as compound and relative to institutional settings. Therefore, the subject's understanding of a concept, at a given moment and under certain circumstances, will imply the appropriation of the different elements composing the corresponding institutional meanings:

1. extensional elements (recognition of prototype situations of use of the object);
2. intentional elements (different characteristic properties and relationships with other entities);
3. expressions and symbolic notations used to represent situations, properties, and relationships.

Furthermore, recognizing the systemic complexity of the object's meaning implies a dynamical, progressive, though nonlinear nature of the process of appropriation by the subject, due to the different domains of experience and institutional contexts in which he/she participates.

The conception of mathematics underlying our theoretical model is characterized primarily by considering mathematics as a human activity. Concepts and mathematical procedures emerge from a persons' acts for solving some problem fields. This activity is mediated by the semiotic instruments provided by the culture and by our capacity for deductive logical reasoning. Secondly, mathematics is a socially shared and logically structured conceptual system. Consequently, the process axis for personal understanding must contain the following categories: intuitive (operative), declaratory (communicative), argumentative (validating), and structural (institutionalized). The achievement of these levels of understanding for a concept or conceptual field will require the organization of specific didactic moments or situations, as Brousseau proposes in his didactic situations' theory.

Human action and intentionality

Our theoretical model also includes, as the primary unit of analysis, the notion of meaningful prototype practice, defined as the action that the person carries out in his/her attempts for solving a class of problem-situations and for which he/she recognizes or attributes a purpose (an intentionality). Therefore, this is a situated expressive form involving a problem-situation, an institutional



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context, a person and the semiotic instruments mediating the action. This notion is used to define mathematical objects as emerging from the systems of meaningful prototype practices. Consequently, understanding the object, in its integral or systemic sense, requires the subject, not only the semiotic and relational components, but to identify a role -an intention (Maier, 1988)- in the problem-solving process for the object.

Assessment of understanding

We conceive the processes for assessing understanding as the study of the correspondence between personal and institutional meanings. The evaluation of a subject's understanding is relative to the institutional contexts in which the subject participates. An institution (educational or not) will say that a subject "understands" the meaning of an object - or that he/she 'has grasped the meaning' of a concept, if the subject is able to carry out the different prototype practices that make up the meaning of the institutional object.

It is also necessary to recognize the unobservable construct character of personal understanding. Consequently, an individual's personal understanding about a mathematical object may be deduced from the analysis of the practices carried out by the person in solving problematic tasks, which are characteristics of that object. Since, for each mathematical object, the population of such tasks is potentially unlimited, the analysis of the task variables and the selection of the items to design evaluation instruments become of primary interest. The construct 'meaning of an object' we propose, in its two dimensions, personal and institutional, might be a useful conceptual tool to study the evaluation processes, the achievement of the 'good understanding', and the institutional and evolutionary factors conditioning them.

Summing Up and Concluding remarks

In this report, we explore some elements for a theory of understanding in mathematics, derived from our previous research work into the meaning of mathematical objects. Our proposal is that mental functioning and sociocultural settings be understood as dialectically interacting moments, or aspects of a more inclusive unit of analysis: human action. Nevertheless, it is necessary to recognize the analytical priority of studying institutional meanings. Since each person develops in different institutions and cultural settings, the psychological processes involved in understanding the linguistic or conceptual mathematical objects are mediated by institutional meanings, namely, by situations-problems, semiotic instruments, habits and shared conventions.

Finally, the systemic nature of meaning and understanding highlights the sampling character of teaching and assessment situations, and the inferential problems associated with their study. Therefore, we propose the characterization of the personal and institutional meaning of mathematical objects, and of its mutual interdependence and development as a priority research agenda for Mathematics Education.

References

1. Brousseau, G. (1986). *Fondements et méthodes de la didactique des mathématiques*.
2. *Recherches en Didactique des Mathématiques*, 7 (2): 33-115.
3. Bruner, J. (1990). *Acts of meaning*. Cambridge, MA: Harvard University Press.
4. Chevallard, Y. (1992). Concepts fondamentaux de la didactique: Perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*, 12 (1): 73-112.
5. Dörfler, W. (1991). Forms and means of generalization in mathematics. In: A. Bishop,
6. S. Mellin-Olsen and J. V. Dormolen (Eds), *Mathematical knowledge: Its growth through teaching* (pp.63-85). Dordrecht: Kluwer A.P.
7. Ernest, P. (1991). *The philosophy of mathematics education*. London: The Falmer Press.
8. Hiebert, J. & Carpenter, Th. P. (1992). Learning and teaching with understanding. In: D. W. Grouws (Ed.), *Handbook of research in teaching and learning of mathematics* (pp. 65-97). New York: Macmillan.
9. Godino, J. D. & Batanero, C. (1994). Significado personal e institucional de los objetos matemáticos. *Recherches en Didactique des Mathématiques*, 14 (3): 325-355. [Institutional and personal meaning of mathematical objects. *Journal für Mathematikdidaktik* (in press)].
10. Godino, J. D. & Batanero, C. (1996). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In: A. Sierpiska (Ed.), *What is Research in Mathematics Education, and What Are its Results?* Dordrecht: Kluwer A. P. (in press).
11. Johnson, M. (1987). *The body in the mind. The bodily basis of meaning, imagination, and reason*. Chicago: The University of Chicago Press.
12. Koyama, M. (1993). Building a two axes process model of understanding mathematics. *Hiroshima Journal of Mathematics Education* 1: 63-73



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13. Maier, H. (1988). Du concept de comprehension dans l'enseignement des mathematiques. In: C. Laborde (Ed.), Actes du Premier Colloque Franco-Allemand de Didactique et l'Informatique, (pp. 29-39). Grenoble: La Pensée Sauvage.
14. Pirie, S.E.B. & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? Educational Studies in Mathematics, 26 (3): 165-190.
15. Ruiz-Higueras, L. (1994). Concepciones de los alumnos de secundaria sobre la noción de función: Análisis epistemológico y didáctico. Ph D. Universidad de Granada.
16. Sierpiska, A. (1994). Understanding in mathematics. London: The Falmer Press.
17. Tymoczko, T. (ed.) (1986). New directions in the philosophy of mathematics. Boston: Kerkhauser.
18. Wittgenstein, L. (1953). Philosophical investigations. Oxford: Basil Blackwell [1958].

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