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(*p,q*)-EXTENSION AND UNIFICATION OF HYPERBOLIC FUNCTIONS AND CIRCULAR FUNCTIONS

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Abstract-The unification of the circular and hyperbolic functions is established. The (p,q)-analysis is performed to obtain extended form of the general function. The general function produces circular and hyperbolic functions as merely different states. (1, q)- De Moivre formula and (p,q)- Euler formula for the general function are proposed.

Keywords: (p,q)-extension, generalized trigonometric function, (1,q)- De Moivre formula, (p,q)- Euler formula for the general function

1. Introduction

The all-time burning issue in the theoretical and mathematical physics is the unification of basic laws or theories or definitions. The theorists ever devote their great efforts to formulate the ideas in the concise form. The conciseness is the major spirit of a theoretical framework. In this direction our study goes to unify circular functions and hyperbolic functions. The (p,q)-generalization yields an elegant form of the unified function. As a starting point to such study, Gauss hyper geometric functions proposed by Carl Frederic Gauss [1] plays the crucial role to include many functions as different states of it.

2. q-Generalization of Hyper Geometric Function

The Gauss hyper geometric function is given by

$$ {}_2F_1\left(\begin{matrix} a & , & b \\ & c & \end{matrix}; \; x\right) = 1+\frac{a}{1}\frac{b}{c}x+\frac{a}{1}\frac{(a+1)}{2}\frac{b}{c}\frac{(b+1)}{(c+1)}x^2+\ldots \qquad \ldots (1), $$

where |x|< 1 and c > 0.
Introducing Pochhammer symbols,

$$ (a)_j = a(a+1)(a+2)\ldots(a+j-1), \qquad \ldots (2), $$

the hyper geometric series attains the form

$$ {}_2F_1\left(\begin{matrix} a & , & b \\ & c & \end{matrix}; \; x\right) = \frac{(a)_0}{(1)_0}\frac{(b)_0}{(c)_0}x^0+\frac{(a)_1}{(1)_1}\frac{(b)_1}{(c)_1}x^1+\frac{(a)_2}{(1)_2}\frac{(b)_2}{(c)_2}x^2+\ldots=\sum_{k=0}\frac{(a)_k}{(1)_k}\frac{(b)_k}{(c)_k}x^k \quad \ldots (3). $$

Another way to represent the hyper geometric series in other elegant form is as follows. Let us represent Eq. (1) in the following identical form:

$$ {}_2F_1\left(\begin{matrix} a & , & b \\ & c & \end{matrix}; \; x\right) = 1+\frac{[a]_1}{1}\frac{[b]_1}{[c]_1}x+\frac{[a]_1}{1}\frac{[(a+1)]_1}{2}\frac{[b]_1}{[c]_1}\frac{[(b+1)]_1}{[(c+1)]_1}x^2+\ldots \qquad \ldots (4). $$

Eduard Heine [2] made the q-generalization of Eq. (4), in the following manner

$$ {}_2\phi_1\left(\begin{matrix} a & , & b \\ & c & \end{matrix}; \; x\right) = 1+\frac{[a]_q}{[1]_q}\frac{[b]_q}{[c]_q}x^1+\frac{[a]_q[a+1]_q}{[1]_q[2]_q}\frac{[b]_q[b+1]_q}{[c]_q[c+1]_q}x^2+\ldots \qquad \ldots (5), $$

with the requirement that

$$ \lim_{q\to 1} {}_2\phi_1 = {}_2F_1 \qquad \ldots (6). $$

It quite fruitful to recall that geometric series sum yields

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$$\lim_{q\to 1}\frac{1-q^n}{1-q}=\lim_{q\to 1}\left\{1+q+q^2+\ldots+q^{n-1}\right\}=n \qquad \ldots (7)$$

When $n\to\infty$ the sum becomes simply $\frac{1}{1-q}$

Heine introduced the Nobel idea:

$$[n]_q=\frac{1-q^n}{1-q} \quad \text{and} \quad [n]_q!=\prod_{j=1}^{n}[j]_q \qquad \ldots (8)$$

The (p,q) generalization of above equation is

$$[n]_{(p,q)}=\frac{p^n-q^n}{p-q}=p^{n-1}+p^{n-2}q+\ldots+pq^{n-2}+q^{n-1} \qquad \ldots (9)$$

$$[n]_{(p,q)}=\frac{p^n-q^n}{p-q}=p^{n-1}\frac{1-(q/p)^n}{1-(q/p)}=p^{n-1}[n]_{q/p} \qquad \ldots (10)$$

In accordance with Eq. (7), we again recall that

$$\lim_{n\to\infty}[n]_{(p,q)}=\frac{1}{p-q}$$

$$[n]_{p,q}!=\begin{cases}\prod_{j=1}^{n}[j]_{p,q}; & n\geq 1\\ 1 \quad ; & n=0\end{cases} \qquad \ldots (11)$$

An analogue of Heine extension should be

$${}_2\Phi_1\left(\begin{matrix}a & , & b\\ & c & \end{matrix};\ x\right)=1+\frac{[a]_{(p,q)}}{[1]_{(p,q)}}\frac{[b]_{(p,q)}}{[c]_{(p,q)}}x^1+\frac{[a]_{(p,q)}[a+1]_{(p,q)}}{[1]_{(p,q)}[2]_{(p,q)}}\frac{[b]_{(p,q)}[b+1]_{(p,q)}}{[c]_{(p,q)}[c+1]_{(p,q)}}x^2+\ldots \ldots (12)$$

3-Tsallis' q-Analysis[3]

Tsallis considered the following differential equation:

$$\frac{dy}{dx}-y^q=0, \quad y(0)=1 \qquad \ldots (13)$$

A wise inspection of Eq. (13) suggests that for q=1, we have the solution

$$y=e^x=e_1^x=e_1(x)$$

Method of separation of variables allows that for any arbitrary value of q, Eq. (13) can be solved to give

$$y=\left[1+(1-q)x\right]^{\left(\frac{1}{1-q}\right)}=e_q^x=e_q(x) \qquad \ldots (14)$$

Eq. (13) can alternatively be written as

$$\frac{dx}{dy}-y^{-q}=0, \quad y(0)=1 \qquad \ldots (15).$$

The direct observation yields that for q=1, Eq. (15) has the solution

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$$x = \ln y = \ln_1 y$$

Whence, for q ≠1, we have the solution

$$x = \ln_q y = \frac{y^{1-q} - 1}{1-q} \quad \ldots (16).$$

It can be verified that $ln_q x$ and e_q^x are inverse functions of each other. Moreover, we see that such extension leads to nonlinearity.

$$\ln_q (uv) = \ln_q (u) + \ln_q (v) + (1-q).\ln_q (u).\ln_q (v) \quad \ldots (17)$$

This nonlinear relation becomes linear only when q=1.

4- q-Exponential as Solution of Nonlinear Oscillator Equation [4]-[6]

Eq. (13) with solution as e_q^x suggests that:

$$\frac{d}{dx} e_q^x = \left(e_q^x\right)^q \quad \ldots (18)$$

Eq. (13) with substitution $z = y^a$ suggests that:

$$\frac{d}{dx} z^{\frac{1}{a}} = z^{\frac{q}{a}} \Rightarrow \frac{dz}{d(ax)} = z^{1-\left(\frac{1-q}{a}\right)} \Rightarrow z = e^{ax}_{1-\left(\frac{1-q}{a}\right)}$$

$$\left(e_q^x\right)^a = e^{ax}_{1-\left(\frac{1-q}{a}\right)} = e_{q'}^{ax} \quad \ldots (19)$$

Combining Eq. (18) and Eq. (19), we obtain an elegant chain of solutions:

$$\partial_x \left[e_q^{ax}\right] = a.e_{q'}^{\xi'}$$

$$\partial_x^2 \left[e_q^{ax}\right] = a^2.q.e_{q''}^{\xi''}$$

$$\partial_x^3 \left[e_q^{ax}\right] = a^3.q^2.q'.e_{q'''}^{\xi'''} \quad \ldots (20)$$

$$q' = 2 - \frac{1}{q}; \qquad q'' = 2 - \frac{1}{q'}; \qquad q''' = 2 - \frac{1}{q''}$$

$$\xi = ax; \qquad \xi' = q\xi; \qquad \xi'' = q'\xi'; \qquad \xi''' = q''\xi''$$

Using Eq. (20) we can obtain nonlinear oscillator differential equation

$$\frac{d^2}{dx^2}\left[e_q^{ikx}\right]^\nu + \gamma^2 \left[e_q^{ikx}\right]^\mu = 0, \qquad q = \frac{\mu - \nu}{2} + 1, \quad k^2 = \frac{2\gamma^2}{\nu(\mu + \nu)} \quad \ldots (21)$$

5-Taylor 'S Series For q-Exponential

Eq. (13) on successive differentiation yields

$$\frac{d^{n+1} y}{dx^{n+1}} = (1.q - 0)(2.q - 1)(3.q - 2)\ldots\{nq - (n-1)\} y^{\left[(n+1)q - n\right]} = Q_n . y^{\left[(n+1)q - n\right]} \quad \ldots (22)$$

$$e_q^x = 1 + \sum_{n=0}^{\infty} Q_n . \frac{x^{n+1}}{(n+1)!} \quad \ldots (23)$$

This expansion on replacing x by ix and then separating into real and imaginary parts, we get Euler formula for q-generalization

$$e_q^{ix} = \cos_q x + i \sin_q x \quad \ldots (24)$$

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Similarly, Eq. (23) can be expressed for hyperbolic entities

$$e_q^x = \cosh_q x + \sinh_q x \qquad \ldots (25)$$

6- p,q-Extension[7-9]

$$(x \oplus y)_{p,q}^n = \begin{cases} \left(p^0 x + q^0 y\right)\left(p^1 x + q^1 y\right)\left(p^2 x + q^2 y\right)\ldots\left(p^{n-1} x + q^{n-1} y\right) & ;\ n \geq 1 \\ 1 & ;\ n = 0 \end{cases} \qquad \ldots (26)$$

$$(x \oplus y)_{p,q}^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} p^{\binom{r}{2}} q^{\binom{n-r}{2}} x^r y^{n-r} \qquad \ldots (27)$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}; \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[r]_{p,q}![n-r]_{p,q}!} \qquad \ldots (28)$$

Keeping similarity with Eq. (14) and factors in Eq. (27), we introduce (p,q)- generalization of the exponential function with following choices:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} = \cosh_{p,q} x + \sinh_{p,q} x$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} = COSh_{p,q} x + SINh_{p,q} x$$

$$\tilde{e}_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!} = \tilde{c}osh_{p,q} x + \tilde{s}inh_{p,q} x \qquad \ldots. (29$$

6- Unification of Circular and Hyperbolic Functions

Let us introduce an $e_{p,q}[\ldots]$ symbol as follows

$$e_{p,q}\begin{bmatrix} r \\ s \end{bmatrix} ; \quad x = \frac{e_{p,q}\left(x(-1)^s\right) + r e_{p,q}\left(-x(-1)^s\right)}{2r^s}$$

$$E_{p,q}\begin{bmatrix} r \\ s \end{bmatrix} ; \quad x = \frac{E_{p,q}\left(x(-1)^s\right) + r E_{p,q}\left(-x(-1)^s\right)}{2r^s}$$

$$\tilde{e}_{p,q}\begin{bmatrix} r \\ s \end{bmatrix} ; \quad x = \frac{\tilde{e}_{p,q}\left(x(-1)^s\right) + r \tilde{e}_{p,q}\left(-x(-1)^s\right)}{2r^s} \qquad \ldots (30)$$

To ensure analyticity we impose that r≠0.

This newly introduced function gives

$$e\begin{bmatrix} \pm 1 \\ 2 \end{bmatrix} ; \quad x = \begin{matrix} \cosh \\ \sinh \end{matrix} x \qquad \ldots (31)$$

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$$e\begin{bmatrix} \pm 1 \\ \frac{1}{2} \end{bmatrix} ; \quad x = \begin{matrix} \cos \\ \sin \end{matrix} x \qquad \ldots (32)$$

Euler formulas can be expressed as

$$e_{p,q}(x) = e_{p,q}\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} ; \quad x + ie_{p,q}\begin{bmatrix} -1 \\ 1/2 \end{bmatrix} ; \quad x \qquad \ldots (33)$$

$$e_{p,q}(x) = e_{p,q}\begin{bmatrix} 1 \\ 2 \end{bmatrix} ; \quad x + e_{p,q}\begin{bmatrix} -1 \\ 2 \end{bmatrix} ; \quad x \qquad \ldots (34)$$

De Moivre Theorem for p=1 case can be given as

$$\left\{ \sum_{r=\pm 1} r^s . e_q \begin{bmatrix} r \\ s \end{bmatrix} ; \quad x \right\}^n = \sum_{r=\pm 1} r^s . e_{q'} \begin{bmatrix} r \\ s \end{bmatrix} ; \quad nx : \; 1.(1-q) = n.(1-q') \qquad \ldots (35)$$

7. Discussion and Conclusion

In case of (p,q)- extension, our choice to define generalized exponential is somewhat modified and we introduce the exponential with numerator depending on p only, depending on q only and free from p and q while we keep the same structure of denominator. We regard different states of newly introduced (p,q)-extended function as circular and hyperbolic functions in the limit p→q→1. The generalized functions $e_{p,q}(x)$, $E_{p,q}(x)$, $\tilde{e}_{p,q}(x)$ need not to satisfy the nonlinear harmonic oscillator equation unlike $e_q(x)$. The amplitude of oscillations remains constant in linear regime. The Euler formula is subjected to the choice of generalized function. Hopefully, the proposed unification and (p, q)-extension might play a role of immense use in theoretical physics and associated mathematics.

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